Electrical Engineering

A.S.Kasatkin
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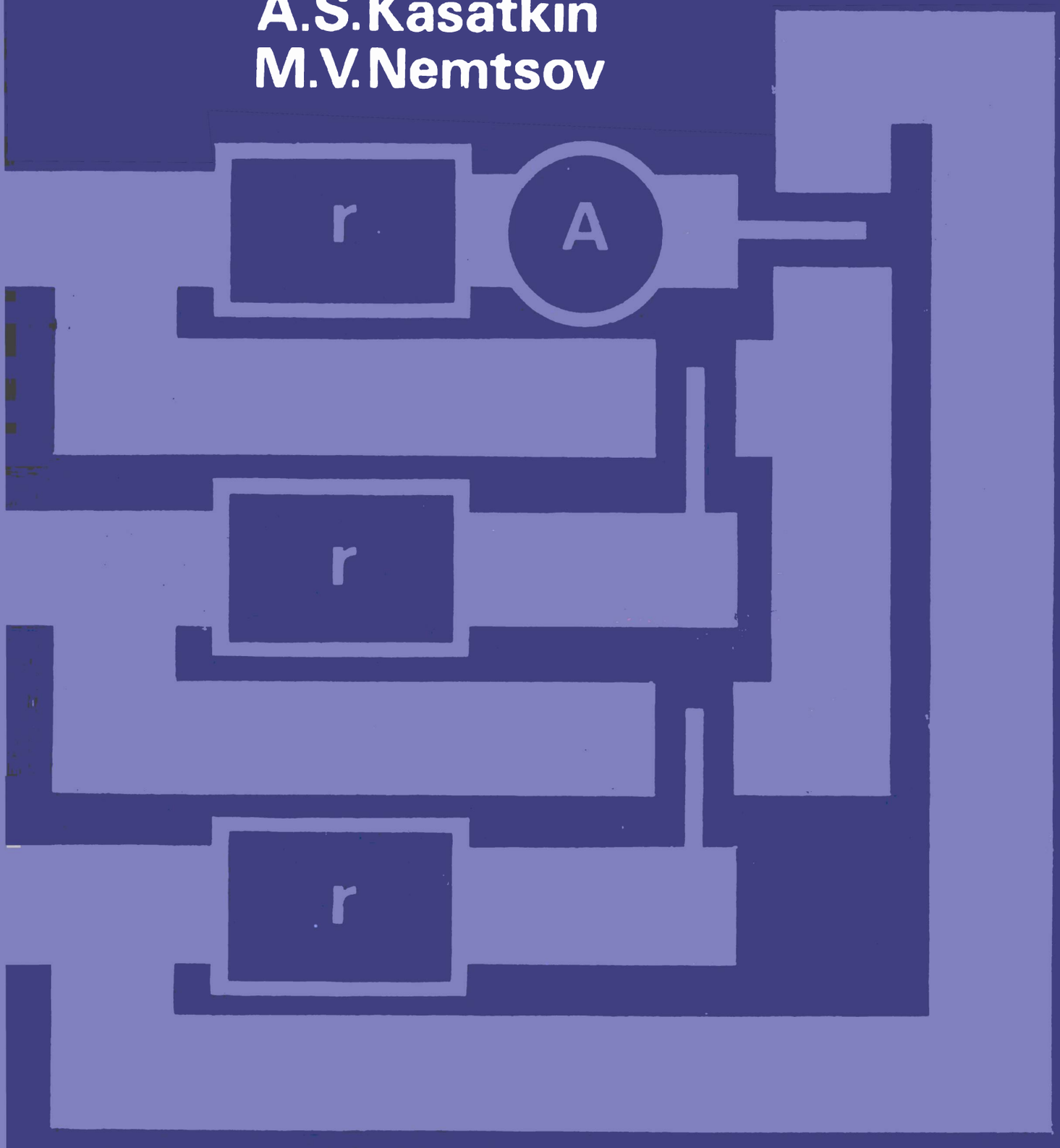

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ELECTRICAL ENGINEERING

А. С. Касаткин, М. В. Немцов

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A.S. Kasatkin and M.V. Nemtsov

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Preface

This is an English translation of the fourth Russian edition, revised and enlarged in comparison with the previous editions to bring it abreast with the requirements of a new curriculum on electrical engineering, and also to embody the experience accumulated in the use of the previous editions.

The book has been written to cover the course in electrical engineering as taught in the Soviet Union to students majoring in other departments. It is largely based on the third edition (1974) of the book by late Professor A. S. Kasatkin and also on the lectures his co-author has been reading at the Moscow Physical Engineering Institute for a long time now.

The course includes basic electricity and theory, design, principles of operation, characteristics, parameters, and applications of the most-used electrical circuits and apparatus.

Although no approach to teaching the subject can afford to be wholly non-mathematical, every effort has been made to restrict the use of mathematics to cases where a mathematical approach to the problem is either unavoidable or actively helpful to the student. On the whole, the book is intended as a framework, sound and accurate within its limits, into which the more advanced and detailed work can be fitted.

I would like to express my deep gratitude to the faculty members of the Chair of Theoretical Electricity and Electrical Measurements at the Ivanovo Power Engineering Institute, the Chair of Electrical Engineering (held by Professor V. G. Gerasimov) at the Moscow Power Engineering Institute, and also to Professor A. V. Netushil of the Moscow Chemical Engineering Institute, Associate Professor B. Ya. Zhukhovitsky, and Associate Professor V. N. Grasevich, who reviewed the manuscript, for valuable suggestions and ideas which have been incorporated in the book and improved the presentation.

M. V. Nemtsov

1 Direct-Current Circuits

1.1 Direct-Current Equipment

A systematic research into electricity and its practical applications historically began at the turn of the 18th century with studies of direct current, an essentially constant-value current that flows in only one direction. It was promoted by the availability of direct-current (d.c.) sources, at first primary cells (A. Volta, 1745-1827), then secondary or storage cells and batteries, and by early successes in the use of electricity for lighting (P. N. Yablochkov, 1847-1894), electrolysis and electroplating (B. S. Yakobi, 1801-1874).

Experimental investigations of the properties of direct current had revealed and explained a wide range of relations and concepts (A. M. Ampere, 1775-1836; G. S. Ohm, 1787-1854; C. A. Coulomb, 1736-1806; and others). Subsequent investigations (M. Faraday, 1791-1867; E. H. Lentz, 1804-1865; J. Henry, 1797-1878; W. Siemens, 1816-1892; J. P. Joule, 1818-1889; W. E. Weber, 1804-1891; J. C. Maxwell, 1831-1879; H. R. Hertz, 1857-1894; and others) had proved that most of the relations initially deduced from the analysis of d.c. circuits are the fundamental laws of electrical engineering.

To-day we have a wide range of direct-current *electric equipment* intended to perform some specific function in the generation, distribution, control, transformation and utilization of electric power. It covers storage batteries, d.c. power transmission lines, ammeters, rheostats, etc. The uses of direct current are as varied and include the manufacture of aluminium by electrochemical processes, electric traction in towns and on railways, electronics, and medicine, to name but a few fields of application.

As often as not thermal and hydraulic electric power stations have to be built away from industrialized areas. This poses the problem of devising economic methods for the conveyance of electric power over long and extremely long distances. As it turns out, this purpose can best be served by direct-current high-voltage power transmission lines and suitable power-handling facilities.

Of late impressive headway has been made in the development and improvement of a large variety of d.c. power sources. Among them are solar batteries which convert the Sun's radiant energy into electricity with the aid of photoelectric cells. They are the basic source of power supply for space vehicles in their flight. Another

example is magnetohydrodynamic (MHD) generators. When used on a fully commercial basis, they will substantially improve the efficiency of power generation.

1.2 The Elements of a D.C. Circuit

In the general case, a d.c. *electric circuit* contains a power source (or sources), pieces of power-consuming apparatus collectively called the load, instruments, switches, and connecting lines or wires.

A *power source* converts energy in some non-electric form into electric energy. For example, primary and secondary cells convert

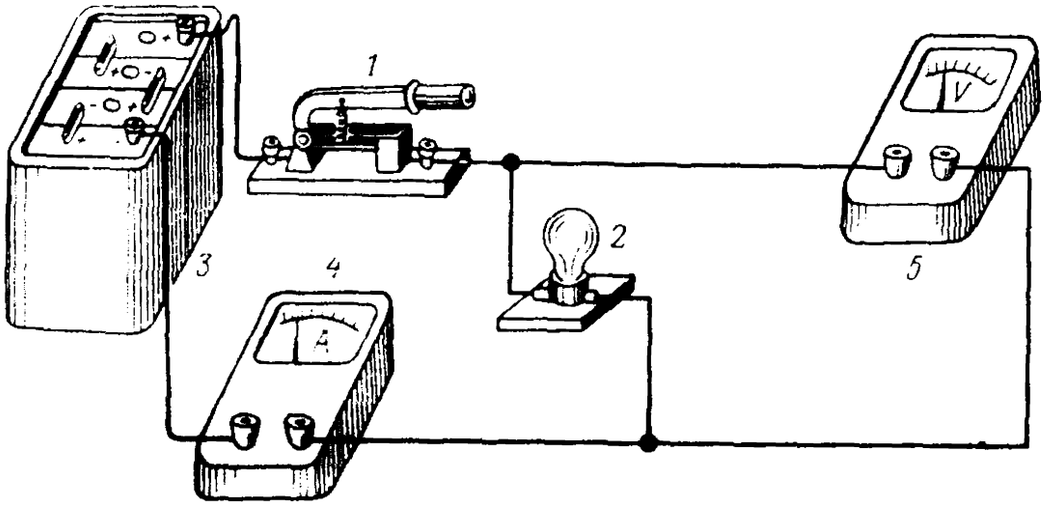


FIGURE 1.1

the energy of chemical processes into electricity, whereas thermocouples and their larger assemblies convert thermal into electrical energy.

The *load* converts electric energy into, say, mechanical (such as by d.c. motors), thermal (such as by electric furnaces), or chemical (such as in electrolysis cells).

The switches, connecting wires, and instruments serve to convey electricity from the power supply to distribute it among the load(s), and to give indications about the status of the circuit as a whole and of its elements.

It is usual to present circuits graphically in the form of *diagrams* of which there may be several types. The one shown in Fig. 1.1 is a pictorial diagram of a simple d.c. electric circuit. Closure of the knife-blade switch 1 connects the electric lamp (load) to the d.c. power supply, the storage battery 3. The current in the circuit is indicated by the ammeter 4, and the voltage across the circuit by the voltmeter 5.

Obviously, pictorial representation of an electric circuit yields a bulky and labour-consuming drawing. The task can substantially

be simplified if each circuit element is replaced with its adopted symbol (Fig. 1.2). Symbols schematically depict the functions performed by the elements, and so the drawings using them are quite aptly called *schematic circuit diagrams*. The schematic diagram of an electric circuit shows the purpose and interaction of all the elements involved, but it is not sufficient for the operating conditions

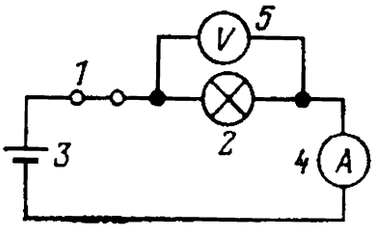


FIGURE 1.2

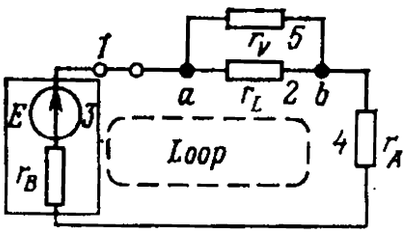


FIGURE 1.3

(voltages, currents, resistances, and the like) to be properly calculated. This need is filled by drawing up an equivalent circuit for each element involved.

The *equivalent circuit* of a real electric circuit is its quantitative model. It consists of an assemblage of idealized circuit elements chosen so that the processes in the actual network can be described to a sufficient degree of approximation. Interconnection of these idealized circuit elements produces an equivalent circuit of a network as a whole.

Equivalent circuits for various pieces of d.c. electric equipment will be discussed in detail in the sections that follow. For the time being, we will limit ourselves to the equivalent circuit (Fig. 1.3) of the elementary network shown in Fig. 1.1.

The configuration of an equivalent circuit is specified by the geometrical (topological) concepts of a branch, a node, and a loop (or mesh). A *branch* consists of one or several series-connected idealized elements each of which has two terminals (input and output), with the input terminal of a succeeding element connected to the output terminal of the previous one. A *node* is a point where three or more branches meet. A *loop* (or *mesh*) is a closed path passing over several branches so that any branch or any node is encountered only once.

The equivalent circuit in Fig. 1.3 has three branches of which two contain one element each, and the third contains three elements. The equivalent circuit also specifies the element parameters, namely the lamp resistance r_L, the voltmeter resistance r_V, the ammeter resistance r_A, the emf E of the storage battery, and the internal resistance r_B of the battery. The three branches meet at two nodes a and b.

If we know the parameters of all the elements in an equivalent circuit, we can, using the laws of electrical engineering, determine the operating conditions or status of all the real circuit components and of the network. That is why it is called the *equivalent circuit.*

1.3 The Positive Directions of Currents and Voltages

If in the electric circuit of Fig. 1.1 we close the knife-blade switch 1, a direct current will begin to flow through all the circuit components. In the equivalent circuit of Fig. 1.3, too, a flow of direct current will be indicated.

Let us take a closer look at what then happens in, say, the lamp filament, the connecting wires, and the coils of the instruments. To this end, we single out of a homogeneous cylindrical conductor a portion of length l (Fig. 1.4).

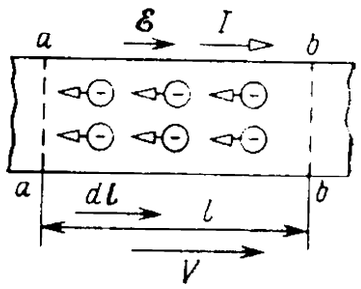


FIGURE 1.4

By the electron theory of electric conduction, valence electrons in metals can readily break loose from their parent atoms, thereby turning them into positive ions. In a solid, the ions form a crystal lattice whose structure shows spatial periodicity. Free (or roaming) electrons chaotically move between the atoms in the lattice (thermal motion) and collide with them.

The longitudinal electric field of strength $\mathscr{E}$, produced in the conductor by the power source, imparts the free electrons an additional velocity (drift velocity) and forces them to move additionally in one direction (along the conductor in Fig. 1.4). Thus, the direct current in metals is a relatively slow drift of free electrons, superimposed on the chaotic thermal motion occurring at a relatively high velocity. In the general case, the *direct current* in a conducting medium is an ordered motion of positive and negative charges under the action of an electric field. For example, in electrolytes and gases, these charges are positive and negative ions moving towards one another.

Since the positive and negative charges move in opposite directions, it is essential to agree on which direction should be taken for the direction of current flow. At present, the flow of current is assumed to be in the direction of motion of positive charges.

Direct current is defined as

$$I = Q/t$$

where t is the time of the uniform travel of the total positive charge Q through the cross-sectional area of the circuit portion in question;

The basic unit of current in the International System of Units is the *ampere*, symbolized A.

If we take two parallel straight conductors of infinite length and of an infinitesimal circular cross-sectional area, place them 1 m apart in a vacuum, and let a direct current of 1 A flow through them, there will be a force of interaction between the conductors, equal to 2×10^{-7} N m^{-1} (newton per metre).

In practice, use is often made of current units which are submultiples or multiples of one ampere. They are the microampere (μA):

$1\ \mu A = 1 \times 10^{-6}\ A$

the milliampere (mA):

$1\ mA = 1 \times 10^{-3}\ A$

the kiloampere (kA):

$1\ kA = 1 \times 10^{3}\ A$

and the megaampere (MA):

$1\ MA = 1 \times 10^{6}\ A$

The basic unit of charge (quantity of electricity) is the *coulomb*, C, defined as the quantity of electricity which passes any point in an electric circuit in one second (1 s) when the current is maintained constant at 1 A.

Voltage is a scalar quantity equal to the linear integral of the electric field strength. *Potential difference* is the voltage in an irrotational electric field where the voltage is independent of the path of integration. (The electric field in a d.c. circuit is irrotational.)

Direct voltage across a portion of a conductor (see Fig. 1.4) is defined as

$$V = \int_a^b \mathcal{E}\, dl = \varphi_a - \varphi_b$$

or

$$V = \int_a^b \mathcal{E}\, dl = \frac{1}{q} \int_a^b \mathbf{F}\, dl = W/q$$

where $\mathbf{F} = q\mathcal{E}$ is the force acting on the charge in a uniform d.c. electric field of strength $\mathcal{E}$; $W = \int_a^b \mathbf{F}\, dl$ is the work done by the electric field in moving the positive charge along the portion of the conductor; φ_a and φ_b are the potentials of the uniform d.c. electric field at sections a and b of the conductor.

The basic SI unit of voltage is the *volt* (V). It is defined as the voltage between the ends of a conductor in which a work of 1 joule (J) is done in moving a positive charge of 1 coulomb (C). There are also the submultiple and multiple units of voltage. They are the microvolt (μV):

$$1\ \mu V = 1 \times 10^{-6}\ V$$

the millivolt (mV):

$$1\ mV = 1 \times 10^{-3}\ V$$

the kilovolt (kV):

$$1\ kV = 1 \times 10^{3}\ V$$

and the megavolt (MV):

$$1\ MV = 1 \times 10^{6}\ V$$

In network calculations, the true directions of currents in the circuit elements are not known in the general case. Therefore, one has to adopt an assumed positive direction of current flow in all circuit elements.

The *assumed positive direction of current flow* in an element (of resistance r in Fig. 1.5) or a branch is chosen arbitrarily and is indicated by an arrow. If, with a given assumed positive direction of current flow, network calculations end up in the positive direction of current flow in the element involved, then the assumed direction is the same as the actual one. Otherwise, the assumed direction of current flow must be reversed.

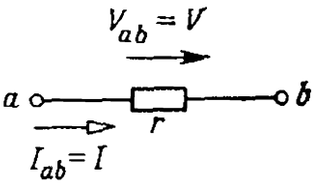


FIGURE 1.5

The *assumed positive direction of voltage* across a circuit element (see Fig. 1.5) is likewise chosen arbitrarily and is indicated by an arrow.

If the terminals of a circuit element are labelled (say a and b, as in Fig. 1.5) and the arrow points from a to b, the assumed positive direction implies that the voltage defined is

$$V = V_{ab}$$

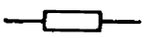
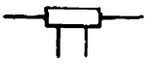
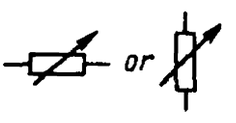
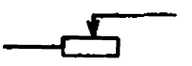
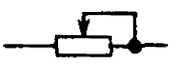
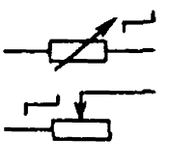
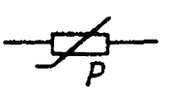
Similar notation is employed for current. For example, the symbol I_{ab} indicates the positive direction of current flow in a circuit element (see Fig. 1.5) from terminal a to terminal b. In a load, the positive directions are usually assumed the same for both current and voltages (see Fig. 1.5).

1.4 Resistors and Resistive Elements

Collisions between free electrons and the atoms in the crystal lattice of a conductor retard their translational (drift) motion. How frequently such collisions take place depends on the structure and properties of the conductor material. This opposition to the directed travel of free electrons, that is, to direct current, is the rationale of the property of *resistance* that any conductor has. The opposition to the flow of direct current in electrolytes and gases can be explained in similar terms.

An electric device which has the property of resistance and is used to limit the flow of current is called a *resistor*. A variable resistor which has one fixed terminal and a movable contact is called a *rheostat*. As an example, Fig. 1.6 shows a wire-wound rheostat with a sliding contact. The symbols for the various types of resistors are given in Table 1.1.

TABLE 1.1 Symbols for Resistors

Description	Symbol
Fixed resistor	
Same, tapped	
Variable resistor (rheostat):	
general symbol	
with break in circuit	
without break in circuit	
Variable resistor (rheostat) adjustable stepwise	
Nonlinear resistor self-adjustable according to an ambient parameter	

In equivalent circuits and in calculations, actual resistors are represented by their idealized models called *resistive elements*. An example of a resistive element is shown in Fig. 1.5. These models neglect the currents through the outer insulation, the former, and the other parts of an actual resistor (Fig. 1.6).

In fact, resistive elements are used as models not only for resistors, but also for other components which present an opposition to

the flow of direct current, irrespective of the physical nature of the opposition.

The parameter of a resistive element is *resistance*, r. The basic SI unit of resistance is the *ohm* (Ω). A conductor is said to have a resistance of 1 Ω, if a voltage of 1 V between the terminals of the conductor maintains a current of 1 A. There are also multiple units of resistance. They are the kilohm (kΩ):

$$1\ \text{k}\Omega = 1 \times 10^3\ \Omega$$

and the megohm (MΩ):

$$1\ \text{M}\Omega = 1 \times 10^6\ \Omega$$

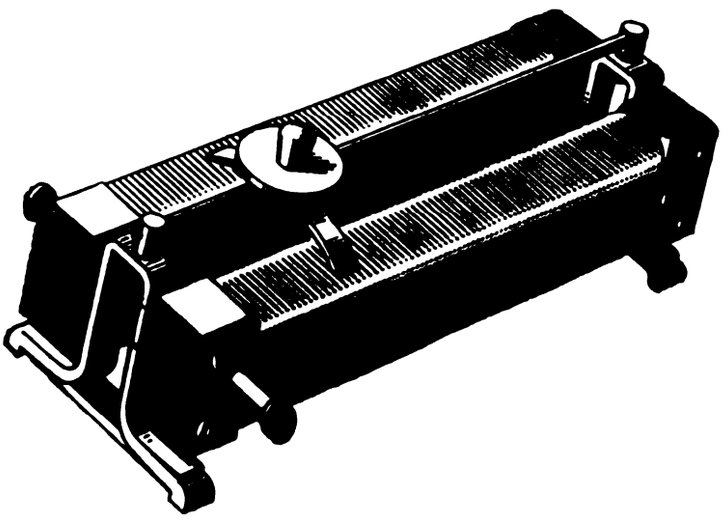


FIGURE 1.6

The conducting properties of materials are usually described in terms of *resistivity* which is defined as the resistance between the opposite faces of a unit cube of a given material. The symbol is ρ_v, and the sample of the material is usually taken to have an edge of 1 m.

Since the resistance of a conductor is directly proportional to its resistivity and length and inversely proportional to the cross-sectional area, the unit of resistivity is

$$1\ \Omega \times 1\ \text{m}^2/1\ \text{m} = 1\ \Omega\ \text{m}$$

The conductors of power-transmission lines and of coils are very long in comparison with their cross-sectional area. For this reason, it is usual to define the resistivity ρ of the materials for such applications as the resistance of a conductor 1 m long and 1 mm² in cross-sectional area. Therefore, their resistivity is expressed in $\Omega\ \text{mm}^2\ \text{m}^{-1}$ or, which is the same, μΩ m (Table 1.2).

The reciprocal of resistivity is called *conductivity*:

$$\gamma_v = 1/\rho_v$$

The reciprocal of resistance is called *conductance*:

$$g = 1/r$$

The basic SI unit of conductance is the *siemens* (S):

$$1\ \text{S} = 1\ \Omega^{-1}$$

and the unit of conductivity is S m^{-1}.

The theory of electric conduction tells us that the drift of free electrons in a conductor is mainly impeded by the vibrating atoms

TABLE 1.2 Resistivity and Temperature Coefficient of Resistance for Some Conductor Materials

Material	Resistivity, μΩ m, at 20°C	Temperature coefficient of resistance (per deg C)
Silver	0.016	0.0035
Commercial copper	0.0172-0.0182	0.0041
Aluminium	0.0295	0.0040
Steel	0.125-0.146	0.0057
Iron	0.09-0.11	0.0060
Cast iron	0.15	0.001
Lead	0.218-0.222	0.0039
Tungsten	0.0503	0.0048
Carbon	10-60	—0.005
Manganin (an alloy used in precision wire-wound resistors and instruments)	0.40-0.52	0.00003
Constantan (thermocouple alloy)	0.44	0.00005
Nichrome (20%Cr-80%Ni alloy for wire-wound resistors and heating elements)	1.02-1.12	0.0001

located at the sites of the crystal lattice, and not atoms as such. Because of this, the resistivity and, in consequence, the resistance of a conductor to direct current depend on temperature. In the general case, the relation is fairly complex. If, however, the temperature is varied within relatively narrow limits (about 200 degrees C), the dependence can be expressed as

$$r_2 = r_1 [1 + \alpha (\Theta_2 - \Theta_1)]$$

where r_1 and r_2 are the resistance at the temperatures Θ_1 and Θ_2, respectively, and α is the temperature coefficient of resistance equal to the relative change in resistance per degree of change in temperature (see Table 1.2).

Most pure metals have a positive temperature coefficient of resistance. For electrolytes and carbon or graphite products it is negative. The temperature coefficient of resistance of alloys used in precision wire-wound resistors (mainly for instruments) is close to zero. An example is manganin, a copper-nickel-manganese alloy (85% Cu, 12% Mn, 3% Ni).

The resistance of a conductor depends not only on the ambient temperature, but also on the current it carries. In the simplest case, this is because a rise in current leads to a rise in the velocity of free electrons and in the number of their collisions with the atoms occupying sites in the crystal lattice. In the final analysis, this leads to a rise in the temperature and, as a corollary, in the resistance of the conductor.

A resistive element of resistance which depends on current only slightly is said to have a *linear resistance* r = const. Otherwise, the resistive element has a *nonlinear resistance* r (I).

In a resistive element of resistance r, the current and the voltage are connected by a simple relation known as *Ohm's law*. Given the same positive direction for the current and voltage (see Fig. 1.5) we have

$$V_{ab} = rI_{ab} \text{ or } V = rI \tag{1.1}$$

For a uniform conductor of length l (in metres), of cross-sectional area A (in square metres), and with resistivity ρ_v, the voltage is given by

$$V = (\rho_v l/A)\, I = rI$$

where $r = \rho_v l/A$ is the resistance of the conductor in ohms.

1.5 Sources of Direct Current

Consider the principal processes taking place in primary cells, the most commonly used sources of direct current.

The simplest primary cell (Fig. 1.7 a) has two plates, or electrodes, one of copper and the other of zinc, immersed in a solution of sulphuric acid, $H_2SO_4 \rightarrow 2H^+ + SO_4^{--}$. By virtue of chemical processes, positive zinc ions Zn^{++} pass into solution, leaving an excess of negative free charges at the zinc plate. In solution, the heavy and sluggish positive zinc ions Zn^{++} crowd the light and mobile positive hydrogen ions H^+ towards the copper plate. At the surface of the copper plate, the hydrogen ions are restored to neutral hydrogen atoms. As this happens, the copper plate loses its free negative charges and acquires a positive charge. Thus, the zinc plate is charged negatively, and the copper plate, positively. As a result, a uniform electric field of strength $\mathscr{E}$ V m^{-1} is set up between them and opposes the directed travel of ions in solution. At some electric field strength $\mathscr{E} = \mathscr{E}_0$, the build-up of charge at the plates ceases. The voltage, or potential difference, between the cell plates corresponding to that instant gives a measure of the extraneous force (in our case, this is a chemical force) that tends to build up charge.

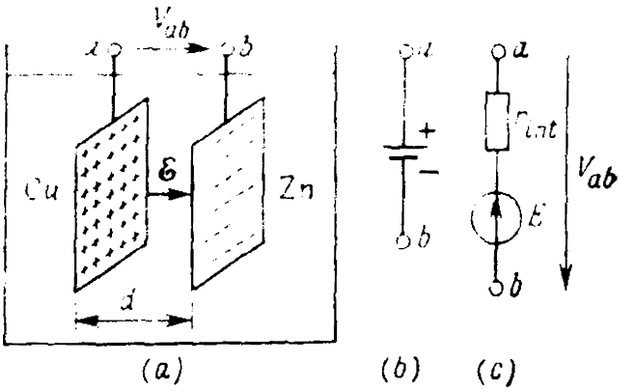


FIGURE 1.7

The quantitative measure of the extraneous force is called the *electromotive force*, or *emf* for short. For a primary cell, the emf is given by

$$E = \mathscr{E}_0 d = V_{ab,\,oc}$$

where E is the emf in volts if the spacing d between the plates is in metres, and $V_{ab,\,oc} = \varphi_{a,\,oc} - \varphi_{b,\,oc}$ is the voltage equal to the potential difference between the plate terminals with the external circuit open, that is, when no load is connected to the terminals a and b of the primary cell and no current is flowing in it.

If we connect a load, say, a resistor, to the terminals of the primary cell, the external circuit will be completed, and a current will begin to flow around it. The charge at each plate will diminish, and there will appear a directed flow of ions in the electrolyte. This directed flow is accompanied by collisions between the ions, and this gives rise to the *internal resistance* of the primary cell to direct current.

In sketch form, a primary cell is depicted as shown in Fig. 1.7 a, and its schematic symbol (the one used in schematic circuit diagrams) is given in Fig. 1.7 b. Its equivalent circuit appears in Fig. 1.7 c. As is seen, a primary cell can be visualized as a series combination of an emf (voltage) source E and a resistive element of resistance r_{int} equal to the internal resistance of the cell. The arrow at E defines the direction of flow for the positive charge inside the cell under the action of extraneous forces.

The equivalent circuit in Fig. 1.7 c is applicable to any other d.c. sources differing from primary cells in the physical nature of emf (voltage) and internal resistance.

1.6 The EMF (Voltage) Source and the Current Source

Let us see what happens in a simple electric circuit containing a power source connected to a load resistor. To begin with, we replace the power source with the equivalent circuit of Fig. 1.7 c, and the load resistor with a resistive element of fixed resistance r_{load}. The resultant equivalent circuit appears in Fig. 1.8 a where the letters a and b label the terminals of the power source.

The properties of a power source as an element of the equivalent circuit can be specified by giving its *external characteristic* — the dependence of the voltage between its terminals $V_{ab} = V$ on the source current I.

If the voltage and internal resistance of the power source remain constant (a linear source), its external characteristic can be written as

$$V_{ab} = \varphi_a - \varphi_b = E - r_{int} I \qquad (1.2)$$

represented by the straight line *a* in Fig. 1.9. The decrease in the source voltage between terminals *a* and *b* (see Fig. 1.8 a) with an increase in current is due to the increase in the voltage drop across the internal resistance r_{int} of the source (that is, the increase in the voltage across the resistive element r_{int}).

In many cases the internal resistance of the source is small in comparison with r_{load}, and the inequality $r_{int}I \ll E$ holds. In such

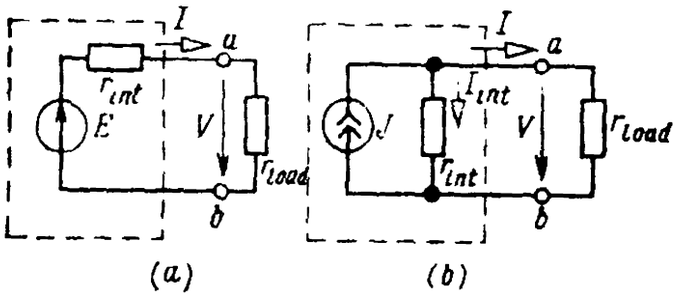


FIGURE 1.8

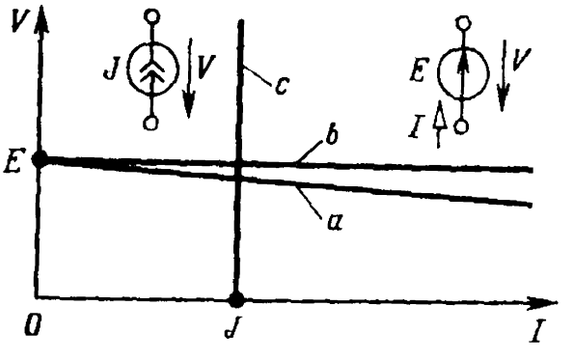


FIGURE 1.9

cases the voltage between the terminals of the source is practically constant and independent of the current flowing through it:

$$V_{ab} \approx E = \text{const}$$

A source with a small internal resistance may be replaced with an idealized model for which $r_{int} = 0$. This model is called an *ideal voltage generator* with only one parameter E. The voltage between the terminals of an ideal voltage generator is independent of the current through it. The external characteristic of an ideal voltage generator is defined as

$$V_{ab} = \varphi_a - \varphi_b = E = \text{const} \tag{1.3}$$

represented by the straight line *b* in the plot of Fig. 1.9. Also shown there is the diagram symbol for an ideal voltage generator.

In a number of special cases, notably in many circuits containing electron tubes and semiconductor devices, the internal resistance of the source is many times the load resistance r_{load} (which is external to the source). Subject to $r_{int} \gg r_{load}$, the source current in such circuits is defined as

$$I = E/(r_{int} + r_{load}) \approx E/r_{int} = I_{sc} = J$$

that is, it is practically equal to the current with the source short-circuited. A source with a large internal resistance may be replaced with an idealized model for which r_{int} tends to infinity and E to zero, and for which the equality

$$E/r_{int} = J$$

holds. This idealized source is called an *ideal current generator* with only one parameter $J = I_{sc}$. The current through an ideal current generator is independent of the external-circuit, or load, resistance r_{load}. A change in r_{load} brings about a change in the terminal voltage of the current generator:

$$V_{ab} = r_{load} J$$

The external characteristic of an ideal current generator is the straight line *c* in Fig. 1.9 which also gives the diagram symbol for it.

The equivalent circuit of a real source depicted as a series connection of an emf source E and a resistive element of resistance r_{int} (see Fig. 1.8 a) can be readily transformed into an equivalent circuit containing an ideal current generator. To this end, we divide every term in Eq. (1.2) by the internal resistance r_{int} of the source and obtain

$$V_{ab}/r_{int} = E/r_{int} - I$$

or

$$E/r_{int} = J = V_{ab}/r_{int} + I = I_{int} + I \tag{1.4}$$

Equation (1.4) may be interpreted as follows: The source current J is the sum of the current I through the resistive element r_{load} (in the external circuit) and the current I_{int} through the resistive element r_{int} connected between the terminals *a* and *b* of the source. The corresponding equivalent circuit appears in Fig. 1.8 b.

It is to be noted that the representation of real sources as ideal voltage generators or ideal current generators is equivalent with respect to the external circuit: In either case, the terminal voltages and the currents in the external circuit are the same.

However, the power relations in the two forms of equivalent circuit are not the same. The power delivered by the ideal voltage generator, EI (Fig. 1.8 a) is not equal to that developed by the ideal current generator, VJ (see Fig. 1.8 b). The power loss defined by Joule's law (the Joule loss) is not the same, either:

$$r_{int} I^2 \neq r_{int} I^2_{int}$$

1.7 Ohm's Law for a Simple Series Circuit

In its simplest form, Ohm's law defines the relation (1.1) between the current through and the voltage across the resistive element of a circuit. In the general case, Ohm's law defines the relation between the voltage across and the current through a simple series circuit.

Figure 1.10 shows a diagram of a simple series circuit containing a series combination of two resistive elements r_1 and r_2 and emf sources E_1 and E_2. The letters a, b, c, d and e label the junctions between the circuit elements. Taking advantages of Eqs. (1.1) and (1.3) and assuming that the positive direction of current flow is from junction a towards junction e, we may write expressions for the potentials at all the junctions.

According to Eq. (1.3),

$$V_{ed} = \varphi_e - \varphi_d = E_2$$

that is,

$$\varphi_e = \varphi_d + E_2 \tag{1.5a}$$

Similarly,

$$V_{bc} = \varphi_b - \varphi_c = E_1$$

that is,

$$\varphi_c = \varphi_b - E_1 \tag{1.5b}$$

According to Eq. (1.1),

$$V_{cd} = \varphi_c - \varphi_d = r_2 I$$

that is,

$$\varphi_d = \varphi_c - r_2 I \tag{1.5c}$$

and similarly,

$$\varphi_b = \varphi_a - r_1 I \tag{1.5d}$$

On eliminating the potentials at the junctions b, c and d between equations (1.5) by consecutive substitution, we can define the potential difference between the start (junction a) and the finish (junction e) of the simple series circuit:

$$\varphi_a - \varphi_e = V_{ae} = E_1 - E_2 + (r_1 + r_2) I_{ae}$$

Hence the current in the simple series circuit is given by

$$I = I_{ae} = (V_{ae} - E_1 + E_2) / (r_1 + r_2)$$

For a simple series circuit containing an arbitrary number of resistive elements and voltage sources, the *generalized Ohm's law* gives:

$$I = (V + \Sigma E)/\Sigma r \tag{1.6}$$

where the positive directions for the current and voltage within the circuit are chosen the same, Σr is the arithmetic sum of resistances of all resistive elements, and ΣE is the algebraic sum of all emfs.

The emfs for which the directions coincide with the assumed positive direction for the current are written with a "+" sign, and the emfs for which the actual direction is opposite to the adopted one are written with a "—" sign.

The electric condition of each element in the circuit is graphically presented by a *potential diagram*. In this diagram, the resistances between each pair of adjacent junctions are plotted as abscissae, and the potentials at the respective junctions as ordinates.

As an example, consider the potential diagram of the simple series circuit between junctions *a* and *e*, shown in Fig. 1.10. In order to construct the diagram, we first choose the origin of potentials, or

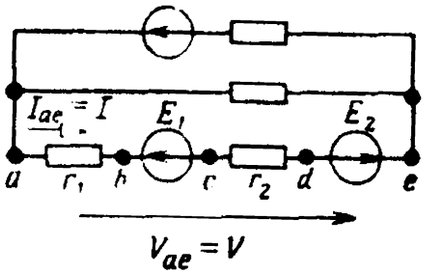


FIGURE 1.10

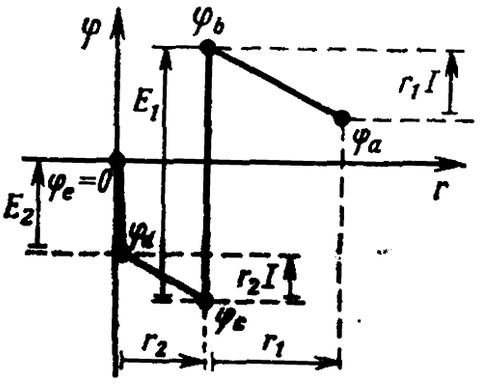


FIGURE 1.11

the datum potential which is assumed to be zero. This datum potential may be taken the one at any (arbitrary) junction. If one of the junctions is grounded (connected to earth), this grounded junction is taken as the datum point.

In our case, let the junction *e* be at zero potential, which fact is shown in Fig. 1.11. On moving along the circuit from the junction *e* to the junction *a*, we obtain the distribution of the potentials found by Eqs. (1.5). If we take some other junction as the datum (zero) potential, the diagram will only be shifted along the axis of potentials so that the datum potential lies on the axis of abscissae.

A potential diagram can be constructed not only for a simple series circuit, but for any circuit, provided we know the currents flowing through the resistive elements and, of course, all the emfs and the source resistances.

1.8 Kirchhoff's Laws

Kirchhoff's laws are the basic laws governing the analysis of all electric circuits. They were formulated by Kirchhoff from a multitude of experiments.

By *Kirchhoff's current (or first) law, the algebraic sum of the currents at any node in an electric circuit is zero*:

$$\sum_{k=1}^{n} I_k = 0 \tag{1.7}$$

Or, stated differently, the algebraic sum of the currents entering a node is equal to the algebraic sum of the currents leaving it. As an example, for the circuit in Fig. 1.12

$$I_1 + I_2 + I_4 = I_3 + I_5$$

or

$$I_1 + I_2 - I_3 + I_4 - I_5 = \sum_{k=1}^{5} I_k = 0$$

This law stems from the fact that electric charge cannot accumulate at the nodes of a d.c. circuit, since otherwise the node potentials and the branch currents would change.

By *Kirchhoff's voltage (second) law, in any closed loop of an electric circuit the algebraic sum of the voltages across the individual resistive*

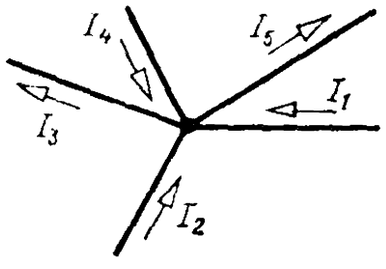


FIGURE 1.12

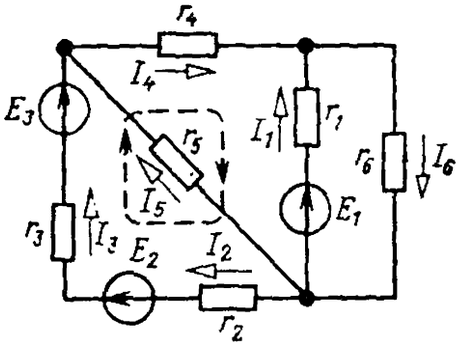


FIGURE 1.13

elements forming the loop is equal to the algebraic sum of the emfs in that loop. Referring to the equivalent circuit in Fig. 1.13, we can see that the algebraic sum of all voltages, which are the products of the currents by the resistances of the respective parts of the loop, is equal to the algebraic sum of all emfs:

$$\sum_{k=1}^{m} V_k = \sum_{k=1}^{m} r_k I_k = \sum_{k=1}^{n} E_k \tag{1.8}$$

where m is the number of resistive elements in the loop, and n is the number of emfs.

The currents and emfs enter Eq. (1.8) with a "+" sign if their directions are the same as that arbitrarily chosen for passing around

the loop. As an example, for the loop shown in Fig. 1.13 and for the direction of passing around the loop indicated by the arrow, we have

$$-r_1I_1+r_2I_2+r_3I_3+r_4I_4=\sum_{k=1}^{4} r_kI_k=\sum_{k=1}^{3} E_k=-E_1+E_2+E_3$$

1.9 Work, Energy and Power in D.C. Circuits. Power Balance

The *work* done in moving an electric charge Q from terminal a to terminal b in a simple series circuit is equal to the product of the charge by the terminal voltage $V_{ab} = V$:

$$W = QV$$

If the charge is in a uniform motion for a time t, that is, if the current is constant, $I_{ab} = I$, then the *charge* (*quantity of electricity*) is given by

$$Q = It$$

Then the work done over time t will be

$$W = VIt$$

The basic SI unit of work is the *joule* (J):

$$1 \text{ J} = 1 \text{ V A s}$$

The time rate of doing work is called *power*:

$$P = W/t = VI \tag{1.9}$$

The basic SI unit of power is the *watt* (W):

$$1 \text{ W} = 1 \text{ J s}^{-1} = 1 \text{ V A}$$

It is the power required to do work at the rate of 1 joule per second. There are also submultiple and multiple units of power. They are the milliwatt (mW):

$$1 \text{ mW} = 1 \times 10^{-6} \text{ W}$$

the kilowatt (kW):

$$1 \text{ kW} = 1 \times 10^{3} \text{ W}$$

and the megawatt (MW):

$$1 \text{ MW} = 1 \times 10^{6} \text{ W} = 1 \times 10^{3} \text{ kW}$$

Energy is the ability to do work, and its basic SI unit is likewise the joule. However, it is too small for practical use. Therefore the practical unit of energy is the *kilowatt hour* (kW h). It is equal to the

work done at a constant power of 1 kW over 1 h. Since 1 W s = 1 J, then

1 W h = 3600 W s = 3600 J

and

1 kW h = 3 600 000 J

In the case of resistive elements, the expression for power in a d.c. circuit, Eq. (1.9), can be transformed by replacing $V = rI$ or $I = V/r = gV$ on the basis of Ohm's law. This yields three expressions for the power in a resistive element of resistance r which is part of a d.c. circuit:

$$P_r = VI = rI^2 = gV^2 \qquad (1.10)$$

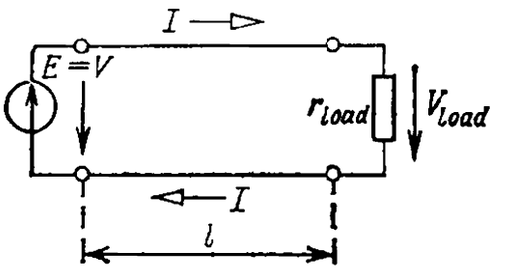


FIGURE 1.14

A very important fact is that the same power can be generated at a low and a high voltage. Let us see how the level of voltage affects the power distribution in a simple power transmission circuit consisting of an ideal voltage generator E, a power transmission line of length l and of resistance r_l, and a load of resistance r_{load} (Fig. 1.14).

By Kirchhoff's voltage law, the sum of the voltages across the line and load resistance is equal to the emf of the ideal voltage source:

$$r_l I + r_{load} I = r_l I + V_{oad} = E = V$$

where V_{load} is the load voltage, and V is the terminal voltage of the ideal voltage generator.

On multiplying all terms of the above equation by the current I, we obtain an equation defining the distribution of power in a circuit:

$$r_l I^2 + V_{load} I = VI$$

where $r_l I^2$ is the power lost (as heat) in the line conductors, $V_{load} I$ is the load power, and $VI = EI$ is the power of the emf source.

Now suppose that, while keeping the load power unchanged, we double the load voltage V_{load} by increasing its resistance r_{load} four-fold and using a larger emf source. As a result, the current in the circuit will fall to

$$I' = I/2$$

In consequence, the power lost in the line conductors will be cut down to one-quarter of the previous amount:

$$r_l (I')^2 = r_l I^2/4$$

Thus, in order to reduce the loss of power in power transmission lines, it will be a good plan to transmit power at the highest feasible voltage.

In any circuit, it is important to maintain the power balance: *The algebraic sum of the powers due to all sources (notably current and voltage generators) must be equal to the arithmetic sum of the powers of all the loads (notably, resistive elements)*:

$$\Sigma V_{\text{source}} I_{\text{source}} = \Sigma r I_r^2$$

or

$$\Sigma P_{\text{source}} = \Sigma P_r \tag{1.11}$$

If we take into consideration the internal resistances r_{int} of the sources, the power of each source, $V_{\text{source}} I_{\text{source}}$, will be less than the power developed by each source, EI_{source}, by the power loss $r_{\text{int}} I^2_{\text{source}}$.

The power of a source should be taken positive and written in the power balance equation (1.11) with a "+" sign, if the positive direction of I_{source} is the same as that of the emf. Otherwise, the power must be taken negative and written in the power balance with a "—" sign (such as when a storage battery is being charged).

As an example, let us write a power balance for the circuit of Fig. 1.13 on the assumption that all the sources are ideal voltage generators, that is, $V_{\text{source}} = E$:

$$E_1 I_1 + E_2 I_2 + E_3 I_3 = r_1 I_1^2 + r_2 I_2^2 + r_3 I_3^2 + r_4 I_4^2 + r_5 I_5^2 + r_6 I_6^2$$

It is to be noted that ideal voltage and current generators can develop an infinitely large power. To demonstrate, let us connect a load of resistance r_{load} to each source. In the case of an ideal voltage generator, if r_{load} tends to zero, the current I tends to infinity, and the power $P = EI$ tends to infinity, too. In the case of an ideal current generator, if r_{load} tends to infinity, the voltage V also tends to infinity, and the power $P = VJ$ tends to infinity.

1.10 Use of Ohm's Law and Kirchhoff's Laws in Circuit Calculations

Using Ohm's law, Eq. (1.1), and Kirchhoff's laws, Eqs. (1.7) and (1.8), we can calculate the conditions (voltages, currents, powers, and resistances) in any electric circuit, however complex.

The general problem in circuit calculations is to find the currents in all parts of a circuit, given the circuit parameters and configuration.

So that we can write equations by Ohm's and Kirchhoff's laws, we should above all adopt (arbitrarily) the positive directions for the currents in all the branches of the circuit in question.

In writing equations for the nodes of the circuit by Kirchhoff's current law, it should be borne in mind that the number of *independent* equations should be by one fewer than the total number n of nodes. In other words, we are to write $n - 1$ equations. The point is that if we wrote equations for all the n nodes, the current in each branch would appear twice in the node equations, because each branch joins two nodes of which one is positive and the other is negative (the current in each branch enters one of the nodes and, in consequence, leaves the other). On adding together the left-hand sides of all the equations set up by Kirchhoff's current law, Eq. (1.7), we get zero identically.

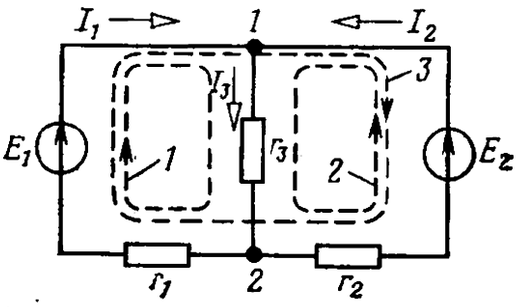


FIGURE 1.15

In writing equations on the basis of Kirchhoff's voltage law, it is likewise required to develop an independent set of equations. Notably, an independent system can be obtained, if the loops are chosen so that every next contains at least one branch not contained in the loops for which the equations have already been written. Such loops are called independent, because their equations are mutually independent.

The number of unknown currents is equal to the number b of branches. Since, by Kirchhoff's current law, we have $n - 1$ independent equations, then, on the basis of Kirchhoff's voltage law, we should have $b - (n - 1)$ equations.

As an example, for the circuit of Fig. 1.15, we could write two equations on the basis of Kirchhoff's current law (1.7) for two nodes (the assumed positive directions of current flow are indicated by the arrows). For node 1:

$$I_1 + I_2 - I_3 = 0 \tag{1.12}$$

For node 2:

$$I_3 - I_1 - I_2 = 0 \tag{1.13}$$

On the basis of Kirchhoff's voltage law, Eq. (1.8), and Ohm's law, Eq. (1.1), we could write three equations for three loops. For loop 1:

$$r_1 I_1 + r_3 I_3 = E_1 \tag{1.14}$$

For loop 2:

$$r_2 I_2 + r_3 I_3 = E_2 \tag{1.15}$$

For loop 3:

$$r_1 I_1 - r_2 I_2 = E_1 - E_2 \tag{1.16}$$

However, the circuit in Fig. 1.15 has only three branches, so only three independent equations are needed for its calculation. For this circuit, the number of nodes is $n = 2$, which means that on the basis of Kirchhoff's law only one independent equation must be written: Eq. (1.12) or Eq. (1.13). Indeed, one of the two equations can be derived from the other by multiplying it by -1.

Of the three loops shown in Fig. 1.15, we may take as independent, say, loops 1 and 2 (loop 2 contains branch 2 which does not enter loop 1). Then Eq. (1.16) can be set up by taking the differences of the right-hand and left-hand sides of Eqs. (1.14) and (1.15), respectively. Alternatively, any other pair of loops may be taken as independent for the circuit in Fig. 1.15.

For planar circuits, that is, those which can be drawn with no branches intersecting, an independent set of equations by Kirchhoff's voltage law can be obtained, for example, if for writing the equation we choose meshes, that is, loops which do not enclose the branches that do not enter the meshes. An example of a planar circuit is shown in Fig. 1.15 where loops 1 and 2 are meshes, and loop 3 encloses branch 3 which is not part of that loop.

Solving the set of independent equations simultaneously yields the branch currents. If any one current turns out to be negative, this is an indication that its actual direction, or sense, is opposite to the assumed positive direction.

As often as not, it takes a good deal of time in order to solve simultaneously the set of equations written on the basis of Kirchhoff's laws. Therefore, wherever possible, simpler methods ought to be used. These simpler methods, likewise based on Ohm's and Kirchhoff's laws, are examined in the sections that follow.

1.11 Series-Parallel Connection of Resistive Elements

When a circuit contains only one source, that part of the circuit which is external to the source may be looked upon as a series-parallel connection of resistive elements. In such a circuit, there may be several resistive elements connected to one another in parallel and in series with other resistive elements as shown in Fig. 1.16.

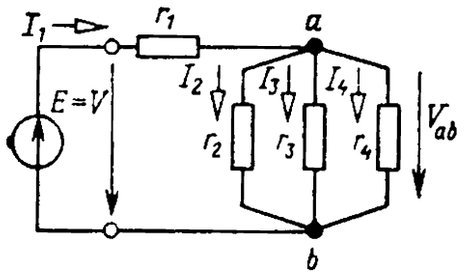


FIGURE 1.16

In calculating a series-parallel circuit, the first step is to determine the equivalent conductance g_{eq} of each parallel connection. This is done by adding together the conductances of all the resistive elements connected in parallel, that is, to the same pair of nodes. In the circuit of Fig. 1.16, three resistive elements of resistances r_2, r_3 and r_4 are connected between two nodes a and b.

Since $g_2 = 1/r_2$, $g_3 = 1/r_3$ and $g_4 = 1/r_4$, the equivalent conductance is found to be

$$g_{eq} = 1/r_2 + 1/r_3 + 1/r_4$$

On replacing the parallel connection of resistive elements with an equivalent resistive element of resistance $r_{eq} = 1/g_{eq}$, we obtain an equivalent circuit containing a series connection of two resistive elements, r_1 and r_{eq}.

The current in the common circuit is

$$I_1 = V/(r_1 + r_{eq})$$

In order to find the currents in the parallel branches, we should first determine the voltage between nodes a and b:

$$V_{ab} = r_{eq} I_1$$

then the branch currents by Ohm's law:

$$I_2 = V_{ab}/r_2$$
$$I_3 = V_{ab}/r_3$$
$$I_4 = V_{ab}/r_4$$

1.12 The Delta-Star Transformation

In some cases, the calculation of a complex electric network can be simplified if in its equivalent circuit we replace some group of resistive elements connected in a particular way with another, but equivalent group using a different form of connection. The equivalence of the two groups of resistive elements is based on the fact that, upon interchange, the operating conditions in the remaining part of the network do not change. In a simple way, this form of transformation was considered in Sec. 1.11 where several parallel resistive elements were replaced with one equivalent element.

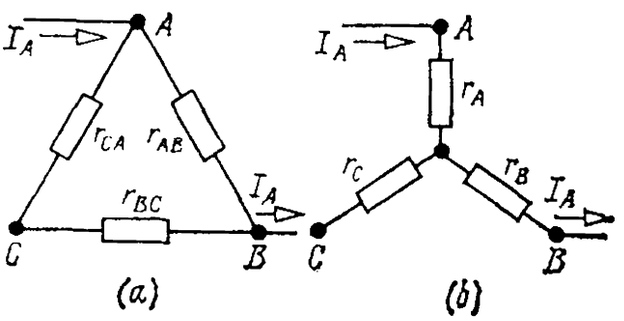


FIGURE 1.17

In the general case a collection of resistive elements connected in an n-arm star can be transformed into an n-sided polygon. The inverse transformation is possible in a limited number of cases. The transformation works both ways between the resistive elements connected in a delta and a three-arm star. This delta-star transformation is frequently used in the analysis of d.c. circuits, but it is especially important in the analysis and design of three-phase networks (see Chap. 3).

The simplest way to determine the condition of equivalence between a delta and a star (Fig. 1.17) is by equating the resistances and

conductances between the like nodes (or terminals) in the two networks. Suppose that a source is first connected between nodes A and B in the delta (see Fig. 1.17 a), then between the same nodes in the star (see Fig. 1.17 b). If the delta and the star are equivalent, the source current I_A must be the same in each case. This will happen if the conductance and, in consequence, the resistance between nodes A and B are the same in the delta and in the star. The same reasoning must hold when a source is connected between nodes B and C, then between C and A.

The conductance between nodes A and B in the delta is

$$\frac{1}{r_{AB}}+\frac{1}{r_{BC}+r_{CA}}=\frac{r_{AB}+r_{BC}+r_{CA}}{r_{AB}r_{BC}+r_{CA}r_{AB}}$$

The resistance between nodes A and B is the reciprocal of the conductance between the same nodes, that is,

$$(r_{AB}r_{BC}+r_{CA}r_{AB})/(r_{AB}+r_{BC}+r_{CA})$$

In a star, the resistance between the same nodes A and B is equal to the sum of the resistances in the two respective branches, $r_A + r_B$.

According to the condition of equivalence, the following equality must be satisfied:

$$r_A+r_B=\frac{r_{AB}r_{BC}+r_{CA}r_{AB}}{r_{AB}+r_{BC}+r_{CA}}=\frac{r_{AB}r_{BC}+r_{CA}r_{AB}}{\Sigma r_\Delta} \tag{1.17}$$

where Σr_Δ is the sum of the resistances in all the branches of the delta.

The structure of a delta and of a star is symmetrical with respect to the nodes. Therefore, the equations defining the equality of the resistances between nodes B and C, and then between nodes C and A can be derived from Eq. (1.17) by a simple cyclic replacement of the indexes:

$$r_B+r_C=\frac{r_{BC}r_{CA}+r_{AB}r_{BC}}{\Sigma r_\Delta} \tag{1.18}$$

$$r_C+r_A=\frac{r_{CA}r_{AB}+r_{BC}r_{CA}}{\Sigma r_\Delta} \tag{1.19}$$

In order to find the branch resistance r_A in a star, we need to add together Eqs. (1.17) and (1.19), to subtract Eq. (1.18) from the sum, and to divide the remainder by 2. The result will be

$$r_A = r_{AB}r_{CA}/\Sigma r_\Delta \tag{1.20}$$

The remaining two expressions for the branch resistances in a star are derived by a simple cyclic replacement of the indexes:

$$r_B = r_{BC}r_{AB}/\Sigma r_\Delta \tag{1.21}$$

$$r_C = r_{CA}r_{BC}/\Sigma r_\Delta \tag{1.22}$$

When all the branches in a delta have the same resistance, that is,

$$r_{AB} = r_{BC} = r_{CA} = r_{\Delta}$$

the branch resistances in the equivalent star will also be the same, such that

$$r_Y = r_{\Delta}/3 \tag{1.23}$$

Sometimes, the need may arise to go back from a star to a delta. In order to set up an equation for this transformation, we multiply pairwise Eqs. (1.20) through (1.22) and add together the products formed. The result will be

$$r_A r_B + r_B r_C + r_C r_A = r_{AB} r_{BC} r_{CA} / (r_{AB} + r_{BC} + r_{CA})$$

On dividing the above equation by Eq. (1.22), we find the branch resistance in the equivalent delta:

$$r_{AB} = r_A + r_B + r_A r_B / r_C \tag{1.24}$$

Expressions for the resistances of the other two branches can be found by a simple cyclic replacement of indexes in Eq. (1.24):

$$r_{BC} = r_B + r_C + r_B r_C / r_A \tag{1.25}$$

$$r_{CA} = r_C + r_A + r_C r_A / r_B \tag{1.26}$$

A typical example of how circuit calculations can be simplified through the delta-star transformation is shown in Fig. 1.18 a and b.

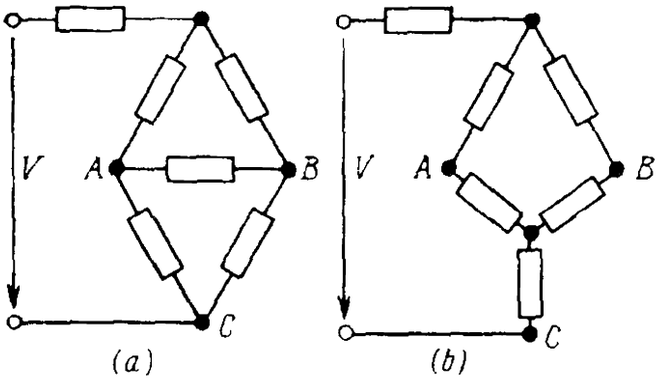


FIGURE 1.18

After one of the deltas in a bridge circuit is replaced with an equivalent star connection, the entire network can be regarded as a series-parallel connection of resistive elements.

1.13 The Node-Pair Method

Frequently, the circuit under analysis contains only two nodes or can be readily transformed to such a network. If so, the simplest way to analyse the network is by the node-pair (or nodal voltage) method.

Figure 1.19 shows a network which has nodes a and b and n branches. Since the branches between nodes a and b are connected in parallel, the potential difference, or voltage, between the nodes can be expressed in terms of the emf E_k, the current I_k and the resistance r_k of any branch. By the generalized Ohm's law (1.6):

$$I_k = (V_{ba} + E_k)/r_k = (E_k - V_{ab})/r_k$$

Hence,

$$V_{ab} = \varphi_a - \varphi_b = E_k - r_k I_k$$

where V_{ab} is the *node-pair voltage*.

From the above relations, the current

$$I_k = (E_k - V_{ab})\, g_k \qquad (1.27)$$

where $g_k = 1/r_k$ is the conductance of the kth branch.

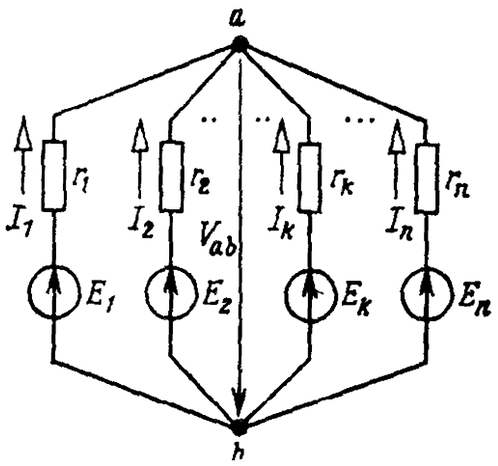


FIGURE 1.19

In the network of Fig. 1.19 it is assumed that all emfs are directed towards one node (node a) and that the positive direction of each current is the same as the sense of the emf in the respective branch or, which is the same, all currents enter that node. Actually, however, some of the emfs may be directed towards node b. Then in finding the branch currents having the same positive sense (that is entering node a) by Eq. (1.27), the emfs directed towards node b must be written with a "—" sign.

By Kirchhoff's current law, the algebraic sum of the currents entering and leaving a node is equal to zero. Therefore,

$$I_1 + I_2 + \ldots + I_k + \ldots + I_n = \sum_{k=1}^{n} I_k = 0$$

and, in consequence,

$$\sum_{k=1}^{n} (E_k - V_{ab})\, g_k = 0$$

Hence, the node-pair voltage can be written in terms of the circuit parameters as follows:

$$V_{ab} = \sum_{k=1}^{n} E_k g_k \Big/ \sum_{k=1}^{n} g_k \qquad (1.28)$$

where the emfs directed towards the node a are written with a "+" sign.

Once we know the node-pair voltage, we can readily find the current in any branch by Eq. (1.27).

1.14 The Loop-Current Method

This method can be used to analyse any linear circuit. With it, fewer simultaneous equations have to be solved than when the equations are written by Kirchhoff's law.

To analyse a complex network (Fig. 1.20 a) by the loop-current method, we choose independent loops in the same manner as the meshes are chosen when the equations are written by Kirchhoff's voltage law (Fig. 1.20 b). Then in each loop, the positive direction of current is chosen—this is the *loop current*; it is the same in all the branches of the loop. The symbol of the unknown loop current bears a repeated loop index, for example I_{kk}. The currents in the branches that are common to two or more loops are found on the basis of Kirchhoff's current law as the algebraic sums of the respective loop currents.

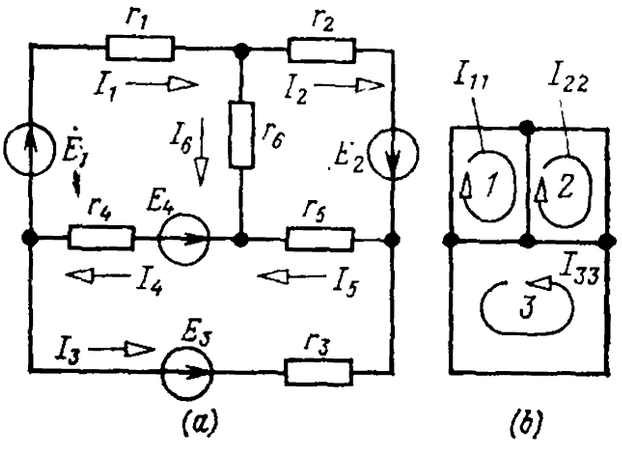


FIGURE 1.20

The algebraic sum of the emfs in all the branches entering each of the chosen independent loops is called the *loop emf*, and its symbol likewise takes a repeated loop index, for example E_{kk}. For any kth loop, it is defined as

$$E_{kk} = \Sigma E \tag{1.29}$$

In equation (1.29), the branch emfs whose sense is in the direction in which we pass around the loop are written with a "+" sign; otherwise, they are written with a "—" sign.

The arithmetic sum of the resistances of all resistive elements entering each of the chosen loops is called the *self-resistance of a loop*, and is designated as r_{kk} (it takes the repeated index of the respective loop), that is,

$$r_{kk} = \Sigma r \tag{1.30}$$

The arithmetic sum of the resistances of the resistive elements in the branches common to two loops m and l is called the *mutual resistance* of those loops. It is obvious that

$$r_{ml} = r_{lm}$$

As is the case with branch currents, the loop currents must obey Kirchhoff's voltage law. Therefore, we may write a set of loop equations for loop currents on the basis of Kirchhoff's voltage law for a network having n independent loops:

$$\begin{aligned}
&r_{11}I_{11} + r_{12}I_{22} + \ldots + r_{1n}I_{nn} = E_{11}\\
&r_{21}I_{11} + r_{22}I_{22} + \ldots + r_{2n}I_{nn} = E_{22}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&r_{n1}I_{11} + r_{n2}I_{22} + \ldots + r_{nn}I_{nn} = E_{nn}
\end{aligned} \tag{1.31}$$

The set of equations (1.31) is a mathematical statement for the loop current method. Since the loop currents (Fig. 1.20 b) are always fewer in number than the branch currents (Fig. 1.20 a), analysis by the loop current method involves fewer unknowns in the set of equations being solved. This fact is especially important in the analysis of complex networks.

The set of equations (1.31) can be resolved for loop currents in the general form, using determinants:

$$I_{kk} = \frac{\Delta_{k1}}{\Delta}E_{11} + \frac{\Delta_{k2}}{\Delta}E_{22} + \ldots + \frac{\Delta_{kk}}{\Delta}E_{kk} + \ldots + \frac{\Delta_{kn}}{\Delta}E_{nn} \tag{1.32}$$

where Δ is the determinant of the system

$$\Delta = \begin{vmatrix}
r_{11}r_{12} & \ldots & r_{1k} & \ldots & r_{1n}\\
r_{21}r_{22} & \ldots & r_{2k} & \ldots & r_{2n}\\
\ldots & \ldots & \ldots & \ldots & \ldots\\
r_{k1}r_{k2} & \ldots & r_{kk} & \ldots & r_{kn}\\
\ldots & \ldots & \ldots & \ldots & \ldots\\
r_{n1}r_{n2} & \ldots & r_{nk} & \ldots & r_{nn}
\end{vmatrix} \tag{1.33}$$

and Δ_{kp} ($p = 1, 2, \ldots, n$) is the signed minor, or cofactor, obtained from the determinant by eliminating the kth row and the pth column and multiplying the resultant determinant by $(-1)^{k+p}$.

As an example, we will analyse the network of Fig. 1.20 a which contains six branches and the three elementary loops shown in Fig. 1.20 b. The procedure is as follows. We assume that all the emfs and the resistances of the resistive elements are specified, assign suitable indexes to the elements of each branch and all the branch currents, and find the six unknown branch currents by the loop current method. On assigning the positive sense to each loop current so that it is in the same direction as we pass around the respective loop, write the following equations:

$$r_{11}I_{11} - r_{12}I_{22} + r_{13}I_{33} = E_{11}$$

or

$$r_1 + r_4 + r_6) I_{11} - r_6 I_{22} + r_4 I_{33} = E_1 - E_4 \quad (1.34a)$$

$$- r_{21} I_{11} + r_{22} I_{22} + r_{23} I_{33} = E_{22}$$

or

$$- r_6 I_{11} + (r_2 + r_5 + r_6) I_{22} + r_5 I_{33} = E_2 \quad (1.34b)$$

$$r_{31} I_{11} + r_{32} I_{22} + r_{33} I_{33} = E_{33}$$

or

$$r_4 I_{11} + r_5 I_{22} + (r_3 + r_4 + r_5) I_{33} = E_3 - E_4 \quad (1.34c)$$

where I_{11}, I_{22} and I_{33} are the loop currents; $E_{11} = E_1 - E_4$, $E_{22} = E_2$ and $E_{33} = E_3 - E_4$ are the loop emfs; $r_{11} = r_1 + r_4 + r_6$, $r_{22} = r_2 + r_5 + r_6$ and $r_{33} = r_3 + r_4 + r_5$ are the self-resistances of the loops; and $r_{12} = r_{21} = r_6$, $r_{13} = r_{31} = r_4$ and $r_{23} = r_{32} = r_5$ are the mutual resistances of the first and second, of the first and third, and of the second and third loops.

Using Eq. (1.32), we can write the solution of Eqs. (1.34) in the general form as follows:

$$I_{kk} = \frac{\Delta_{k1}}{\Delta} E_{11} + \frac{\Delta_{k2}}{\Delta} E_{22} + \frac{\Delta_{k3}}{\Delta} E_{33}$$

where the third-order determinant is

$$\Delta = \begin{vmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix}$$

$$= r_{11} r_{22} r_{33} + r_{21} r_{32} r_{13} + r_{31} r_{12} r_{23} - r_{11} r_{32} r_{23} - r_{21} r_{12} r_{33} - r_{31} r_{22} r_{13}$$

$$= r_{11} \Delta_{11} + r_{21} \Delta_{21} + r_{31} \Delta_{31}$$

whereas the signed minors, or cofactors, of the determinant are:

$$\Delta_{11} = \begin{vmatrix} r_{22} & r_{23} \\ r_{32} & r_{33} \end{vmatrix} = r_{22} r_{33} - r_{23} r_{32}$$

$$\Delta_{12} = \Delta_{21} = - (r_{12} r_{33} - r_{32} r_{13}), \text{ etc.}$$

Once the loop currents are found, we can determine all the branch currents by Kirchhoff's current law:

$$I_1 = I_{11}$$

$$I_2 = I_{22}$$

$$I_3 = I_{33}$$

$$I_4 = I_{11} + I_{33}$$

$$I_5 = I_{22} + I_{33}$$

$$I_6 = I_{11} - I_{22}$$

Thus, with the loop-current method, it suffices to calculate the determinant and its cofactors, after which the loop currents are readily found by Eq. (1.32).

The loop-current method can be used to analyse a network if it contains both emf (voltage) and current sources. In such a case, the current of each loop containing a source is known and equal to the source current. Therefore, no equations need to be developed for such loops in the set of equations (1.31). Equations are set up for all the remaining independent loops.

Another universal method for the analysis of complex networks with any number of nodes is the nodal voltage method. For circuits with two nodes, this method was examined in Sec. 1.13.

1.15 The Superposition Theorem

This is a fundamental theorem arising from the linearity of the circuits being considered. It may be stated as follows: The current in any branch of a network is the algebraic sum of the currents in that branch due to each voltage source acting separately, with all the other sources removed and their internal resistances left in the network. In a different form, it is also stated as follows: The response of a linear network with several sources is equal to the algebraic sum of the responses produced by each source acting alone, all others being reduced to zero.

In mathematical form, the superposition theorem is, for example, the solution for the set of loop-current equations (1.32). On replacing in (1.32) each of the n loop emfs with the algebraic sum of branch emfs, (1.29), contained in the respective loop, and on collecting the like terms, we obtain (on the assumption that each kth loop contains one kth branch not contained in the other loops):

$$I_{kk} = I_k = g_{k1}E_1 + g_{k2}E_2 + \ldots + g_{kk}E_{kk} + \ldots + g_{kf}E_f + \ldots + g_{km}E_m \tag{1.35}$$

where I_k is the current of the kth branch entering only the kth loop; g_{kk} is the self-conductance of the kth branch; g_{kf} is the mutual conductance of the kth and fth branches; and m is the number of branches containing sources.

It is to be noted that the self-conductance g_{kk} and the mutual conductance g_{kf} are not the reciprocals of the self-resistance and the mutual resistance of the loops (see Sec. 1.14).

As follows from Eq. (1.35), the *self-conductance of a branch* is the ratio of the branch current to the branch emf (voltage) on the condition that the emfs (voltages) due to the remaining sources are zero.

The *mutual conductance between two branches* is the ratio of the current in one branch to the source emf (voltage) in the other on the condition that the emfs (voltages) due to the remaining sources are zero.

If a network contains sources whose internal resistances may not be neglected for purposes of analysis, these sources must be replaced with resistive elements whose resistances are equal to the internal resistances of the sources.

Networks containing linear resistive elements always satisfy the condition of reciprocity which is stated as

$$g_{kf} = g_{fk}$$

The superposition theorem is also applicable to cases where a network contains an ideal current generator (or generators). In

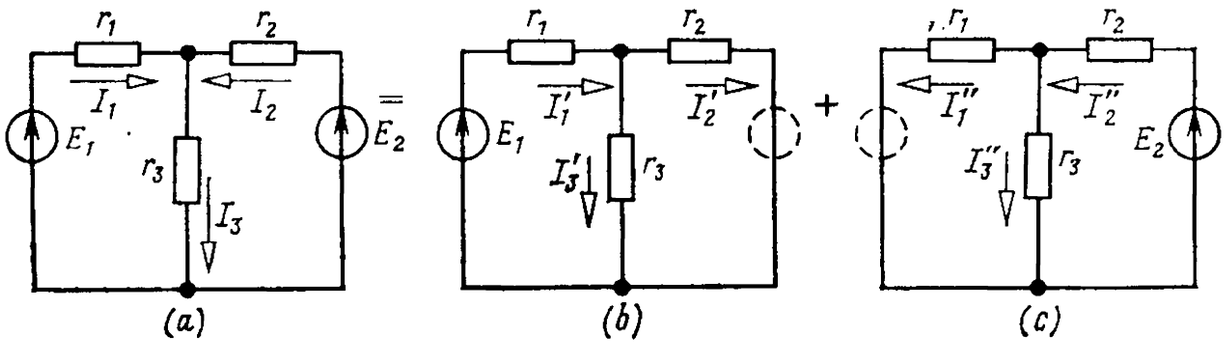


FIGURE 1.21

such a case, the current in any branch is the algebraic sum of the partial currents due to the action of each current generator acting individually. In determining each partial current, the remaining current generators must be replaced with resistive elements of an infinitely large resistance, that is, the respective parts of the network must be open-circuited.

Figure 1.21 a illustrates the procedure for the analysis of a network containing two emf (voltage) generators by the superposition theorem.

The first step is to remove E_2. Removal of E_2 leaves a simple series-parallel connection of resistive elements (see Fig. 1.21 b). The current in the common part of the network is

$$I_1' = \frac{E_1}{r_1 + r_2 r_3/(r_2 + r_3)}$$

The currents in the two parallel branches are

$$I_2' = (E_1 - r_1 I_1')/r_2$$

$$I_3' = (E_1 - r_1 I_1')/r_3$$

The next step is to remove E_1 (see Fig. 1.21 c). Following that, the currents I_1'', I_2'' and I_3'' are found in a similar way.

On the basis of the superposition theorem, the branch currents in the network of Fig. 1. 21 a are:

$$I_1 = I_1' - I_1''$$
$$I_2 = I_2'' - I_2'$$
$$I_3 = I_3' + I_3''$$

A major limitation of the superposition method is the fact that it calls for special care when the partial currents are opposite in direction and are close in value. Failure to do so may result in an appreciable error in the final result although the error in finding a partial current is small.

Using the superposition theorem, it is possible to find partial currents due not only to individual sources, but to groups of sources as well.

1.16 The Compensation Theorem

The *compensation theorem* may be stated as follows: If in a particular branch of a network containing a resistive element of resistance r, through which a current I flows, the resistance is increased, the

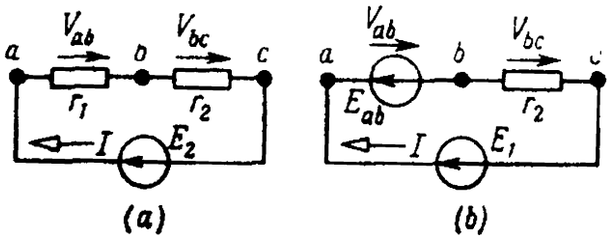


FIGURE 1.22

current may be restored to its original value by inserting a voltage generator of emf

$$E = rI = V \tag{1.36}$$

in series opposition with the network.

The compensation theorem stems directly from equation (1.8) written on the basis of Kirchhoff's voltage law, in which any term in the sum of voltages across the resistive elements may be transposed with the opposite sign to the right-hand side of the equation. This is equivalent to replacing the corresponding resistive element with a voltage generator acting in opposition to the current in that resistive element.

As an example, let us apply the compensation theorem to the network in Fig. 1.22 a. To begin with, we write an equation by Kirchhoff's voltage law:

$$V_{ab} + V_{bc} = r_1 I + r_2 I = E_1$$

Now we transpose the term V_{ab} with the opposite sign to the right-hand side of the equation:

$$V_{bc} = E_1 - V_{ab}$$

Then we connect a voltage generator (Fig. 1.22 b)

$$E_{ab} = V_{ab} = r_1 I$$

between terminals a and b instead of the resistive element. This will leave the current unchanged because the potential difference $\varphi_a - \varphi_b$ between the terminals a and b does not change. In the circuit of Fig. 1.22 a,

$$\varphi_a - \varphi_b = V_{ab}$$

and in the circuit of Fig. 1.22 b the potential difference

$$\varphi_a - \varphi_b = E_{ab}$$

is the same as

$$E_{ab} = V_{ab}$$

In consequence, the circuit in Fig. 1.22 b is equivalent to the original network in Fig. 1. 22 a.

1.17 The Equivalent Generator (Active Two-Terminal Network) Method

This method is warranted in cases where one is to find the current in only one branch ab of a complex electric network, such as in a load connected to an amplifier. Similar problems also arise in the analysis and design of systems intended to measure nonelectric quantities by electric methods, automatic control systems, etc.

Let there be a complex network which may contain voltage and/or current generators, and resistive elements with fixed resistance. We remove a branch, ab, and symbolize the remainder of the network by a box and label it with the letter A (for active) because it acts as an *active two-terminal network* with regard to the removed branch ab connected to it at the terminals a and b. In contrast, when a network contains no sources, it is called *passive*, and the box symbolizing it is labelled with the letter P (for passive).

If we connect a load of resistance r_{load} to the terminals a and b of the active two-terminal network (in doing so, we load the network), a current, $I_{ab} = I$, will begin to flow in the removed branch ab (Fig. 1.23 a).

The equivalent generator method makes it possible to find the current in the load r_{load} by treating the active two-terminal network as an *equivalent generator*. This is the reason why the technique is called the equivalent generator method in the USSR*.

In order to determine the parameters of the equivalent generator, we need to carry out some preliminary transformations in the network of Fig. 1.23 a. The procedure is as follows.

We connect two identical voltage generators $E_1 = E_2$ in series with the load and in opposition to each other (see Fig. 1.23 b). Since

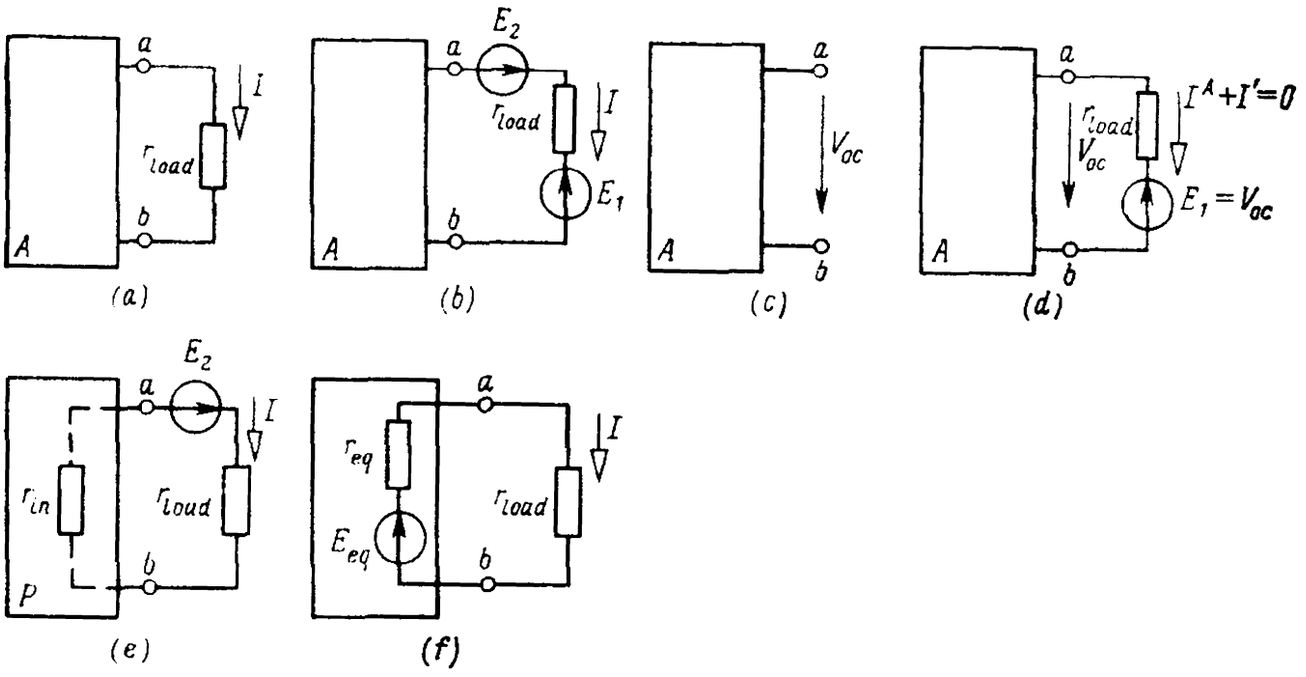


FIGURE 1.23

the two sources are identical and act in opposition, the current in the branch ab, $I_{ab} = I$, remains unchanged. In consequence, the conditions in the active two-terminal network do not change, either, and the network in Fig. 1.23 b is equivalent to the original network in Fig. 1.23 a.

The current $I_{ab} = I$ in the network of Fig. 1.23 b can be found by the superposition theorem, namely by writing the current I as the sum of three partial currents, namely the current I^A due to all the sources in the active two-terminal network, the current I' due to E_1, and the current I'' due to E_2:

$$I = I^A + I' + I''$$

We choose $E_1 = E_2$ to be equal to the open-circuit voltage of the active two-terminal network, $V_{ab,\,oc} = V_{oc}$. This is the voltage between the terminals a and b with the load disconnected ($I = 0$), as

* Outside the Soviet Union, it is known as Thévenin's (Helmholtz's) theorem in cases involving voltage generators, and Norton's theorem in cases involving current generators.—*Translator's note.*

shown in Fig. 1.23 c. Now, with all the sources within the active two-terminal network and the source $E_1 = V_{oc}$ acting together, the current in the load resistance (Fig. 1.23 d), is zero ($I^A + I' = 0$).

Hence, in finding the current I, we may remove all of these sources and replace them with resistive elements whose resistances are equal to the internal resistances of the sources. As a result of this transformation (Fig. 1.23 e), the active two-terminal network is converted into a passive one containing only resistive elements. The parameter of the passive two-terminal network is its driving-point, or input, resistance r_{in} as measured between the terminals a and b. The current I in the load due to only $E_2 = V_{oc}$ (see Fig. 1. 23 e) is the same as in the original network of Fig. 1.23 a.

From comparison of Figs. 1.23 a and e, we can see that the equivalent circuit for the two-terminal active network can be drawn up by using an ideal voltage generator of emf $E_{eq} = E_2 = V_{oc}$ and of internal resistance $r_{eq} = r_{in}$, that is, an equivalent generator (Fig. 1.23 f).

For the equivalent network of Fig. 1.23 f,

$$I = E_{eq}/(r_{load} + r_{eq}) = V_{oc}/(r_{load} + r_{in}) \tag{1.37}$$

Equation (1.37) shows that in order to find the current in the load, it suffices to know two parameters of the active two-terminal network, namely the voltage which would be produced between its terminals on open circuit, and the driving-point, or input, resistance which can be measured with suitable instruments.

The parameters of an active two-terminal network are determined by a combination of an *open-circuit* and a *short-circuit test*. The procedure is this. First the open-circuit voltage V_{ab} is measured between the terminals of the network with a voltmeter. Then the terminals a and b are short-circuited ($r_{load} = 0$), and the short-circuit current of the network is measured with an ammeter. Finally, the input resistance $r_{in} = V_{oc}/I_{sc}$ is found directly from Eq. (1.37).

As an example, let us find the current I in the electric network of Fig. 1.24 a, using the equivalent-generator method. To find the open-circuit voltage V_{oc} between the terminals a and b of the active two-terminal network, we open the branch containing the resistive element r_{load} (Fig. 1.24 b). Using the superposition theorem and noting the symmetry of the network, we get

$$V_{oc} = rJ/2 + E/2$$

On replacing the sources (in our example, they are a voltage generator and a current generator) in the active two-terminal network with resistive elements whose resistances are equal to the in-

ternal resistances of the respective sources (in our example equal to zero for the voltage generator and to infinity for the current generator), we obtain a passive two-terminal network (Fig. 1.24 c)

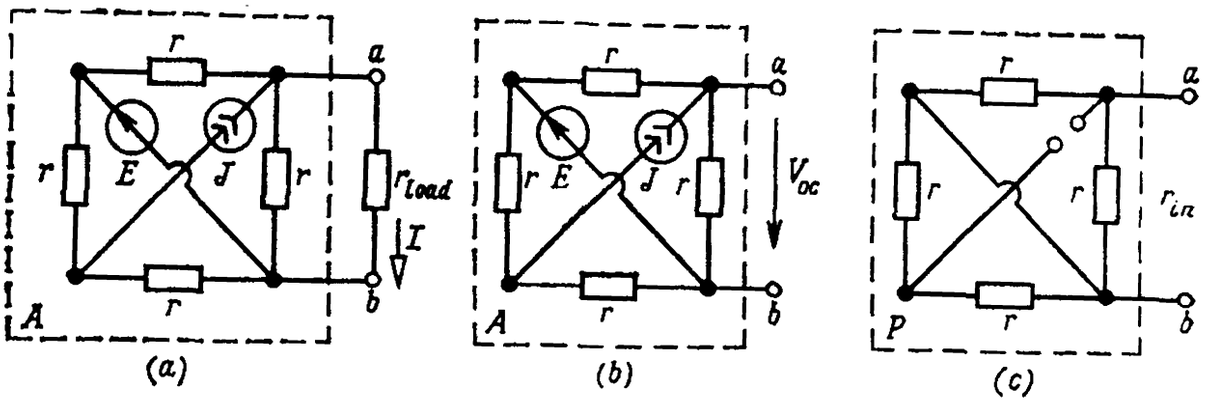


FIGURE 1.24

whose input resistance (the one measured between the terminals a and b) is $r_{in} = r/2$. On the basis of Eq. (1.37), the sought current is

$$I = \frac{rJ/2 + E/2}{r_{load} + r/2}$$

1.18 The Maximum Power Transfer Theorem

In communications, electronics, automatic control and other applications, it is often desired that the source should transfer to the associated load an amount of power which is maximally attainable in the circumstances, whereas the efficiency of power transfer is of minor importance.

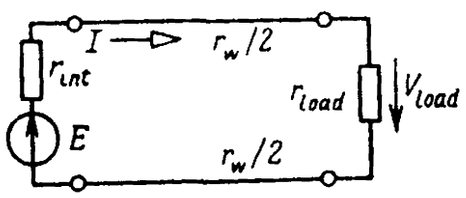


FIGURE 1.25

As an example of maximum power transfer, consider a load of resistance r_{load} which is supplied by a voltage generator of emf E and internal resistance r_{int}. The generator is set up at some distance from the load and the two are interconnected with a two-wire line whose total resistance is r_w (Fig. 1.25).

Let r be the sum of the source resistance r_{int} and of the wire resistance r_w, that is,

$$r = r_{int} + r_w$$

By Ohm's law, the current in the circuit is

$$I = E/(r + r_{load})$$

and (by Joule's law) the power dissipated in the load is

$$P_{load} = r_{load} I^2 = r_{load} E^2/(r + r_{load})^2$$

For the two limiting values of load resistance, namely $r_{load} = 0$ and $r_{load} = \infty$, the power dissipated in the load will be zero because in the former case the voltage between the load terminals is zero, while in the second case the current in the circuit is zero. Therefore the maximum attainable load power (for the given E and r) should correspond to some definite value of r_{load}. In order to find it, we equate to zero the first derivative of the power P_{load} with respect to r_{load}:

$$dP_{load}/dr_{load} = E^2 [(r + r_{load})^2 - r_{load} (2r + 2r_{load})]/(r + r_{load})^4 = 0$$

Since the denominator of the above expression is not equal to infinity, then

$$(r + r_{load})^2 - 2r_{load}r - 2r^2_{load} = 0$$

Hence, the power transferred to the load will be a maximum subject to

$$r_{load} = r \tag{1.38}$$

This maximum power is given by

$$P_{load,\ max} = r\,(E/2r)^2 = E^2/4r$$

Equation (1.38) is the mathematical statement of the *maximum power transfer theorem.*

Figure 1.26 shows the current dependences of the load power $P_{load} = r_{load}I^2$, the source power $P_E = (r_{load} + r)\,I^2 = EI$, the load voltage V_{load}, and of the efficiency of power transfer

$$\eta = P_{load}/P_E$$

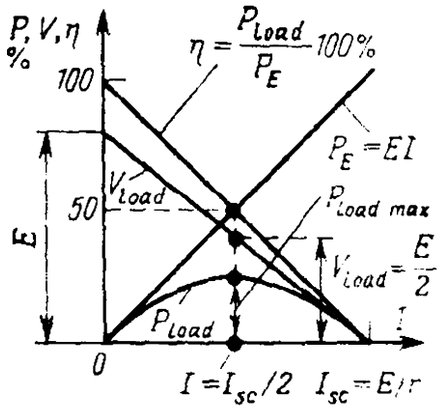


FIGURE 1.26

On substituting for P_{load} and P_E in the expression for the efficiency, we get the efficiency as a function of the load resistance: $\eta = r_{load}I^2/(r_{load} + r)\,I^2 = r_{load}/(r_{load} + r)$

At maximum power transfer ($r_{load} = r$), the efficiency is $\eta = 50\%$. In other words, half the power supplied by the source is lost in the internal resistance of the source and in the intervening connections. For power transmission lines which handle large blocks of power, this is an intolerably low efficiency. (In power transmission lines, the power lost on its way to the loads does not exceed 10% of the available power.) In applications involving small load power of the order of a few milliwatts (such as in communications, automatic control and the like), the low efficiency is immaterial. What is

important is that the source should deliver the largest possible power to the load (which may be, say, a telephone handset).

We have defined the condition for maximum power transfer to the load in a simple circuit containing a transmission line. However, its applicability is much wider. As will be recalled, any complex linear network containing sources and connected to the load by two terminals may be replaced with an equivalent voltage generator of emf E_{eq} and internal resistance r_{eq} (see Sec. 1.17). This equivalent generator obeys the maximum power transfer theorem as well. The power transferred by the equivalent generator to the load will be a maximum if the load resistance connected to its terminals is equal to the internal resistance of the equivalent generator, that is, $r_{load} = r_{eq}$.

1.19 Four-Arm Bridges

Automatic control and instrumentation systems often use the four-arm bridge with a meter, a relay coil or any other actuator connected across a pair of opposite junctions which we will call the diagonal. So long as the bridge is at balance, there will be no current flowing in the diagonal. As soon as the bridge is brought out of balance by, say, changing the parameter of one of the arms, a current will begin to flow in the diagonal. From its magnitude and direction it is an easy matter to judge on the degree of out-of-balance and, in the final analysis, the magnitude of the factor that has upset the balance.

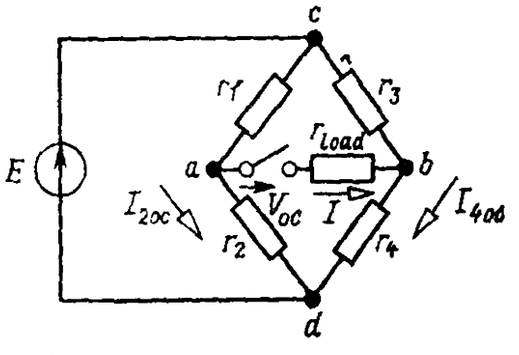


FIGURE 1.27

In schematic form, a four-arm d.c. bridge is shown in Fig. 1.27. It is connected to a source of emf E, and contains resistive elements r_1, r_2, r_3 and r_4 in its arms.

In order to determine the current $I = I_{ab}$ flowing in the bridge diagonal when the switch is closed, we will use the equivalent generator method. To this end, we represent the four-arm bridge with respect to two terminals a and b as an active two-terminal network (Fig. 1.23 a) with a resistive element r_{load} connected to the terminals a and b.

Before we can find the current by Eq. (1.37), we should determine the voltage which will be produced between the terminals a and b on open circuit (that is, with the bridge diagonal open). In the open-circuit condition,

$$I_{2,\,oc} = E/(r_1 + r_2)$$

$$I_{4,\,oc} = E/(r_3 + r_4)$$

and

$$V_{oc} = r_2 I_{2,\,oc} - r_4 I_{4,\,oc} = \frac{r_2 E}{r_1 + r_2} - \frac{r_4 E}{r_3 + r_4}$$

The input resistance of the two-terminal network, seen at he terminals a and b when looking into the network (with the voltage source open-circuited and the terminals c and d short-circuited, that is, with the resistive elements $r_1 r_2$ and $r_3 r_4$ connected in parallel) is given by

$$r_{in} = \frac{r_1 r_2}{r_1 + r_2} + \frac{r_3 r_4}{r_3 + r_4}$$

By Eq. (1.37), the current in the bridge diagonal is

$$I = V_{oc}/(r_{in} + r_{load}) = \frac{[r_2 (r_3 + r_4) - r_4 (r_1 + r_2)]\, E}{r_1 r_2 (r_3 + r_4) + r_3 r_4 (r_1 + r_2) + r_{load} (r_1 + r_2) (r_3 + r_4)}$$

If the resistances of any three bridge arms and the emf of the source are held constant, then the change in the current through the bridge diagonal will be solely dependent on changes in the resistance of the fourth arm. If the resistance of the fourth arm is made to vary by some environmental factor (temperature, pressure, humidity, etc.), the variations in the current flowing in the bridge diagonal will represent on a certain scale the changes in the surroundings. With this arrangement, a four-arm bridge can be used as a transducer for measuring nonelectric quantities.

In automatic control applications, four-arm bridges are used as elements of feedback loops. The magnitude and sense of the current in the bridge diagonal provide information on the basis of which changes are initiated in the position of some final control element so as to maintain the controlled variable at its preset value (the one that should normally exist with no current flowing in the bridge diagonal).

1.20 Nonlinear D.C. Circuits

In the general case, the current in any piece of d.c. electric equipment is a nonlinear function of the voltage across the terminals of that piece, or symbolically: $I\,(V)$. If this nonlinearity is negligible over a wide range of current and voltage values, the nonlinear device in question may be represented by a linear equivalent circuit. In practice, however, the current-voltage relation is substantially nonlinear for a good many devices.

As an example, Fig. 1.28 shows the nonlinear $I\,(V)$ characteristics of three d.c. devices, namely (a) an incandescent lamp, (b) a semiconductor diode, and (c) a vacuum tube. As is seen, the difference is striking.

For purposes of analysis, a network containing nonlinear elements should be replaced with its nonlinear equivalent circuit. The simplest element in a nonlinear equivalent circuit is a *nonlinear resistive element.* Its diagram symbol is shown in Fig. 1.29. In the equivalent circuits for nonlinear d.c. networks, it can replace any of the devices whose *current-voltage* (or *external*) *characteristics* are shown in Fig. 1.28. These characteristics can be specified analytically, graphically (see Fig.1.28), or in tabular form.

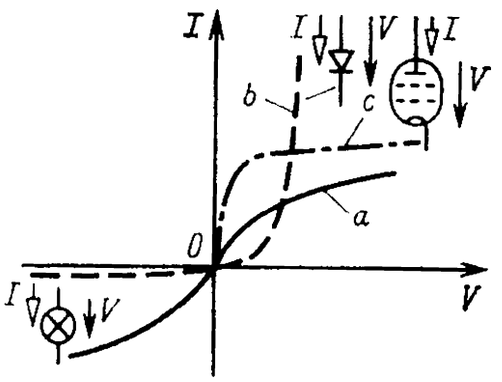


FIGURE 1.28

In the general case, the nonlinear equivalent circuit of a d.c. device or network will contain, in addition to nonlinear resistive elements, also nonlinear voltage and current generators whose behaviour and properties are likewise specified by giving their current-voltage (external) characteristics.

In the subsequent discussion, we will limit ourselves to the analysis of nonlinear networks containing solely nonlinear resistive elements.

Because of the nonlinearity of the current-voltage characteristics of nonlinear resistive elements, the superposition theorem is not applicable to nonlinear electric networks. For the same reason, all the

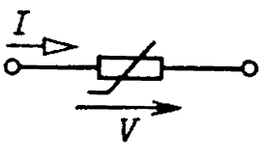


FIGURE 1.29

other methods and techniques based on the superposition theorem are likewise inapplicable completely or to a certain degree. In the general case, we cannot analyse nonlinear electric networks with the aid of the loop-current method, the nodal-voltage method, or the circuit-transformation method. The equivalent generator method (the Thévenin theorem) may be used to analyse a nonlinear electric network on the condition that the active two-terminal network does not contain any nonlinear resistive elements.

1.21 Analysis of Nonlinear Networks

We will consider two basic methods used to analyse nonlinear electric networks, graphical and analytical.

The graphical method. With this method, the sequence of steps is about the same as it is in the case of linear networks, except that

the addition and subtraction of voltages or currents is replaced with the addition and subtraction of the abscissae and ordinates of the respective current-voltage characteristics.

Let us take a closer look at the graphical analysis of networks in which the nonlinear resistors are connected in series, in parallel, and in series-parallel.

The series case. Figure 1.30 shows a network containing two series-connected nonlinear resistive elements, and also the respective current-voltage characteristics, $I\ (V_1)$ and $I\ (V_2)$.

If we connect a current generator to the network shown, the analysis procedure is trivial. To demonstrate, if we know the source cur-

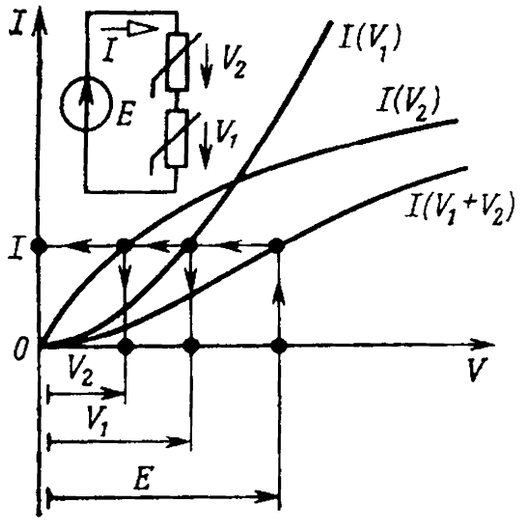


FIGURE 1.30

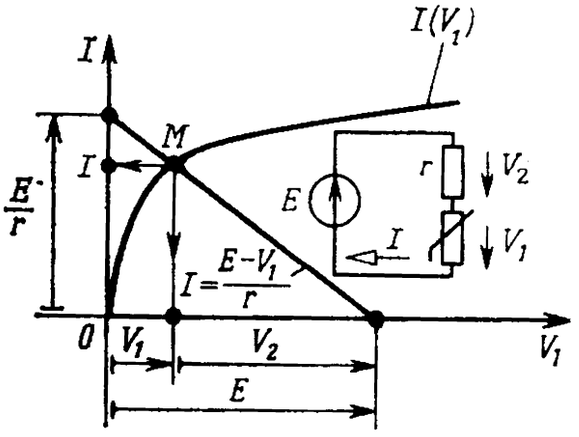


FIGURE 1.31

rent and, as a consequence, also the current in the nonlinear resistive elements, the voltages across them can be found directly from the respective current-voltage characteristics.

The situation will be far more complex if we connect a voltage generator of emf E to the network. In this case, we need to carry out an equivalent transformation of the network before we can proceed with the actual network analysis.

Let us replace the network containing two series-connected nonlinear resistive elements with a network containing only one equivalent resistive element. Since the currents through the original and the transformed elements are the same, the current-voltage characteristic of the equivalent nonlinear resistive element can be constructed by graphically combining the two given current-voltage characteristics, $I\ (V_1)$ and $I\ (V_2)$ in terms of voltage. The procedure for this construction will be clear from Fig. 1.30.

With the source emf E specified in advance, we can determine the current I in the network immediately from the current-voltage characteristic of the equivalent resistive element thus constructed. Once the network current is found, we can go back to the current-voltage

characteristics of the original resistive elements, $I(V_1)$ and $I(V_2)$, and to determine the respective voltages, V_1 and V_2.

The analysis procedure remains the same if a network contains more than two series-connected nonlinear resistive elements whose current-voltage characteristics are known.

In practice, it is often the case that a network contains a series connection of a nonlinear and a linear resistive element. This is true, for example, of various amplifiers where the role of the nonlinear element is played by a transistor or some other device. Such networks should preferably be analysed graphically by the *load-line method*. We begin by writing the equation for the network of Fig. 1.31 on the basis of Kirchhoff's voltage law:

$$rI + V_1 = E$$

Hence,

$$I = (E - V_1)/r \tag{1.39}$$

With E and r held constant, it follows from Eq. (1.39) that the current I in the linear resistive element is a linear function of the voltage V_1 across the nonlinear element, that is $I = F(V_1)$. This relation is represented by a straight line called the *load line*. With respect to the terminals of the nonlinear element it is the external characteristic of the active two-terminal network.

The load line is constructed on the basis of Eq. (1.39) on setting first $I = 0$ so that $V_1 = E$, and then $V_1 = 0$ so that $I = E/r$. As is seen, the load line passes through a point on the voltage axis and through another point on the current axis.

The elements connected in series carry the same current which must lie both on the load line $I = F(V_1)$ and on the current-voltage characteristic of the nonlinear resistive element, $I(V_1)$. Therefore, the intersection of the two characteristics (point M in Fig. 1.31) defines the network variables, that is the current I and the voltages V_1 and V_2.

The load-line method can be applied to a network containing a linear active two-terminal network with a nonlinear resistive element connected to its terminals. To demonstrate, the linear active two-terminal network may be replaced with an equivalent generator made up of a series combination of an ideal voltage generator E_{eq} and a linear resistive element r_{eq} as shown in Fig. 1.23 f. If r_{load} is nonlinear, the network is similar to that shown in Fig. 1.31.

The parallel case. Figure 1.32 shows a network which consists of two nonlinear resistive elements connected in parallel. Connection of a voltage source to this network leads to a trivial problem. With the applied voltage specified in advance, the current in each of the nonlinear resistive elements can be found directly from the respective current-voltage characteristic, $I_1(V)$ or $I_2(V)$.

If we connect a current source J (Fig. 1.32) to the same parallel combination of nonlinear resistive elements, the network can be analysed graphically. To this end, we replace the two parallel-connected nonlinear resistive elements with one equivalent nonlinear resistive element. Since the voltage across the original nonlinear elements must be the same as that across the equivalent nonlinear resistor, we may plot the current-voltage characteristic of the equivalent element by graphically combining $I_1(V)$ and $I_2(V)$ in terms of current.

With the source current J known in advance, the voltage V applied to the equivalent nonlinear resistive element can be found

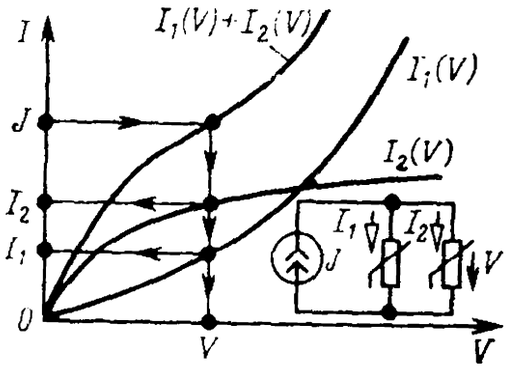


FIGURE 1.32

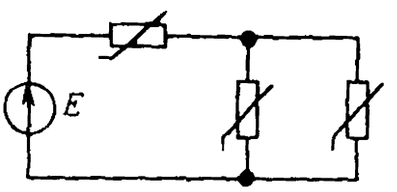


FIGURE 1.33

directly from its characteristic. As has already been noted the voltage V is the same across each of the original resistive elements, and we can readily find the currents I_1 and I_2 in the original elements from their current-voltage characteristics $I_1(V)$ and $I_2(V)$ (Fig. 1.32).

When a network contains more than two parallel-connected nonlinear resistive elements whose current-voltage characteristics are known, the analysis procedure is as already explained.

The series-parallel case. If a series-parallel network contains a source (Fig. 1.33), there is nothing fundamentally new in its analysis. To begin with, the parallel combination of nonlinear resistive elements is replaced with an equivalent nonlinear resistive element as already explained. After this transformation, the network contains two nonlinear resistive elements connected in series. Its analysis has already been examined.

The analytical method. Of the many analytical methods used to analyse nonlinear networks, the one most commonly used is based on the *linearization of the current-voltage characteristics.* To this end, the current-voltage characteristic of each element is divided into several portions, and each portion is then approximated with a straight line segment. Taken together, the segments yield a current-voltage characteristic approximated by a broken line. Within each linearized portion, the current-voltage relation can be defined by a

linear equation (with constant coefficients). As a result, each nonlinear element is represented by an equivalent circuit containing constant-parameter elements, that is, by a linear equivalent circuit.

On representing all the nonlinear resistive elements with their linear equivalents within the operating limits of the current-voltage characteristics, we obtain a complete linear equivalent circuit for the original nonlinear net work. This complete linear equivalent circuit can be analysed by all the methods applicable to linear networks with the proviso that the operating point defining the conditions in each nonlinear element is located within the linearized portion of its current-voltage characteristic.

Consider the linear equivalent circuits of nonlinear resistive elements and find all the circuit variables involved.

If a portion of a nonlinear current-voltage characteristic can be approximated with a straight-line segment, then within that portion the ratio of voltage change ΔV to current change ΔI will be a constant that is

$$\Delta V/\Delta I = \text{const}$$

or, in the limit,

$$\mathrm{d}V/\mathrm{d}I = \text{const}$$

The ratio $\mathrm{d}V/\mathrm{d}I$ is called the *dynamic resistance*, r_d, of the nonlinear resistive element being approximated.

In contrast to the dynamic resistance, the ratio V/I at every point of the same portion of the current-voltage characteristic is called the *static resistance*, r_s. The static resistance does not remain constant even within the linearized portion of the current-voltage characteristic. The exception is the characteristics in which the linearized portion or its extension passes through the origin of coordinates. Because of this, the static resistance is used in analysis only seldom.

Figure 1.34 a shows a nonlinear current-voltage characteristic in which the portion above point *a* may be regarded linear and, in consequence, may be represented by a straight-line segment. Let us extend this linearized portion until it cuts the voltage axis. Assume also that this intercept on the voltage axis represents on a scale some fictitious emf, $E_0 > 0$.

The linearized portion of the current-voltage characteristic corresponds to the dynamic resistance

$$r_\mathrm{d} = \mathrm{d}V/\mathrm{d}I = m_r \tan\alpha = (V - E_0)/I \qquad (1.40)$$

where $m_r = m_V/m_I$ is the coefficient determined by the scales of construction along the axis of abscissae, m_V, and along the axis of ordinates, m_I.

From Eq. (1.40) we derive an expression relating the voltage across and the current in the nonlinear resistive element within the operating portion (above point a in Fig. 1.34 a):

$$V = E_0 + r_d I \tag{1.41}$$

As follows from Eq. (1.41), at $V > V_a$ the nonlinear resistive element may be replaced with a linear equivalent circuit consisting of the source of constant emf E_0 which acts in opposition to the current, and a linear resistive element of resistance r_d (Fig. 1.34 a).

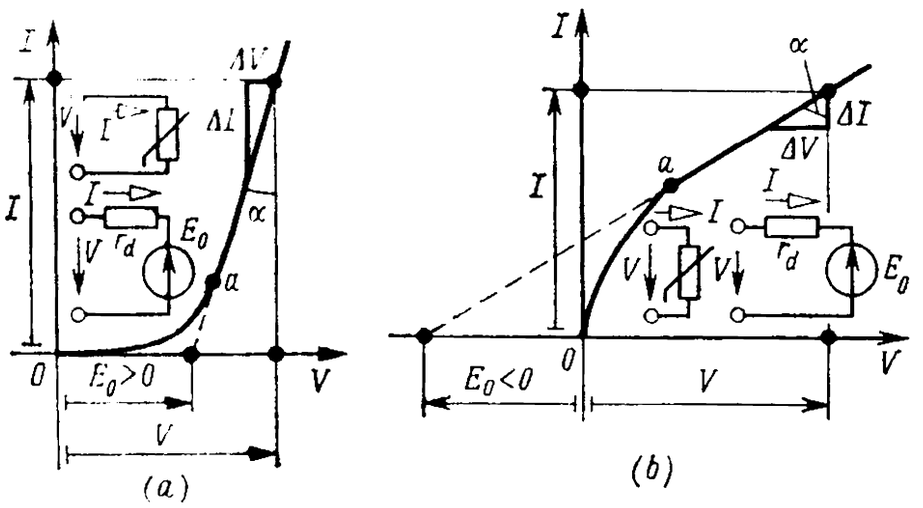


FIGURE 1.34

Alternatively, the linearized portion of the current-voltage characteristic above the point a may be as shown in Fig. 1.34 b. On extending the linearized portion of the current-voltage characteristic, we obtain an intercept which represents on the adopted scale the emf $E_0 < 0$ on the negative voltage half-axis. Now,

$$r_d = \mathrm{d}V/\mathrm{d}I = m_r \tan \alpha = (V - E_0)/I \tag{1.42}$$

That is, within the linearized portion of the current-voltage characteristic we have

$$V = E_0 + r_d I \tag{1.43}$$

which is the same as Eq. (1.41). In consequence, the equivalent circuits of the nonlinear resistive element in Fig. 1.34 a and b are likewise the same, with the only difference that in the former case $E_0 > 0$, and in the latter case, $E_0 < 0$.

2 Sinusoidal A.C. Circuits

2.1 Sinusoidal A.C. Devices

Sinusoidal a.c. devices are widely used in the most diverse fields of application, such as the generation, transmission, transformation and distribution of electric power, in electric drive, household appliances, industrial electronics, radio engineering, etc. There are a number of factors behind the popularity of sinusoidal a.c. devices.

To-day, energy is conveyed over large distances primarily in the form of electric current. A vital condition for long-distance electric power transmission and distribution is the possibility of power transformation in a simple manner entailing minimum losses. This can only be done in sinusoidal a.c. devices known as transformers. This explains why the present-day electric power industry is predominantly using the sinusoidal current. The exceptions seem to be superhigh-voltage d.c. power transmission lines and some process installations, but they too are part of sinusoidal a.c. systems.

Advances in the design and use of sinusoidal a.c. equipment have to a great extent been stimulated by the possibility of building very large power-generating and utilizing units. At present, many thermal power plants have turboalternators rated at 1.5 GW per unit. This is also true of hydroelectric power stations.

Among the simplest and cheapest electric motors are sinusoidal induction motors free from moving electric contacts.

The standard (or power) frequency for electric plants and systems (notably, for all electric power stations) is 50 Hz in the Soviet Union and most other countries and 60 Hz in the United States. This frequency has been chosen for simple reasons. A lower frequency is inacceptable because the human eye begins to note the flicker in incandescent lamps. A higher frequency is undesirable, because the emf of self-induction builds up in proportion to frequency, and this has a detrimental effect on power transmission over wires and the operation of many pieces of electric equipment.

The above considerations do not limit, however, the use of sinusoidal current at other frequencies in many industrial and scientific applications. For example, electric furnaces used in the manufacture of refractory and especially pure metals operate at frequencies in the range 500 Hz-50 kHz. In ultrasonic equipment, on the other hand, the frequency may be as low as several hertz.

Advances in communications, especially in radio, have brought with them the development of devices, such as antennas, oscillators, frequency converters, and the like, which operate in the megahertz region of the frequency spectrum. Many of these devices depend for their operation on the ability of alternating current to set up an alternating electromagnetic field with which power can be conveyed in a directed manner without resort to wires.

In the pages that follow, we will deal primarily with sinusoidal a.c. devices operating at commercial (power) frequency and with the methods used to analyse their operation.

2.2 The Elements of a Sinusoidal A.C. Circuit

In addition to what we would find in a d.c. circuit (such as sources, instruments, switches, etc.), sinusoidal a.c. circuits will usually contain devices specific only to them, namely transformers, inductors, etc.

All the components of a sinusoidal a.c. circuit can be depicted in an easy-to-grasp and compact manner by means of a schematic circuit diagram similar to that shown in Fig. 1.2.

For purposes of a.c. network analysis, it is advisable to change from a schematic diagram to an equivalent circuit which is a quantitative model of the processes that take place in the original network. The equivalent circuit of a network is obtained if all the network components are represented by their idealizations each of which describes the behaviour and properties of the associated real component with sufficiently good approximation. The currents and voltages observed in real sinusoidal networks and in their equivalent circuits at particular instants of time, and also in other networks where the currents and voltages are time functions, are called *instantaneous* and symbolized with the letters i and v.

The elements of equivalent circuits for sinusoidal a.c. networks are sinusoidal current generators, sinusoidal emf (voltage) generators, and also resistive, inductive and capacitive elements. We have already examined the concepts of ideal emf (voltage) and current generators, and resistive elements in connection with the analysis of d.c. circuits. Inductive and capacitive elements are specific features of sinusoidal a.c. networks.

If the parameters of elements are independent of the current flowing in them and the applied voltage, those are linear elements. Otherwise, they will be nonlinear.

2.3 The Inductive Element

There is always a magnetic field around any conductor carrying a current i. In sinusoidal a.c. devices, such as transformers, electric motors and instrument coils, special effort is made to set up strong magnetic fields around them.

We will consider the properties of the varying magnetic field in such devices, taking as an example two inductors differing in the direction of winding, and neglecting the resistance of the wires.

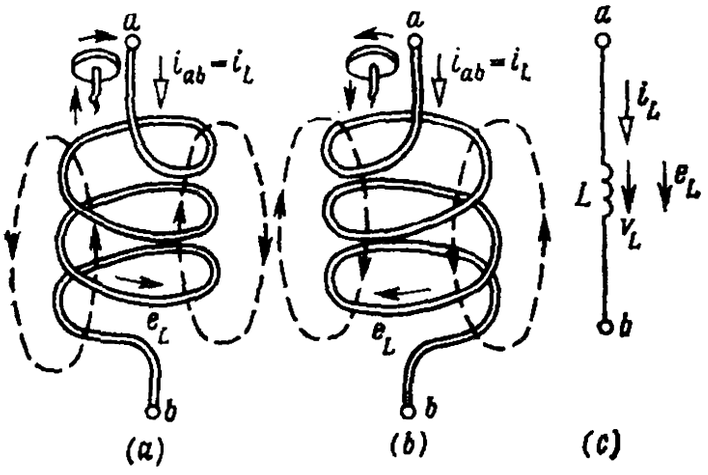


FIGURE 2.1

If the current in a coil, $i_{ab} = i_L$, is constant, the magnetic field around the coil will likewise be constant, and it can be defined in terms of the *magnetic flux* Φ, an assemblage of continuous magnetic lines of force, or the lines of induction vector **B**, threading the surfaces bounded by a closed contour. The direction of magnetic lines of force depends on the direction of winding and the direction of current. Inside the coil, it is the same as the progressive movement of a right-handed corkscrew when its handle is rotated in the direction of the current flowing in the coil (the right-handed screw rule, Fig. 2.1 a and b). In the general case, the magnetic field around a current-carrying coil is fairly complex in configuration. Fortunately, in order to characterize an inductor as a network element we need not know the distribution of the magnetic field in the space surrounding the coil. It suffices to find the *flux linkage* Ψ with all w turns:

$$\Psi = \Phi_1 + \Phi_2 + \ldots + \Phi_k + \ldots + \Phi_w = \sum_{k=1}^{w} \Phi_k$$

where Φ_k is the magnetic flux linking the kth turn.

The basic SI unit of magnetic flux is the *weber* (Wb), and that of flux linkage, the *weber-turns* (Wb-t).

Since in our case the flux linkage with coil turns depends on the current flowing in the coil, it is called *self-flux linkage*. If the same magnetic flux Φ links all the turns of a coil, the self-flux linkage is

$$\Psi = w\Phi$$

The ratio of the self-flux linkage of a coil to the current in the coil. $i_{ab} = i_L$, is called *self-inductance* or, simply, *inductance*:

$$L = \Psi/i_L \tag{2.1}$$

If the self-flux linkage of a coil is proportional to the current flowing in the coil, then $L = \text{const}$. Otherwise, the inductance is a function of current, $L\ (i_L)$. The dependence of inductance on current is well-pronounced, for example, in iron-cored inductors.

Since electric current is always accompanied by a magnetic field, any current-carrying part of an electric device must possess inductance. Of course, in many cases this inductance is very small and may be neglected.

The basic SI unit of inductance is the *henry* (H):

$$1\ \text{H} = 1\ \text{Wb A}^{-1}$$

One henry is a very large unit, so in practice use is often made of its submultiples which are the millihenry (mH):

$$1\ \text{m H} = 1 \times 10^{-3}\ \text{H}$$

and the microhenry (μH):

$$1\ \mu\text{H} = 1 \times 10^{-6}\ \text{H}$$

If the current through a coil increases or decreases, the self-flux linkage will also increase or decrease. Every change in flux linkage gives rise to the emf of self-induction e_L in the coil turns in accord with the *law of electromagnetic induction*. By convention, the positive direction of the emf of self-induction is indicated by the rotation of the handle of a corkscrew advanced progressively in the direction of the magnetic lines of force, and is the same as the assumed (by convention) positive direction of current flow (see Fig. 2.1 a and b). By definition, the emf of self-induction is

$$e_L = -\,\mathrm{d}\Psi/\mathrm{d}t \tag{2.2a}$$

or, subject to Eq. (2.1),

$$e_L = -\,L\,\mathrm{d}i_L/\mathrm{d}t \tag{2.2b}$$

It follows from Eq. (2.2) that the actual direction of the emf of self-induction at any given instant of time may differ from the assumed positive direction and depends on the sign of the time derivative of current. It is also seen that the emf of self-induction always opposes the change of current (Lenz's law).

For an alternating current to flow through an inductor, the voltage between its terminals must be equal and, at each instant of time, opposite in direction to the emf of self-induction (Fig. 2.1. c):

$$v_{ab} = v_L = -\,e_L = L\,\mathrm{d}i_L/\mathrm{d}t = \mathrm{d}\Psi/\mathrm{d}t \tag{2.3}$$

It follows from Eq. (2.3) that

$$1 \text{ Wb} = 1 \text{ V s}$$

and that

$$1 \text{ H} = 1 \text{ V s A}^{-1} = 1 \Omega \text{ s}$$

If the resistance of the coil wire may not be neglected, the equivalent circuit in Fig. 2.1 c must be expanded to include a series-connected resistive element.

Inductance L is the parameter of linear inductive elements, and a *linear inductive element* is an equivalent (idealization) of any part of an electric device in which the self-flux linkage is proportional to current. If this dependence is not linear, the equivalent circuit will contain a *nonlinear inductive element* specified by giving its *weber-ampere characteristic*, $\Psi(i_L)$.

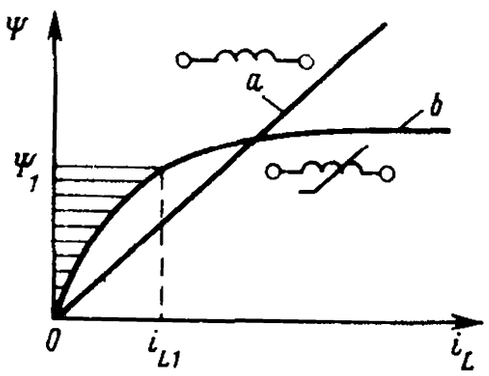


FIGURE 2.2

Figure 2.2 shows the weber-ampere characteristics of (a) a linear inductive element and (b) a nonlinear inductive element, and also their diagram symbols used in equivalent circuits to replace the actual circuit components.

If in time t_1 the current in an inductive element changes from zero to i_{L1}, the magnetic field of the element (see Fig. 2.1 c) will have stored an amount of energy given by

$$W_M = \int_0^{t_1} i_L v_L \, dt$$

or, in view of Eq. (2.3),

$$W_M = \int_0^{\Psi_1} i_L \, d\Psi = \int_0^{i_{L1}} i_L L(i_L) \, di_L \qquad (2.4)$$

where Ψ_1 is the self-flux linkage for $i_L = i_{L1}$ (Fig. 2.2).

As follows from Eq. (2.4), the energy stored in the magnetic field of an inductive element at i_L is proportional to the area enclosed between the weber-ampere characteristic and the axis of ordinates (see Fig. 2.2 where the shaded area is proportional to the energy stored by the magnetic field of the inductive element at i_{L1}).

It follows from Eq. (2.4) and subject to Eq. (2.1) that the *energy stored by the magnetic field* of a linear inductive element at i_L is given by

$$W_M = Li_L^2/2 = \Psi i_L/2 \qquad (2.5)$$

As the current through an inductive element increases (decreases), the energy stored by the magnetic field of the element increases (decreases). Thus, inductive elements may be regarded as devices capable of accumulating and storing magnetic energy.

2.4 The Capacitive Element

The various parts of an electric device carry electric charge, and this charge gives rise to an electric field between the parts. In some devices, such as insulators, capacitors, etc., the electric field may be fairly strong.

Figure 2.3 a shows an elementary flat capacitor whose plates of area A are placed in vacuum a distance d apart. If we apply a volta-

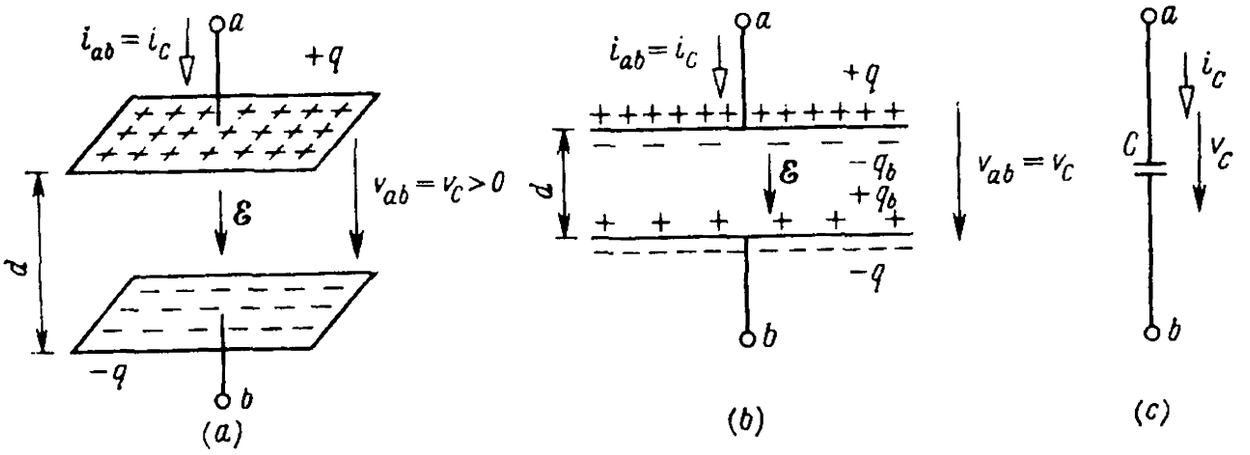


FIGURE 2.3

ge $v_{ab} > 0$ between the upper and lower plates, an identical amount of charge, $\pm q$, will accumulate on each plate. This is an *excess charge.*

If we neglect the edge effect, the accumulated charge will give rise to a uniform electric field between the plates, of strength (or intensity)

$$\mathcal{E} = v_{ab}/d = q/\varepsilon_0 A \tag{2.6}$$

where ε_0 is the *electric constant* (or *permittivity of free space*) equal to 8.854×10^{-12} F m^{-1}.

The charge stored on the capacitor is proportional to the applied voltage $v_{ab} = v_C$:

$$q = Cv_{ab} = Cv_C \tag{2.7}$$

where C is a proportionality factor, called the *capacitance of the capacitor.*

On solving Eqs. (2.6) and (2.7) simultaneously, we get an expression for a flat capacitor with vacuum as the dielectric:

$$C = \varepsilon_0 A/d$$

The capacitance of a flat capacitor can be increased by filling the space between the plates with some dielectric (Fig. 2.3 b). Under the action of the electric field, the randomly oriented dipole molecules of the dielectric are oriented in some particular direction. As they do so, the positive and negative charges inside a uniform dielectric balance each other, whereas a *bound charge*, q_b, is left at the boundaries with the plates of the flat capacitor. This bound charge is negative at the boundary with the positively charged plate, and positive at the boundary with the negatively charged plate. The bound charge tends to diminish the electric field strength $\mathscr{E}$ inside the capacitor

$$\mathscr{E} = v_{ab}/d = (q - q_b)/\varepsilon_0 A$$

whence it follows that, given the same electric field strength and, in consequence, the same applied voltage $v_{ab} = v_C$, the charge must be greater. Therefore, as follows from Eq. (2.7), there will also be an increase in the capacitance of the flat capacitor as compared with that of a vacuum capacitor:

$$C = \varepsilon_r \varepsilon_0 A/d \tag{2.8}$$

where ε_r is the *relative permittivity* or *dielectric constant* of the material filling the space between the plates of the capacitor.

The product of the relative permittivity (or dielectric constant) of the medium, ε_r, by the electric constant ε_0 is called the *absolute permittivity* of the medium:

$$\varepsilon_a = \varepsilon_r \varepsilon_0 \tag{2.9}$$

The properties of some dielectrics are summarized in Table 2.1.

TABLE 2.1 Relative Permittivity, Electric Strength and Resistivity of Some Materials

Material	ε_r	$\mathscr{E}$ str, MV m^{-1}	ρ_V, Ωm
Transformer oil	2.1-2.4	15-20	10^{12}-10^{13}
Petrolatum	2.2-2.6	20-25	5×10^{12}-10^{13}
Polyethylene	2.2-2.4	35-60	10^{13}-10^{15}
Therylene	3.0-3.5	80-120	10^{14}-10^{16}
PVC	6-8	6-15	10^{10}-10^{12}
Paraffin wax	2.0-2.2	22-32	10^{14}-10^{16}
Hard rubber (ebonite)	3.0-3.5	15-20	10^{13}-10^{14}
Paper-base-laminate	6-8	20-40	10^{9}-10^{11}
Mica (muskovite)	6.5-7.2	98-175	10^{13}-10^{15}
Marble	8-9	1.0-4.0	10^{7}-10^{8}
Slate	6-8	0.5-1.5	10^{6}-10^{7}
Asbestos-cement	6-8	2.0-3.0	10^{6}-10^{7}

Note. The electric strength of all the materials is given for the rms value of the applied voltage (see Sec. 2.6).

If the excess charge (on the plates of the capacitor) is proportional to the applied voltage, then the capacitance of the capacitor is constant, C = const. Otherwise, the capacitance of the capacitor is a function of voltage, $C(v_C)$.

The basic SI unit of capacitance is the *farad* (F):

$$1\ \text{F} = 1\ \text{C V}^{-1} = 1\ \text{A s V}^{-1}$$

One farad is a very large unit. In practice, use is ordinarily made of the submultiple units. These are the microfarad (μF):

$$1\ \mu\text{F} = 1 \times 10^{-6}\ \text{F}$$

the nanofarad (nF):

$$1\ \text{nF} = 1 \times 10^{-9}\ \text{F}$$

and the picofarad (pF):

$$1\ \text{pF} = 1 \times 10^{-12}\ \text{F}$$

In Fig. 2.3 c, the capacitor shown in Fig. 2.3 a and b is represented by its model in the form of a *linear capacitive element* of capacitance C. This model is in effect an ideal capacitor, that is, a capacitor in which we neglect the heat-generating losses in the dielectric, the terminal insulation, the plate metal, etc. Otherwise, the model is extended to include a resistive element.

Since an electric field always exists between the various current-carrying (or live) parts of an electric device or piece of equipment, there always must be an amount of capacitance between them. Of course, in some cases, it is so small that it may be neglected.

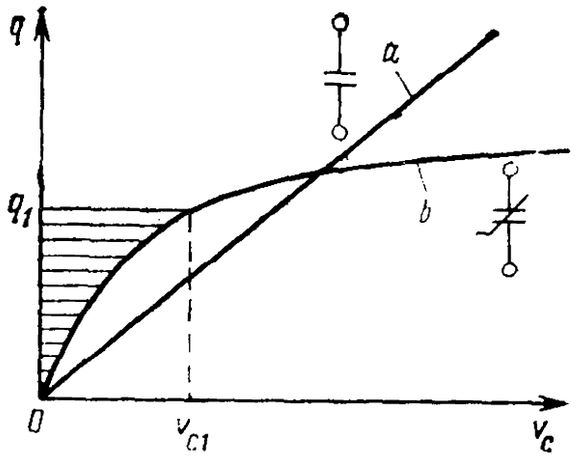


FIGURE 2.4

The linear capacitive element shown in Fig. 2.3 c may serve as a model for any part of an electric device, if the amount of charge on it is proportional to the applied voltage. If the charge is not proportional to the voltage, the model will be a *nonlinear capacitive element* which is specified by giving its *coulomb-volt* (or *charge-voltage*) *characteristic*, $q(v_C)$. Figure 2.4 shows a charge-voltage characteristic for a linear capacitive element (curve *a*) and a nonlinear capacitive element (curve *b*) along with their diagram symbols.

If the voltage applied to a capacitive element is varied (increased or decreased), the charge on the plates will vary, too, and a current will begin flowing in the capacitive element. The positive direction of current flow in a capacitive element is chosen to be the positive

direction of the applied voltage. Then the current in the capacitive element can be defined as

$$i_{ab} = i_C = \mathrm{d}q/\mathrm{d}t \tag{2.10}$$

Subject to Eq. (2.7), the current in a linear capacitive element is

$$i_C = C\,\mathrm{d}v_C/\mathrm{d}t \tag{2.11}$$

If in time t_1 the voltage across a capacitive element changes from zero to v_{C1}, the electric field of the element will store an amount of energy given by

$$W_E = \int_0^{t_1} i_C v_C\,\mathrm{d}t$$

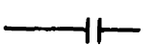
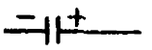
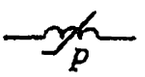
or, subject to Eq. (2.10),

$$W_E = \int_0^{q_1} v_C\,\mathrm{d}q = \int_0^{v_{C1}} v_C C\,(v_C)\,\mathrm{d}v_C \tag{2.12}$$

where q_1 is the excess (free) charge at $v_C = v_{C1}$ (Fig. 2.4).

As follows from Eq. (2.12), the energy stored in the electric field of a capacitive element at v_C is proportional to the area enclosed by the charge-voltage characteristic of the element and the axis of ordi-

TABLE 2.2 Diagram Symbols of Inductors and Capacitors

Component	Diagram symbol
Inductor, air-cored reactor	
Iron-cored inductor	
Fixed capacitor	
Electrolytic capacitor	
Variable capacitor	
Nonlinear capacitor adjustable as a function of parameter P	
Nonlinear inductor adjustable as a function of parameter P	

nates (see Fig. 2.4, where the shaded area is proportional to the energy stored by the electric field of a nonlinear capacitive element at v_{C1}).

From Eq. (2.12) and subject to Eq. (2.7), it follows that in the case of a linear capacitive element at v_C the *energy stored by its electric field* is given by

$$W_E = Cv_C^2/2 = qv_C/2 \tag{2.13}$$

The above analysis shows that, similarly to inductive elements, the capacitive elements of electric networks may be treated as energy accumulators capable of storing an amount of energy.

The diagram symbols for several forms of inductors and capacitors are given in Table 2.2.

In the subsequent analysis of linear sinusoidal a.c. networks we will omit the term "linear".

2.5 Sinusoidal Sources

Commercially, sinusoidal sources are mechanically driven electric generators. In them the mechanical energy developed by steam or hydraulic turbines is converted to electric energy. The construc-

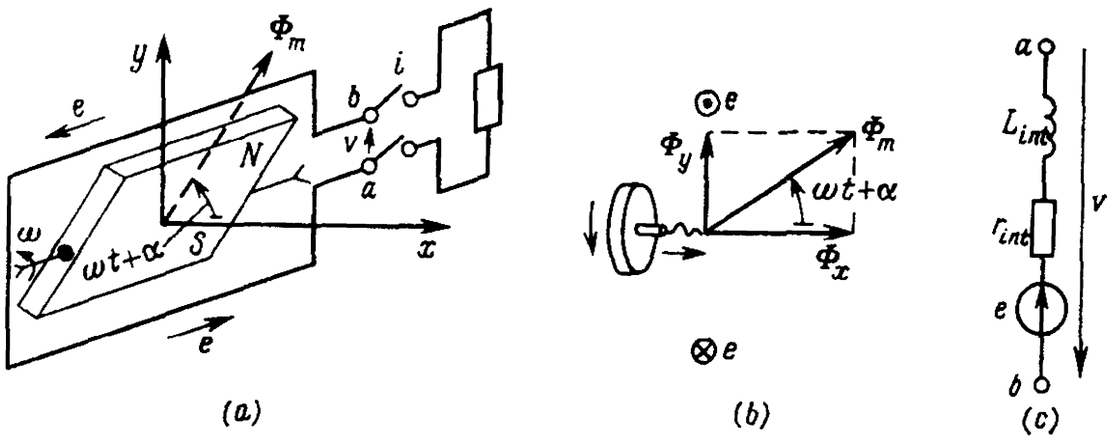


FIGURE 2.5

tion and operation of commercial generators will be examined in detail in later chapters. Here we shall limit ourselves to looking into the principle underlying the operation of such a generator under certain simplifying assumptions.

In sketch form, the arrangement of an elementary generator is shown in Fig. 2.5 a. As is seen, it consists of a fixed flat open turn with terminals *a* and *b*, and a permanent magnet which rotates at a constant *frequency* f, or at a constant *angular frequency* $\omega = 2\pi f$ rad s^{-1}, inside the turn.

The basic SI unit of frequency is the *hertz* (Hz):

$1\ \text{Hz} = 1\ \text{s}^{-1}$

In various fields, use is made of its multiple units:
the kilohertz (kHz):

$1\ \text{kHz} = 1 \times 10^3\ \text{Hz}$

the megahertz (MHz):

$1\ \text{MHz} = 1 \times 10^6\ \text{Hz}$

The reciprocal of the frequency is the *period*

$T = 1/f$

which is measured in seconds.

Let the magnetic flux of the permanent magnet in the simple generator of Fig. 2.5 be Φ_m. From the pattern of the magnetic flux (see Fig. 2.5 b) it follows that the instantaneous value of the magnetic-flux component threading the turn, that is, directed along the x-axis, is

$$\Phi_x = \Phi_m \cos(\omega t + \alpha) = \Phi_m \sin(\omega t + \psi_\Phi) \tag{2.14}$$

where Φ_m is the peak value (amplitude) of the magnetic flux threading the turn; α is the initial angular position (that is, that at time t_0) of the permanent magnet relative to the x-axis; $\psi_\Phi = \pi/2 + \alpha$ is the initial phase of the magnetic flux; and $\omega t + \psi_\Phi$ is the instantaneous phase of the magnetic flux. Here and elsewhere, the *initial phase* defines the value of the *sinewave* function at time $t = 0$.

By the law of electromagnetic induction, a change in the flux linkage of a turn induces an emf in the turn, such that its positive direction (Fig. 2.5 a) is related to the positive direction of the flux Φ_x by the right-handed screw rule (the positive direction of the emf is in the direction of rotation of the screw handle as the screw progressively advances in the direction of the magnetic flux Φ_x). The emf of induction, as it is called, is given by

$$\begin{aligned} e &= -\,\mathrm{d}\Phi_x/\mathrm{d}t = -\,\Phi_m \omega \cos(\omega t + \psi_\Phi) \\ &= E_m \sin(\omega t + \psi_e) \end{aligned} \tag{2.15}$$

where $E_m = \omega \Phi_m$ is the amplitude (peak value) of the induced emf, and $\psi_e = \psi_\Phi - \pi/2 = \alpha$ is the initial phase of the emf.

Figure 2.6 shows plots of the magnetic flux $\Phi_x = \Phi_x(\omega t)$ and of the emf $e = e(\omega t)$ as functions of the phase ωt, that is, time t. It is to be noted that sinusoidal quantities are customarily plotted as functions of ωt. Therefore the initial phase defines the shift of the sinusoidal quantity plotted relative to the origin of coordinates, that is, $\omega t = 0$. The initial phase is always reckoned along the axis

of abscissae from the nearest zero-crossing of the sinusoidal quantity on the left of the axis of ordinates to the origin. If the initial phase is greater than zero, the start of the plot is shifted to the left from the origin of coordinates, as shown in Fig. 2.6. If it is smaller, the

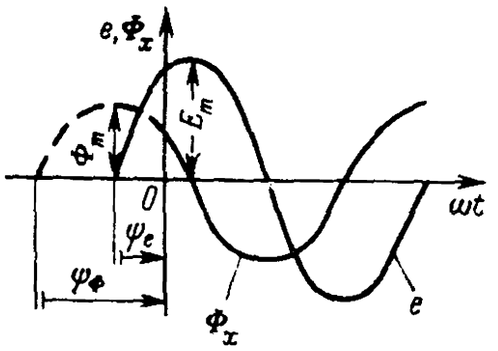


FIGURE 2.6

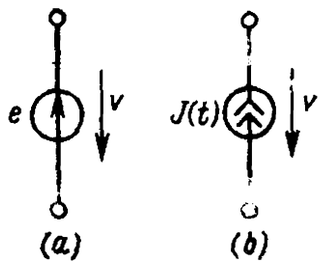


FIGURE 2.7

start of the plot will be shifted to the right of the origin of coordinates.

If we connect a resistor to the terminals *a* and *b* of the generator (see Fig. 2.5 a), a sinusoidal current *i* will be flowing around the circuit thus completed.

Figure 2.5 c shows the equivalent circuit, or model, of a generator. In the diagram, r_{int} represents the resistance of the turn wire, and L_{int} represents the inductance of the turn.

If, in analysis, we may neglect the r_{int} and L_{int} in the model, we will obtain an ideal *sinusoidal emf (voltage) source* (Fig. 2.7 a). If the current in the generator circuit does not practically depend on the external-circuit parameters, we will have an ideal *sinusoidal current source* $J(t)$ (see Fig. 2.7 b), with $J(t) = i_{sc}$ being the generator current with its terminals *a* and *b* short-circuited.

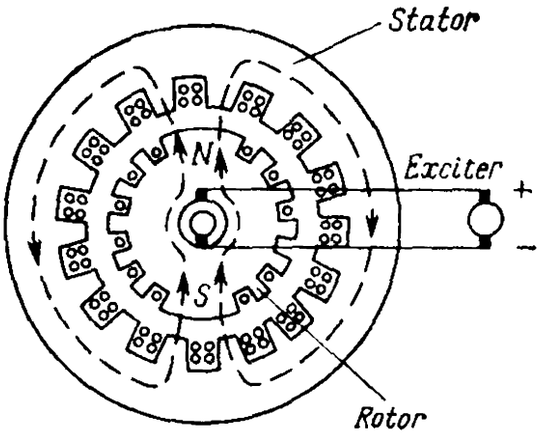


FIGURE 2.8

A typical commercial sinusoidal generator consists of a stationary part called the *stator*, and a rotating part called the *rotor*. The stator is built up of electric sheet steel laminations into a hollow cylinder (Fig. 2.8). On the inside the stator has slots which receive insulated wires appropriately interconnected to form the stator winding where a sinusoidal emf is induced. The rotor is an electromagnet excited by a direct current. The source of the excitation current is an exciter which is a small d.c. generator connected to the rotor winding via the slip rings mounted on the rotor shaft, and brushes.

One revolution of a two-pole (N and S) rotor corresponds to one period of the sinusoidal emf induced in the stator winding. If the stator has p pairs of poles, one revolution of the rotor corresponds to p periods of the sinusoidal emf. If the rotor spins at n revolutions per minute, the generator emf will have pn periods per minute. Thus, the frequency of a sinusoidal emf, or the number of periods per second, is

$$f = pn/60$$

2.6 The Peak, Average and RMS Values of Sinusoidal EMFs, Voltages, and Currents

In a linear circuit driven by a sinusoidal emf, the voltages and currents are likewise sinusoidal:

$$v = V_m \sin(\omega t + \psi_v)$$
$$i = I_m \sin(\omega t + \psi_i)$$

where ω is the angular frequency; ψ_v and ψ_i are the initial phase of the voltage and current respectively; V_m and I_m are the *peak values* (*amplitudes*) of the voltage and current.

The *average value* of a sinusoidal quantity (which may be an emf, a voltage or a current) is the average value over the positive half-period. It is the same as the average absolute value (the value in magnitude). For example, on setting the initial phase equal to zero, the average value of a sinusoidal current is given by

$$I_{av} = \frac{2}{T}\int_0^{T/2} i\,dt = \frac{2}{T}\int_0^{T/2} I_m \sin \omega t\,dt = 2I_m/\pi \qquad (2.16a)$$

Similarly, for an emf and a voltage,

$$E_{av} = 2E_m/\pi,\ V_{av} = 2V_m/\pi \qquad (2.16b)$$

A sinusoidal current flowing in a resistive element of resistance r causes it to rise in temperature because of heat dissipation. The same amount of heat in the same resistive element can be generated by passing through it a certain definite direct current. The value of the direct current defined in such a manner is called the *effective value* of the corresponding sinusoidal current. For example, if a sinusoidal current heats a resistive element in exactly the same way as a direct current of 5 A would, the effective value of the sinusoidal current is 5 A.

When a sinusoidal current flows in a resistive element of resistance r, the thermal energy dissipated there over one period T is given by

$$Q_{\text{heat}} = \int_0^T ri^2 \, dt$$

where i is the instantaneous value of the sinusoidal current.

By the definition of the effective value of a sinusoidal current, the same amount of thermal energy must be dissipated in the same resistive element by a direct current over the same time interval T:

$$Q_{\text{heat}} = rI^2T$$

Therefore,

$$rI^2T = \int_0^T ri^2 \, dt$$

Hence, the effective value of the sinusoidal current is

$$I = \sqrt{\frac{1}{T}\int_0^T i^2 \, dt} \qquad (2.17)$$

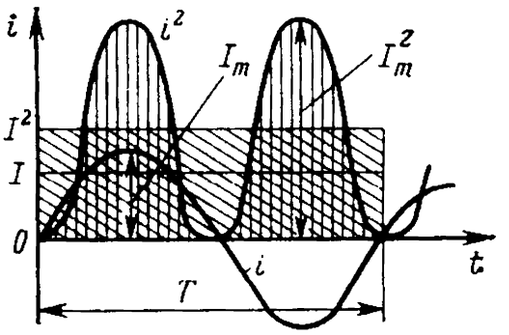


FIGURE 2.9

Thus, the effective value of a sinusoidal current is defined as the root-mean-square value over a period. A plot of sinusoidal current i, time variations in the square of the current i^2, and graphical determination of I^2 (from the equality of the areas $I^2T = \int i^2 \, dt$) and, in consequence, of the rms value I are all shown in Fig. 2.9.

The rms value of a sinusoidal current can readily be defined in terms of its amplitude value:

$$I = \sqrt{\frac{1}{T}\int_0^T I_m^2 \sin^2 \omega t \, dt} = I_m \sqrt{\frac{1}{T}\int_0^T \frac{1-\cos 2\omega t}{2} \, dt}$$

Since $\int_0^T dt = T$, and $\int_0^T \cos 2\omega t \, dt = 0$, then it follows that

$$I = I_m/\sqrt{2} \qquad (2.18a)$$

Or, in words, the rms value of a sinusoidal current is $1/\sqrt{2}$ of its amplitude value.

The rms value of a sinusoidal voltage is defined in a similar manner. The heat dissipated in a resistive element of conductance $g = 1/r$ in time T in the case of a direct voltage is given by

$$Q_{heat} = gV^2T$$

and in the case of a sinusoidal voltage,

$$Q_{heat} = \int_0^T gv^2 \, dt$$

By comparing the two expressions, we can define the rms value fo a sinusoidal voltage as

$$V = V_m/\sqrt{2} \tag{2.18b}$$

Similarly, for any other sinusoidal quantity (emf, magnetic flux, charge, etc.), the rms value is given by

$$A = A_m/\sqrt{2} \tag{2.19}$$

The rms value has been chosen as the basic characteristic of sinusoidal current because in most cases the effect of current is proportional to the square of the rms value (for example, the thermal effect and the force of interaction between the forward and reverse wires of a two-wire line). Several designs of electric measuring instruments (such as the hot-wire type, the electrodynamic type, the moving-iron type, and the electrostatic type) can measure both direct and sinusoidal a.c. currents; when calibrated in terms of direct current and connected in a sinusoidal a.c. circuit, they will indicate the rms value of the sinusoidal current.

In the design of insulation, it is important to remember that twice every period the instantaneous value of a sinusoidal voltage exceeds the rms value by a factor of $\sqrt{2}$. Therefore, the insulation in a sinusoidal a.c. circuit is exposed to more adverse conditions than in a similar d.c. circuit. This is one of the reasons why superlong power transmission lines operate on direct current at very high voltages (up to 1.5 MV).

2.7 Representation of Sinusoidal Quantities

Sinusoidal quantities can be represented in more than one way, namely as trigonometric functions, as plots of time functions, as rotating phasors and, finally, as complex numbers.

In Sections 2.5 and 2.6, for example, we represented them as trigonometric functions, Eqs. (2.14) and (2.15), and as time functions, Fig. 2.6.

Now we shall see how sinusoidal quantities can be represented as rotating phasors and complex numbers.

Representation of sinusoidal quantities as rotating phasors. In order to represent a sinusoidal quantity

$$a = A_m \sin(\omega t + \psi)$$

with an initial phase ψ as a rotating phasor, we draw (Fig. 2.10 a) a radius phasor $\bar{A}_m$ such that its length (on the adopted scale) is equal to the amplitude A_m and its angle with the horizontal axis is ψ. This is the initial position at time $t = 0$. From the tip of the radius

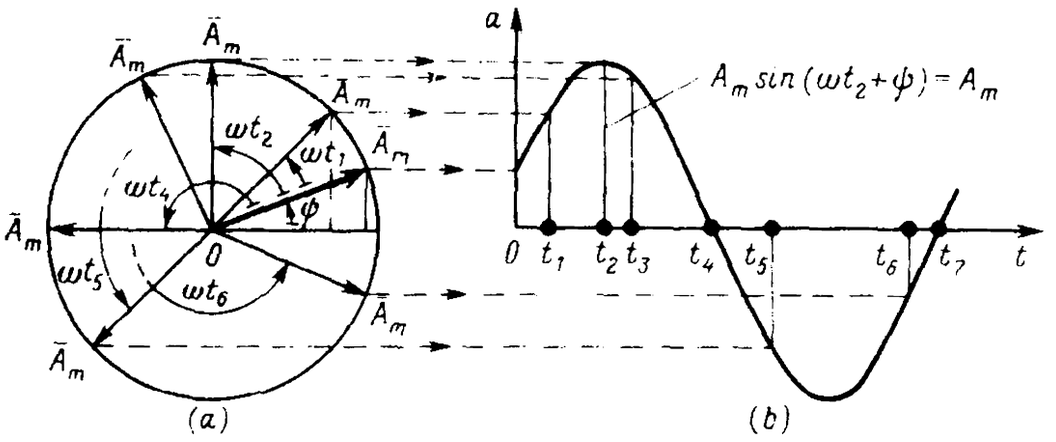


FIGURE 2.10

phasor $\bar{A}_m$ in the initial position we drop a perpendicular of length $A_m \sin \psi$ on the horizontal axis. Suppose that the radius phasor rotates at a constant angular frequency $\omega = 2\pi/T = 2\pi f$ counterclockwise. Here T is the period and f is the cyclic frequency of rotation.

At time t_1 the radius phasor $\bar{A}_m$ turns from the initial position by an angle ωt_1; the length of the perpendicular dropped from its tip is $A_m \sin(\omega t_1 + \psi)$.

Obviously, the length of the perpendicular dropped from the tip of the rotating radius phasor on the horizontal axis is a maximum at time t_2 when $\omega t_2 + \psi = \pi/2$:

$$A_m \sin(\omega t_2 + \psi) = A_m \sin(\pi/2) = A_m$$

Next to the circle described by the tip of the rotating radius phasor, we construct a Cartesian plot of the sinusoidal quantity $A_m \times \sin(\omega t + \psi)$ as a function of the phase ωt or of time t (Fig. 2.10 b). At time t_2 the sinusoidal quantity a is a maximum. As the radius phasor rotates, the sinusoidal quantity $a = A_m (\sin \omega t + \psi)$ decreases while remaining positive, then drops to zero at time t_4, changes sign to become negative at times t_5 and t_6, and again positive beginning at time t_7, and so on.

Rotating phasors offer a compact means for representing a combination of several sinusoidal quantities varying at the same frequency on a single plot, which is an especial convenience in the analysis of complex electric networks.

Representation of sinusoidal quantities as complex numbers. It is easy to change from the rotating-phasor to the complex-number representation of sinusoidal quantities. In order to represent a sinusoidal quantity specified in trigonometric form

$$a = A_m \sin(\omega t + \psi) \tag{2.20}$$

with an initial phase ψ as a complex number, we draw a phasor on a complex plane (Fig. 2.11) from the origin at the angle ψ to the real axis: on the adopted scale the length of the phasor is equal to the amplitude A_m of the sinusoidal quantity, and its tip is located at a point corresponding to a particular complex number — the *complex amplitude* of the sinusoidal quantity

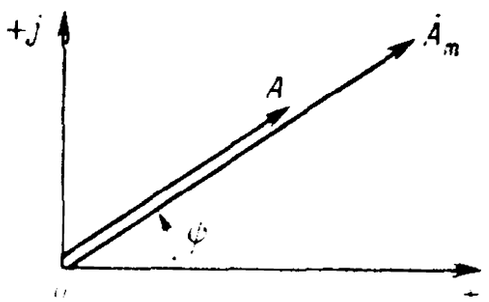


FIGURE 2.11

$$\dot{A}_m = A_m \exp(j\psi) = A_m \angle \psi$$

The same symbol is used to designate the phasor corresponding to the complex amplitude on the complex plane.

As the phase $\omega t + \psi$ of the sinusoidal quantity increases with time, the angle between the phasor and the real axis increases too, and so a rotating phasor results such that

$$\dot{A}_m \exp j(\omega t + \psi) = A_m \cos(\omega t + \psi) + jA_m \sin(\omega t + \psi)$$

It is easy to see that the imaginary part of the rotating phasor is equal to the specified sinusoidal quantity, Eq. (2.20).

In fact, the representation of a sinusoidal quantity as a complex amplitude $\dot{A}_m$ and the corresponding phasor on a complex plane is geometrically similar to the representation of the same quantity as a rotating radius phasor $\overline{A}_m$ at time $t = 0$ (see Fig. 2.10 a). Therefore it might seem that the two representations are practically the same. But this is not so.

With sinusoidal quantities represented as complex numbers, we can use the effective complex method of sinusoidal a.c. circuit analysis which has won general recognition now.

The phasor on a complex plane, whose length on the adopted scale is equal to the rms value of the sinusoidal quantity involved, and the corresponding complex number are referred to as the *complex rms value of that sinusoidal quantity*

$$\dot{A} = \dot{A}_m/\sqrt{2} = A \exp(j\psi) = A \angle \psi \tag{2.21}$$

The same symbol is used to designate the phasor itself on a complex plane (see Fig. 2.11).

The complex value of a sinusoidal quantity can be written in any one of three forms, as follows:

exponential

$$\dot{A} = A \exp(j\psi) = A\angle\psi \tag{2.22}$$

trigonometric

$$\dot{A} = A\cos\psi + jA\sin\psi \tag{2.23}$$

and *algebraic*

$$\dot{A} = A' + jA'' \tag{2.24}$$

where $A' = A\cos\psi$ is the real part and $A'' = A\sin\psi$ is the imaginary part of the complex value of the sinusoidal quantity

$$A = \sqrt{(A')^2 + (A'')^2}$$

and

$$\psi = \arctan(A''/A')$$

We can change from the exponential to the trigonometric form by invoking Euler's formula:

$$\exp(j\psi) = \cos\psi + j\sin\psi \tag{2.25}$$

When $\psi = \pi/2$ and $\psi = -\pi/2$, two frequently used relations follow immediately

$$\exp(j\pi/2) = j \quad \text{and} \quad \exp(-j\pi/2) = -j = 1/j \tag{2.26}$$

In the analysis of sinusoidal a.c. circuits, one mostly uses the complex rms values of sinusoidal quantities. For short they are called the *complex values* of sinusoidal quantities, and the corresponding phasors on a complex plane are referred to as the *phasors of complex values*. For example, in the case of the sinusoidal current

$$i = I_m \sin(\omega t + \psi_i) = 10\sin(\omega t + 45°)$$

the corresponding complex value is

$$\dot{I} = I\exp(j\psi_i) = (10/\sqrt{2})\exp(j45°) = 7.07\angle 45°$$

The collection of phasors representing the complex values of sinusoidal quantities at the same frequency is called a *phasor diagram*. By reference to a phasor diagram, we can replace the addition and subtraction of complex values by the addition and subtraction of the respective phasors. This often simplifies the computations and makes them more instructive.

The relative position of the complex phasors on a phasor diagram will remain unchanged if the initial phases ψ of all the complex

values are increased (decreased) by the same amount. This only implies that all the phasors are turned through the same angle. In the analysis of electric circuit, a phasor diagram is often constructed so that one of the complex phasors is directed along the real axis and is called the *reference* (or *datum*) *phasor*.

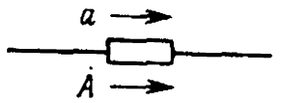


FIGURE 2.12

The sinusoidal quantities (current, voltage, and the like) defining the conditions in an electric circuit periodically vary in direction, but one of them is assumed to be positive. This direction is chosen arbitrarily and indicated by an arrow in the circuit diagram. With the positive direction thus assumed, a sinusoidal quantity can be represented both by its instantaneous value defined as

$$a = A_m \sin(\omega t + \psi)$$

and by the respective complex value

$$\dot{A} = A \angle \psi$$

(see Eq. (2.21)). Thus, the one-to-one correspondence between the representations of sinusoidal currents, voltages and other quantities implies the same positive direction (Fig. 2.12).

2.8 Ohm's Law in Complex Form for Resistive, Inductive and Capacitive Elements

The relations between the currents and voltages associated with resistive, inductive and capacitive elements are governed by the physical processes taking place in them. A mathematical description of the physical processes in each of these elements depends on the choice of an analytical representation for sinusoidal quantities. In our subsequent exposition, when representing sinusoidal currents, voltages and the like, we will use both the respective trigonometric functions and plots, and the complex values.

A. The resistive element. Let the positive direction of the sinusoidal current

$$i_r = I_{rm} \sin(\omega t + \psi_i)$$

in a resistive element of fixed resistance r be the same as the positive direction of the voltage applied to the element (Fig. 2.13). Then the instantaneous values of voltage and current will be connected by a relation similar to Eq. (1.1) and recognized as Ohm's law

$$v_r = r i_r$$

or

$$v_r = r I_{rm} \sin(\omega t + \psi_i) = V_{rm} \sin(\omega t + \psi_v)$$

where the amplitudes of current and voltage are connected by a relation of the form

$$V_{rm} = rI_{rm} \tag{2.27a}$$

and their initial phases are the same

$$\psi_v = \psi_i \tag{2.27b}$$

which means that in a resistive element the current and the voltage vary in phase as shown in Fig. 2.13 for the initial phase $\psi_v = \psi_i > 0$.

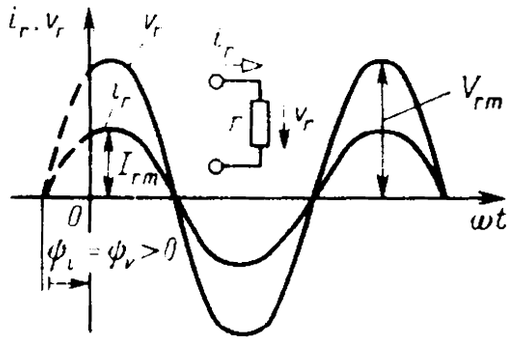


FIGURE 2.13

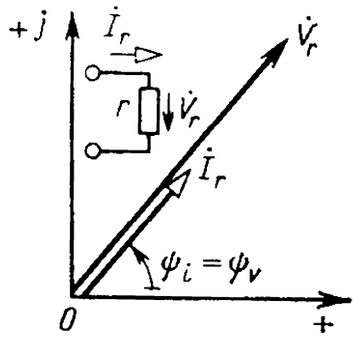


FIGURE 2.14

On dividing the right-hand and left-hand sides of Eq. (2.27a) by $\sqrt{2}$, we obtain the relation defining the rms values of voltage and current in a resistive element

$$V_r = rI_r \tag{2.28}$$

Now we will represent the sinusoidal current in and the sinusoidal voltage across a resistive element in terms of their complex values [see Eq. (2.22)]:

$$\dot{I}_r = I_r \exp(j\psi_i)$$

and

$$\dot{V}_r = V_r \exp(j\psi_v)$$

Since $V_r = rI_r$ (2.28) and $\psi_v = \psi_i$ (2.27b), it follows that the complex current $\dot{I}_r$ in and the complex voltage $\dot{V}_r$ across a resistive element obey Ohm's law in complex form

$$\dot{V}_r = r\dot{I}_r \tag{2.29}$$

The relation between the complex current in and the complex voltage across a resistive element is illustrated in the phasor diagram of Fig. 2.14. From the phasor diagram it is also seen that the complex current and voltage phasors for a resistive element are in phase.

B. The inductive element. If an inductive element carries a sinusoidal current

$$i_L = I_{Lm} \sin (\omega t + \psi_i)$$

then, in accord with the law of electromagnetic induction, Eq. (2.3), the voltage across the inductive element is given by

$$v_L = -e_L = L\, di_L/dt = \omega L I_{Lm} \cos (\omega t + \psi_i)$$
$$= V_{Lm} \sin (\omega t + \psi_i + \pi/2) = V_{Lm} \sin (\omega t + \psi_v)$$

where the amplitudes of voltage and current are connected by a relation of the form

$$V_{Lm} = \omega L I_{Lm} \tag{2.30a}$$

and their initial phases by the relation

$$\psi_v = \psi_i + \pi/2 \tag{2.30b}$$

On dividing the right-hand and left-hand sides of Eq. (2.30a) by $\sqrt{2}$, we obtain the relation for the rms voltage across and the rms current in an inductive element:

$$V_L = \omega L I_L = x_L I_L \tag{2.31}$$

The instantaneous values of sinusoidal current in and sinusoidal voltage across an inductive element appear in the plot of Fig. 2.15 (constructed for $\psi_i > 0$) from which it is seen that the sinusoidal current i_L lags behind the sinusoidal voltage v_L in phase by an angle $\varphi = \psi_v - \psi_i = \pi/2$.

The quantity $x_L = \omega L$ in Eq. (2.31) has the units of ohm and is called the *inductive reactance*, while its reciprocal $b_L = 1/\omega L$ is called the *inductive susceptance*. The values of x_L and b_L are the parameters of inductive elements in sinusoidal a.c. circuits.

Inductive reactance is proportional to the angular frequency of sinusoidal current, so it is zero for direct current ($\omega = 0$). For this reason many pieces of electric equipment designed for operation in a.c. circuits may not be used in d.c. circuits. Their d.c. impedance is relatively small, and the heavy direct current might be destructive (say, for the primary winding of the transformer in a radio receiver).

The sinusoidal current i_L in and the sinusoidal voltage v_L across an inductive element can be represented by the corresponding complex values as follows:

$$\dot{I}_L = I_L \exp (j\psi_i)$$

and

$$\dot{V}_L = V_L \exp (j\psi_v)$$

A phasor diagram for an inductive element is shown in Fig. 2.16. As is seen, the phasor of the complex current $\dot{I}_L$ lags behind the phasor of the complex voltage $\dot{V}_L$ in phase by $\pi/2$, so the phase shift between them is $\varphi = \pi/2$ as shown in Fig. 2.15.

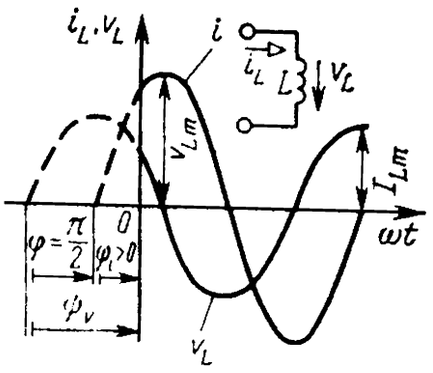


FIGURE 2.15

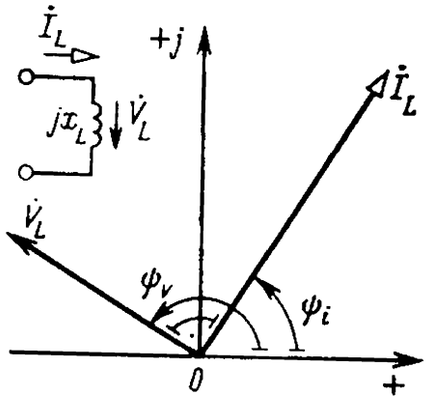


FIGURE 2.16

Using Eqs. (2.31) and (2.26), we deduce Ohm's law in complex form for an inductive element

$$\dot{V}_L = \omega L I \exp(j\psi_v) = \omega L I \exp j(\psi_i + \pi/2)$$

or

$$\dot{V}_L = j\omega L \dot{I}_L = jx_L \dot{I}_L \tag{2.32}$$

The quantity $j\omega L = jx_L$ in Eq. (2.32) is called the *complex inductive reactance*, and its reciprocal $1/j\omega L = -jb_L$ is called the *complex inductive susceptance.*

Alternatively, the complex voltage across an inductive element may be expressed in terms of the complex flux linkage. As follows from Eq. (2.1),

$$\dot{\Psi} = L\dot{I}_L$$

and on the basis of Eq. (2.32),

$$\dot{V}_L = -\dot{E}_L = j\omega\dot{\Psi} \tag{2.33}$$

Equation (2.33) is a mathematical statement of the law of electromagnetic induction, Eq. (2.3), in complex form.

Sometimes, the phasor diagram of an inductive element (Fig. 2.16) may be expanded to include the phasor $\dot{E}_L$ which is opposite in direction and equal in magnitude to $\dot{V}_L$. This convention is, for example, used in the analysis of transformers.

C. The capacitive element. If the voltage between the terminals of a capacitive element is sinusoidal

$$v_C = V_{Cm} \sin(\omega t + \psi_v)$$

then, in accord with Eq. (2.11), the associated sinusoidal current is given by

$$i_C = C\,dv_C/dt = \omega C V_{Cm} \cos(\omega t + \psi_v)$$
$$= I_{Cm} \sin(\omega t + \psi_v + \pi/2) = I_{Cm} \sin(\omega t + \psi_i)$$

where the amplitudes of current and voltage are connected by a relation of the form

$$I_{Cm} = \omega C V_{Cm} \tag{2.34a}$$

and their initial phases by the relation

$$\psi_i = \psi_v + \pi/2 \tag{2.34b}$$

On dividing the right-hand and left-hand sides of Eq. (2.34a) by $\sqrt{2}$, we obtain the relation for the rms values of the voltage across and the current in a capacitive element

$$V_C = (1/\omega C)\, I_C = I_C/b_C \tag{2.35}$$

The quantity $b_C = \omega C$ in Eq. (2.35) has the units of *siemens* (S) and is called the *capacitive susceptance*, and its reciprocal $x_C = 1/\omega C$

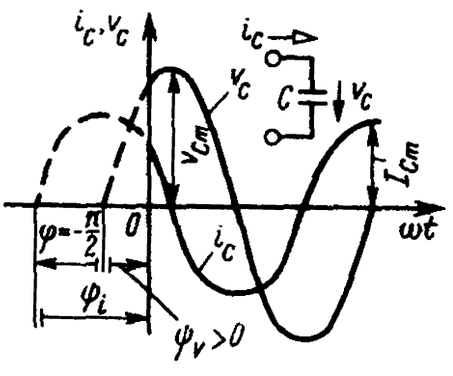


FIGURE 2.17

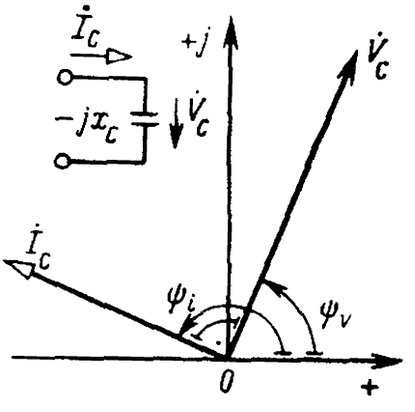


FIGURE 2.18

is called the *capacitive reactance*. The values of x_C and b_C are the parameters of the capacitive elements used in sinusoidal a.c. circuits.

In contrast to inductive reactance, capacitive reactance decreases with increasing frequency of a sinusoidal current. At constant voltage, the reactance of a capacitive element is infinite.

The instantaneous values of the sinusoidal voltage across and the sinusoidal current in a capacitive element are shown in the plot of Fig. 2.17 (constructed for $\psi_v > 0$). As is seen, v_C lags behind i_C in phase by an angle $\psi_i - \psi_v = \pi/2$. In other words, the phase shift between them is $\varphi = \psi_v - \psi_i = -\pi/2$.

In complex notation, the sinusoidal current i_C and the sinusoidal voltage v_C for a capacitive element will be written as

$$\dot{I}_C = I_C \exp(j\psi_i)$$

and

$$\dot{V}_C = V_C \exp(j\psi_v)$$

A phasor diagram for a capacitive element is shown in Fig. 2.18. As is seen, the complex voltage phasor $\dot{V}_C$ lags behind the complex current phasor $\dot{I}_C$ in phase by $\pi/2$.

In view of Eqs. (2.34) and (2.26), Ohm's law in complex form for a capacitive element is

$$\dot{V}_C = (1/j\omega C)\,\dot{I}_C = -jx_C\dot{I}_C \tag{2.36}$$

The quantity $1/j\omega C = -jx_C$ in Eq. (2.36) is called the *complex reactance of a capacitive element*, and its reciprocal $j\omega C = jb_C$ is called the *complex susceptance of a capacitive element.*

2.9 Kirchhoff's Laws for Sinusoidal A.C. Circuits

The mathematical statement of Kirchhoff's two laws depends on the choice of a representation for sinusoidal quantities. We will use trigonometric functions and the corresponding complex values. In the trigonometric form of representation, Kirchhoff's laws relate the instantaneous values of the respective sinusoidal quantities (for any instant of time). In the complex-number form of representation, these laws relate the complex values of the same quantities.

A. Kirchhoff's current (or first) law. This law states that the algebraic sum of currents at any node of an electric circuit at any instant of time is equal to zero. For sinusoidal a.c. circuits this implies that in the branches meeting at any node the algebraic sum of instantaneous currents is zero:

$$\sum_{k=1}^{n} i_k = 0 \tag{2.37}$$

that is,

$$\sum_{k=1}^{n} I_{mk} \sin(\omega t + \psi_{ik}) = 0 \tag{2.38}$$

where n is the number of branches meeting at a node. In our further discussion we will assign a "+" sign to the currents entering a node, and a "—" sign to the currents leaving a node.

As an example, Fig. 2.19 shows a plot of the instantaneous values of three sinusoidal currents at one node:

$$i_1 = I_{m1} \sin(\omega t + \psi_{i1})$$
$$i_2 = I_{m2} \sin(\omega t + \psi_{i2})$$
$$i_3 = I_{m3} \sin(\omega t + \psi_{i3})$$

for the assumed positive direction. By Kirchhoff's current law

$$\sum_{k=1}^{3} i_k = i_1 + i_2 - i_3 = 0$$

for any instant of time.

If we wish to re-cast the mathematical statement of *Kirchhoff's current law in complex form*, we should represent all the currents in (2.38) with their complex values as defined in (2.21)

$$I_k = I_k \angle \psi_{ik}$$

In complex notation Kirchhoff's current law is written as

$$\sum_{k=1}^{n} \dot{I}_k = 0 \tag{2.39}$$

or in words, the algebraic sum of the complex currents in all the branches meeting at a node of a sinusoidal a.c. circuit is zero. Here,

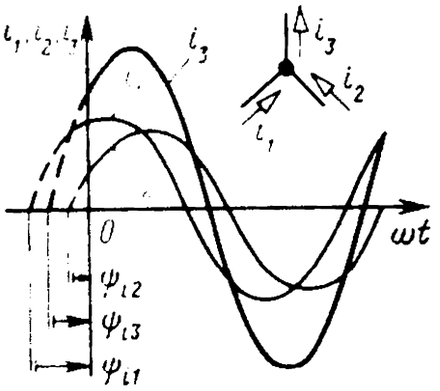


FIGURE 2.19

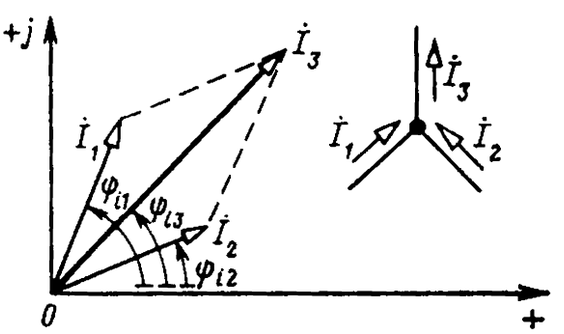


FIGURE 2.20

too, we assign a "+" sign to the currents entering the node, and a "—" sign to the currents leaving the node.

Figure 2.20 shows a phasor diagram for three complex currents: $\dot{I}_1 = I_1 \angle \psi_{i1}$, $\dot{I}_2 = I_2 \angle \psi_{i2}$, and $\dot{I}_3 = I_3 \angle \psi_{i3}$. On the phasor diagram either of the two following equalities must be satisfied:

$$\sum_{k=1}^{3} \dot{I}_k = \dot{I}_1 + \dot{I}_2 - \dot{I}_3 = 0$$

or

$$\dot{I}_1 + \dot{I}_2 = \dot{I}_3$$

B. Kirchhoff's voltage (second) law. This law states that the algebraic sum of the voltages across all the resistive, inductive and capacitive (that is, all passive) elements in any loop (mesh) of an electric circuit is equal at any instant of time to the algebraic sum of all the emfs in that loop (mesh). In sinusoidal a.c. circuits the various emfs and the voltages across the passive elements in any loop (mesh) are varying all the time. Yet the algebraic sums of the voltages and of the emfs are the same:

$$\sum_{k=1}^{n} v_k = \sum_{k=1}^{m} e_k \tag{2.40}$$

or

$$\sum_{k=1}^{n} V_{mk} \sin(\omega t + \psi_{vk}) = \sum_{k=1}^{m} E_{mk} \sin(\omega t + \psi_{ek})$$

where n is the number of passive elements and m is the number of emfs in the loop (mesh). We assume that all the sinusoidal voltages v_k and emfs e_k in (2.40) for which the positive direction coincides with an arbitrarily chosen direction for traversal around the loop are written with a "+" sign; otherwise they are written with a "—" sign. Taking Fig. 2.21 as an example, the loop is traversed clockwise, so, by Kirchhoff's voltage law

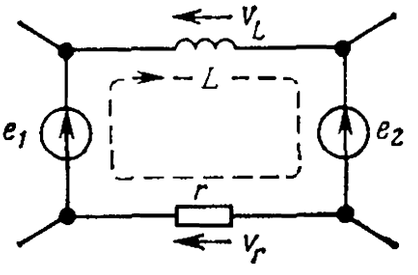


FIGURE 2.21

$$-v_L + v_r = e_1 - e_2$$

If we wish to express *Kirchhoff's voltage law in complex form*, all the sinusoidal voltages v_k and emfs e_k in (2.40) should be replaced with their complex values in accord with Eq. (2.21):

$$\dot{V}_k = V_k \angle \psi_{vk}$$

and

$$\dot{E}_k = E_k \angle \psi_{ek}$$

In complex notation Kirchhoff's voltage law is written as

$$\sum_{k=1}^{n} \dot{V}_k = \sum_{k=1}^{m} \dot{E}_k \tag{2.41}$$

Or in words, the algebraic sum of the complex voltages across all the passive (resistive, inductive and capacitive) elements in any loop (mesh) of a sinusoidal a.c. circuit is equal to the algebraic sum of all complex emfs in that loop (mesh). Here we assign a "+" sign to the complex voltages and emfs whose positive directions are the same as an arbitrary direction for traversal of the loop; otherwise

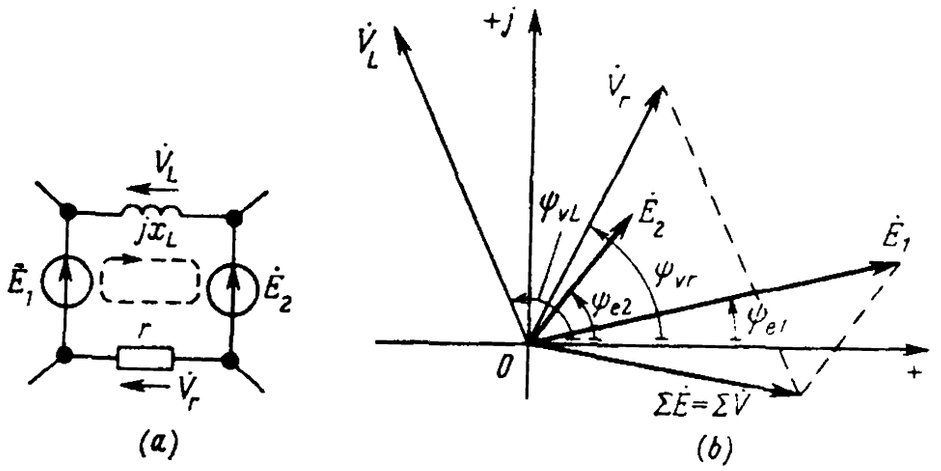


FIGURE 2.22

we assign a "—" sign. Taking the loop of Fig. 2.21, also shown in Fig. 2.22 a, as an example, we may write on the basis of Kirchhoff's voltage law

$$\dot{V}_r - \dot{V}_L = \dot{E}_1 - \dot{E}_2$$

Figure 2.22 b shows a phasor diagram for the emfs and voltages in that loop. It convincingly illustrates Kirchhoff's voltage law in complex form.

2.10 Analysis of Sinusoidal A.C. Circuits by the Complex Method

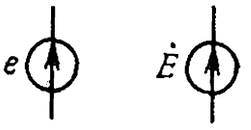
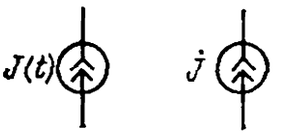
As has been shown in Sec. 2.7, there is a one-to-one correspondence between the instantaneous values of sinusoidal quantities, Eq. (2.20), and their complex values, Eq. (2.21). Therefore any form of representation for sinusoidal quantities may be used in order to define the conditions in a sinusoidal a.c. circuit. When, however, sinusoidal quantities are presented in compex notation, Ohm's laws for resistive elements, (2.29), inductive elements, (2.32), and capacitive elements, (2.36), and the two Kirchhoff's laws, (2.39) and (2.41), are set up as algebraic rather than differential equations. Simultaneous solution of algebraic equations in order to find the complex values of currents in and of voltages across all the elements of an electric circuit, that is, the use of the complex method of analysis, is a relatively simple task. Once the complex values are found, we

can immediately write, if necessary, the corresponding instantaneous values of the sinusoidal quantities involved.

The analysis of a sinusoidal a.c. circuit by the complex method may conveniently be divided into several, logically independent steps.

(1) The initial specifications of all the elements of the circuit involved are presented in complex form. To do so, firstly, the sinusoidal emfs or currents of energy sources specified by their instantaneous values (in trigonometric form) are presented in complex form (see Table 2.3) and, secondly, the corresponding complex reactances

TABLE 2.3 Instantaneous and Complex Values of Sinusoidal EMFs and Currents

Source	Instantaneous value	Complex value	Symbol
EMF	$e = E_m \sin(\omega t + \psi_e)$	$\dot{E} = (E_m/\sqrt{2})\exp(j\psi_e)$	
Current	$J(t) = J_m \sin(\omega t + \psi_i)$	$\dot{J} = (J_m/\sqrt{2})\exp(j\psi_i)$	

or susceptances of all the inductive and capacitive elements of the circuit (see Table 2.4) are found.

TABLE 2.4 Complex Impedances (Resistance and Reactance) and Complex Admittances (Conductance and Susceptance) of Passive Elements

Element	Parameter	Complex impedance	Complex admittance
Resistive	r	r	$1/r = g$
Inductive	L	$j\omega L = jx_L$	$1/j\omega L = -jb_L$
Capacitive	C	$1/j\omega C = -jx_C$	$j\omega C = jb_C$

(2) Positive directions are chosen for the currents in all the branches and labelled with arrowheads in the circuit diagram.

(3) Using Ohm's and Kirchhoff's laws in complex form and noting the chosen positive direction of current in the branches, a set of equations are set up to define the conditions in the circuit.

(4) The set of equations thus obtained is solved, that is, the complex currents in the circuit branches and the complex voltages across the circuit elements are found.

The complex values of currents and voltages thus determined uniquely define the corresponding instantaneous values of sinusoidal currents and voltages.

As an example, we will apply the above procedure to the sinusoidal a.c. circuit of Fig. 2.23 a which contains a source of emf $e = E_m \sin(\omega t + \psi_e)$, a source of current $J(t) = J_m \sin(\omega t + \psi_i)$,

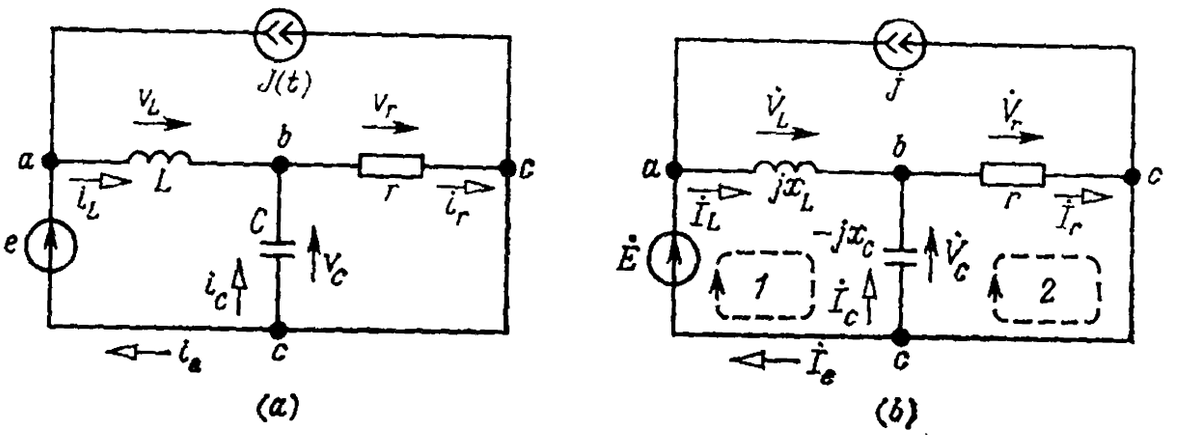


FIGURE 2.23

a resistive element r, an inductive element L, and a capacitive element C.

1. To begin with, we re-write the sinusoidal emf and current specified by their instantaneous values in complex form on the basis of Eq. (2.21) and Table 2.3:

$$\dot{E} = E\angle\psi_e; \quad \dot{J} = J\angle\psi_i$$

Next we find the complex reactance of the inductive element, $j\omega L = jx_L$, and of the capacitive element, $1/j\omega C = -jx_C$, of the circuit from Table 2.4.

The circuit of Fig. 2.23 a with its parameters represented in complex form is drawn up in Fig. 2.23 b.

2. Now we assume positive directions for the unknown currents in the branches (see Fig. 2.23 a) and positive directions for the voltages across the circuit elements so that they are the same as the directions of the currents. The positive directions for the respective complex values are the same (see Fig. 2.23 b).

3. Noting the assumed positive directions for the currents and voltages, we write a complete set of equations for circuit analysis. Since the circuit has three nodes (a, b, and c), we use Kirchhoff's current

law in complex form, Eq. (2.39), and set up an equation for two nodes, say a and b:

$$\dot{J} + \dot{I}_e - \dot{I}_L = 0 \tag{2.42a}$$

$$\dot{I}_L + \dot{I}_C - \dot{I}_r = 0 \tag{2.42b}$$

The circuit has three loops (meshes), so in the general case three independent equations should have been written by Kirchhoff's voltage law. However, the upper loop contains a current source which implies that the current $\dot{J}$ in that loop is known, so we only need to write equations for the remaining two loops labelled 1 and 2 in Fig. 2.23b:

$$\dot{V}_L - \dot{V}_C = \dot{E} \tag{2.42c}$$

$$\dot{V}_r + \dot{V}_C = 0 \tag{2.42d}$$

Applying Ohm's law in complex form for a resistive element, Eq. (2.29), for an inductive element, Eq. (2.32), and for a capacitive element, Eq. (2.36), we write

$$\begin{aligned} \dot{V}_r &= r\dot{I}_r \\ \dot{V}_L &= jx_L\dot{I}_L \\ \dot{V}_C &= -jx_C\dot{I}_C \end{aligned} \tag{2.42e}$$

Hence, we may write Eqs. (2.42c) and (2.42d) as

$$jx_L\dot{I}_L + jx_C\dot{I}_C = \dot{E} \tag{2.42f}$$

$$r\dot{I}_r - jx_C\dot{I}_C = 0 \tag{2.42g}$$

4. By solving the set of four algebraic equations (2.42a), (2.42b), (2.42f) and (2.42g), in four unknown currents, we find their complex values:

$$\dot{I}_r = I_r \exp(j\psi_{ir})$$

$$\dot{I}_L = I_L \exp(j\psi_{iL})$$

$$\dot{I}_C = I_C \exp(j\psi_{iC})$$

$$\dot{I}_e = I_e \exp(j\psi_{ie})$$

The instantaneous values corresponding to the above complex values are as follows:

$$i_r = \sqrt{2}\, I_r \sin(\omega t + \psi_{ir})$$

$$i_L = \sqrt{2}\, I_L \sin(\omega t + \psi_{iL})$$

$$i_C = \sqrt{2}\, I_C \sin(\omega t + \psi_{iC})$$

$$i_e = \sqrt{2}\, I_e \sin(\omega t + \psi_{ie})$$

The complex values of voltages are found by Eq. (2.42e), and their instantaneous values are written in the same manner as for the currents.

As is the case with linear d.c. circuits, linear sinusoidal a.c. networks obey the principle of superposition (see Sec. 1.15). Therefore, the analysis of linear sinusoidal circuits can be simplified through the use of the various techniques applicable to linear d.c. circuits, such as the circuit transformation method (see Sec. 1.12), the node-pair method (see Sec. 1.13), the loop (mesh) current method (see Sec. 1.14), the equivalent generator method (see Sec. 1.17), etc. As a rule, these analysis techniques are used in combination with the complex method. Obviously, the mathematical statements invoked in the analysis of d.c. circuits remain in force for a.c. circuits as well. We only need to replace all the emfs, voltages and currents with the complex values of the respective sinusoidal quantities, and the resistances of the circuit elements with their complex impedances (resistances and reactances).

In our further discussion, the complex values of emfs, voltages, currents, and the like will be referred to for brevity as, say, a complex emf, a complex voltage, a complex current, or still simpler as emfs, voltages, or currents, etc.

2.11 The Single-Loop Sinusoidal A.C. Circuit

In a single-loop sinusoidal a.c. circuit (Fig. 2.24) containing a source of sinusoidal emf

$$= E_m \sin(\omega t + \psi_e)$$

the current is also sinusoidal

$$i = I_m \sin(\omega t + \psi_i)$$

and the voltages across the resistive, inductive and capacitive elements are as follows:

$$v_r = V_{rm} \sin(\omega t + \psi_{vr})$$

$$v_L = V_{Lm} \sin(\omega t + \psi_{vL})$$

$$v_C = V_{Cm} \sin(\omega t + \psi_{vC})$$

The conditions in a single-loop sinusoidal a.c. circuit can conveniently be analysed by the complex method. To this end, we write all the sinusoidal quantities in complex notation in accord with Eq. (2.21):

$$\dot{E} = E \angle \psi_e$$

$$\dot{I} = I \angle \psi_i$$

$$\dot{V}_r = V_r \angle \psi_{vr}$$

$$\dot{V}_L = V_L \angle \psi_{vL}$$

$$\dot{V}_C = V_C \angle \psi_{vC}$$

The positive directions of the currents, emfs and voltages are shown in Fig. 2.24 by arrowheads.

We choose to traverse the loop in the clockwise direction and write an equation by Kirchhoff's voltage law, Eq. (2.41):

$$\dot{V}_L + \dot{V}_r + \dot{V}_C = j\omega L\dot{I} + r\dot{I} - j/\omega C\dot{J} = \dot{E} \qquad (2.43)$$

which takes into account Ohm's law for the resistive element, (2.29), the inductive element (2.32), and the capacitive element (2.36).

From Eq. (2.43) we find the complex current in the circuit

$$\dot{I} = \frac{\dot{E}}{r + j(\omega L - 1/\omega C)}$$

or

$$\dot{I} = \frac{\dot{V}}{r + j(\omega L - 1/\omega C)} \qquad (2.44)$$

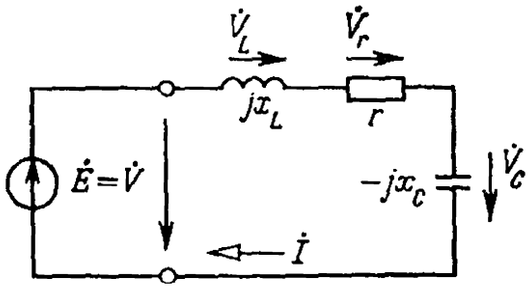


FIGURE 2.24

where $\dot{V} = V \exp(j\psi_v) = \dot{E} = E \exp(j\psi_e)$ is the voltage between the terminals of the single-loop circuit of Fig. 2.24.

The quantity in the denominator of Eq. (2.44) is called the *complex impedance* of a single-loop circuit

$$Z = r + j(\omega L - 1/\omega C) = r + j(x_L - x_C) \qquad (2.45a)$$

and its reciprocal

$$Y = 1/Z$$

is called the *complex admittance* of a single-loop circuit.

Each value of Z, that is, each complex number corresponds to a point in a complex plane. The point is uniquely located by a phasor

in a complex plane (Fig. 2.25). This phasor is a geometric interpretation of the complex impedance and is likewise symbolized by $\underline{Z}$.

The phasor diagram in Fig. 2.25 also shows the terms of the complex impedance as phasors for two cases: $x_L > x_C$ (Fig. 2.25 a) and $x_L < x_C$ (Fig. 2.25 b). In the former case the complex impedance

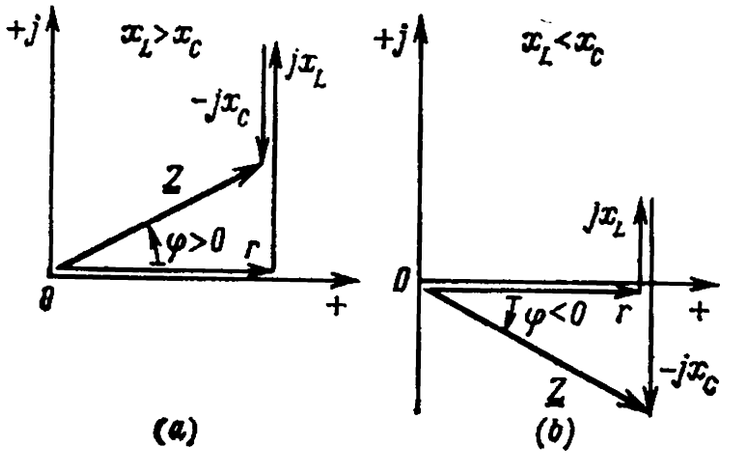


FIGURE 2.25

is inductive in its effect; in the latter case it is capacitive. On the basis of the geometric interpretation of complex impedance we can readily go over from its algebraic representation in Eq. (2.45a) to its trigonometric and exponential forms:

$$\underline{Z} = z \cos \varphi + jz \sin \varphi \tag{2.45b}$$

$$\underline{Z} = z \exp (j\varphi) = z \angle \varphi \tag{2.45c}$$

where

$$z = |\underline{Z}| = \sqrt{r^2 + (x_L - x_C)^2}$$

is the magnitude of complex impedance or simply impedance, and

$$\varphi = \arctan \frac{x_L - x_C}{r}$$

is the argument of complex impedance. Depending on the sign of the difference $(x_L - x_C)$, the argument of complex impedance can be either positive (lagging or inductive) or negative (leading or capacitive). In any case, however, $|\varphi| \leqslant \pi/2$.

On substituting (2.45c) in (2.44), we obtain an expression for the current in the circuit on the basis of *Ohm's law for a single-loop circuit*:

$$\dot{I} = \dot{E}/\underline{Z} = (E/z) \exp j (\psi_e - \varphi)$$

or

$$\dot{I} = I \exp (j\psi_i) = \dot{V}/\underline{Z} = (V/z) \exp j (\psi_v - \varphi) \tag{2.46}$$

that is,

$$I = V/z \text{ and } \psi_i = \psi_v - \varphi \tag{2.47}$$

If we know the values of the resistive, inductive and capacitive elements and the voltage between the terminals of a single-loop circuit (see Fig. 2.24), then, by invoking Ohm's law for such a circuit, as defined in (2.46), we can uniquely determine the complex current in the circuit. If we know the complex current in such a circuit, the complex voltages across the resistive, inductive and capacitive ele-

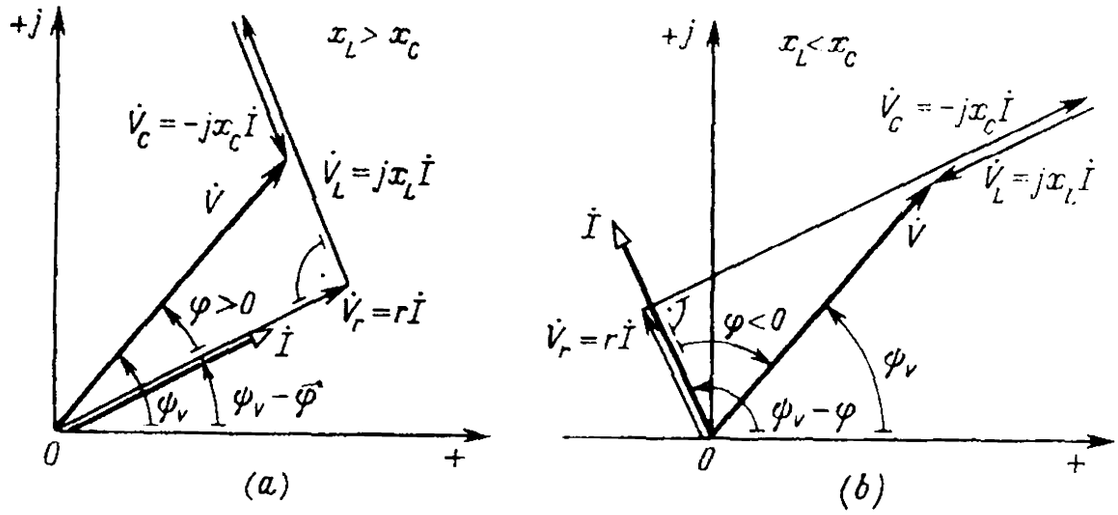


FIGURE 2.26

ments can be calculated by Eqs. (2.29), (2.32), and (2.36), respectively.

Figure 2.26 shows phasor diagrams for the current and voltages in the single-loop circuit of Fig. 2.24 for two cases: $x_L > x_C$ (Fig. 2.26 a) and $x_L < x_C$ (Fig. 2.26 b) for the same specified voltage $\dot{V} = V \angle \psi_v$.

When the complex impedance of a circuit is inductive in its effect, the current $\dot{I}$ lags behind the voltage $\dot{V}$ in phase, because $\varphi > 0$ (Fig. 2.25 a) and, as follows from Eq. (2.47), $\psi_i < \psi_v$. When the complex impedance of a circuit is capacitive in its effect, the current in the circuit leads the voltage in phase, because $\varphi < 0$ (see Fig. 2.25 b) and, as follows from Eq. (2.47), $\psi_i > \psi_v$. On the phasor diagram the positive values of the phase angle φ are counted counterclockwise from the complex current phasor $\dot{I}$; its negative values are counted clockwise.

When a single-loop circuit contains several resistive, inductive and capacitive elements connected in series, the complex impedance is given by

$$\underline{Z} = \Sigma R + j\,(\Sigma x_L - \Sigma x_C) = r + jx \tag{2.48}$$

where $r = \Sigma R$ is the *resistance* and $x = \Sigma x_L - \Sigma x_C$ is the *reactance* of this single-loop circuit. In the resistance, electric energy is irreversibly converted to other forms of energy; in the reactance such a conversion does not happen.

The concepts of resistance and reactance we have just introduced for a single-loop circuit are equally applicable to the more elaborate circuits. In the general case, any passive, two-terminal sinusoidal a.c. circuit has both a resistance and a reactance.

The voltage across the equivalent-circuit element corresponding to the resistance or reactance of a circuit is called the *voltage drop.*

Graphically, Eq. (2.48) can be represented in a complex plane by an impedance triangle as shown in Fig. 2.27. The impedance tri-

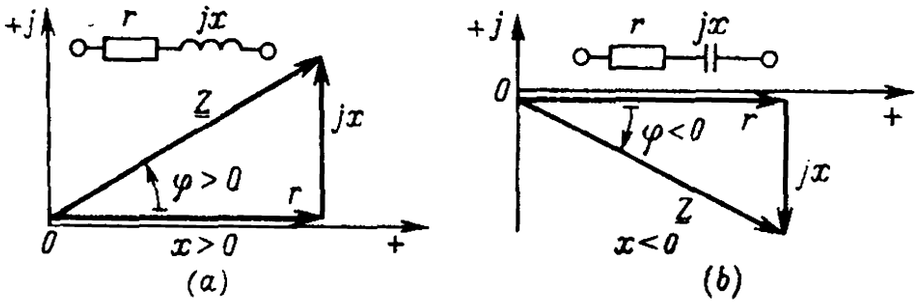


FIGURE 2.27

angle appearing in Fig. 2.27 a corresponds to $x > 0$, that is, when the complex impedance is inductive in its effect, and the impedance triangle in Fig. 2.27 b corresponds to $x < 0$, that is, when the complex impedance is capacitive. The equivalent circuits of the physical networks are given in the same figures.

Referring to the impedance triangle, we can readily derive the trigonometric and exponential expressions for the complex impedance of a single-loop passive network, which are fully identical with Eqs. (2.45), with the magnitude z and the argument φ of the complex impedance defined in (2.48) taking the following form:

$$z = \sqrt{r^2 + x^2} \tag{2.49a}$$

$$\varphi = \arctan (x/r) \tag{2.49b}$$

If we know the complex impedance $\underline{Z}_k$ for any section of a single-loop circuit, then the complex impedance of the complete single-loop circuit will be

$$\underline{Z} = \sum_{k=1}^{n} \underline{Z}_k = \underline{Z}_1 + \underline{Z}_2 + \ldots + \underline{Z}_k + \ldots + \underline{Z}_n$$

where n is the number of sections.

2.12 The Resistance, Reactance and Impedance of a Passive Two-Terminal Network

In Figure 2.28, a passive circuit external to an energy source is shown as a passive two-terminal network (or one-port). The parameter of such a one-port is its input complex impedance, that is, the complex impedance seen looking into terminals a and b:

$$\underline{Z} = \dot{E}/\dot{I} = \dot{V}/\dot{I} = z \angle \varphi = z \cos \varphi + jz \sin \varphi = r + jx$$

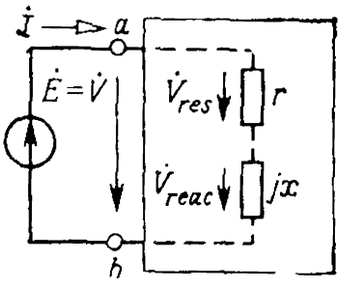


FIGURE 2.28

where $\dot{V} = V \angle \psi_v$ is the complex voltage across the one-port, $\dot{I} = I \angle \psi_i$ is the complex current in the one-port, and $\varphi = \psi_v - \psi_i$ is the argument of the complex impedance such that $|\varphi| \leqslant \pi/2$.

From the above expression it follows that any passive one-port may be represented by an equivalent circuit consisting of a series combination of a resistance r and a reactance x. The impedance of a passive one-port is given by Eq. (2.49a). Depending on the sign of

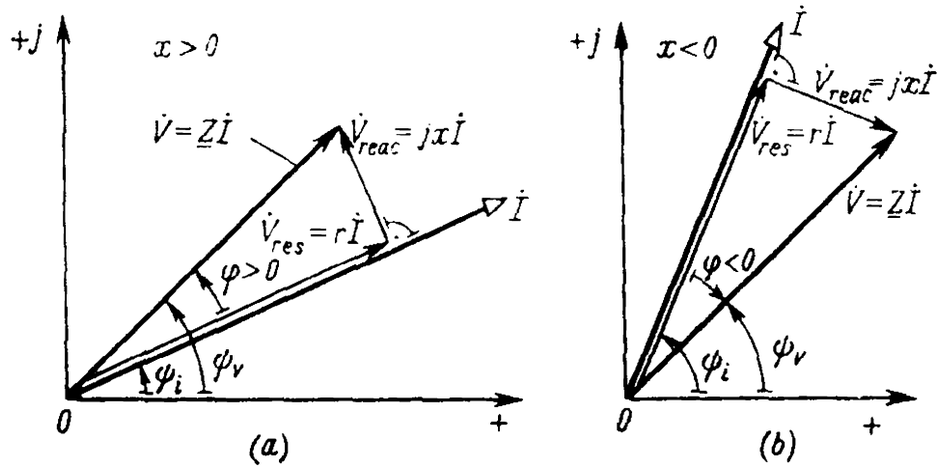


FIGURE 2.29

the reactance x, the complex impedance of a passive one-port may be inductive ($x > 0$, see Fig. 2.27 a) or capacitive ($x < 0$, see Fig. 2.27 b).

Figure 2.29 shows phasor diagrams plotted for the passive one-port of Fig. 2.28, where $\dot{V}_{res} = r\dot{I}$ is the *resistive component* and $\dot{V}_{reac} = jx\dot{I}$ is the *reactive component* of the voltage $\dot{V}$ between the terminals of the one-port.

On a complex plane, the complex voltage phasors $\dot{V}_{res}$, $\dot{V}_{reac}$ and $\dot{V}$ form a *voltage triangle*:

$$\dot{V} = \dot{V}_{res} + \dot{V}_{reac}$$

The magnitude of the resistive voltage component phasor is

$$V_{res} = V \cos \varphi$$

and this phasor runs in phase with current phasor $\dot{I}$. The magnitude of the reactive voltage component phasor is $V_{reac} = V \mid \sin \varphi \mid$, and this phasor is shifted in phase relative to $\dot{I}$ by $\mid \pi/2 \mid$. Thus, the *inductive reactive voltage* leads the current $\dot{I}$ in phase by $\pi/2$ (Fig. 2.29 a), and the *capacitive inductive voltage* lags behind the current $\dot{I}$ in phase by $\pi/2$ (Fig. 2.29 b). From the voltage triangle it follows that

$$V = \sqrt{V_{res}^2 + V_{reac}^2}$$

2.13 Power Relations in Resistive, Inductive and Capacitive Elements

Power relations in sinusoidal a.c. circuits are fairly complex because the physical processes taking place in the various circuit elements are different.

For better insight into the power relations applicable to sinusoidal a.c. circuits, we will consider plots of the instantaneous power, voltage and current separately for a resistive, an inductive, and a capacitive element connected to a voltage source (Fig. 2.30).

If the voltage applied to a resistive element of resistance r is

$$v_r = V_{rm} \sin \omega t$$

the current will be

$$i_r = v_r/r = I_{rm} \sin \omega t$$

that is, it will be in phase with the applied voltage. At any instant, the power in the resistive element (the instantaneous power) will be

$$p_r = v_r i_r = V_{rm} I_{rm} \sin^2 \omega t = \frac{V_{rm} I_{rm}}{2} (1 - \cos 2\omega t)$$

The instantaneous values of current i_r, voltage v_r and power p_r for a resistive element are shown in Fig. 2.30 a. The instantaneous power in a resistive element is positive at any instant. In other words, a resistive element draws energy at any instant, and this energy is irreversibly converted to other forms.

The power in a resistive element averaged over a period, that is, *active power* is given by

$$P_r = \frac{1}{T} \int_0^T p_r \, dt = V_r I_r = r I_r^2 = I_r^2/g = g V_r^2 = V_r^2/r \qquad (2.50)$$

where $V_r = V_{rm}/\sqrt{2}$ is the rms value of voltage, and $I_r = I_{rm}/\sqrt{2}$ is the rms value of current.

In the case of an inductive element L (see Fig. 2.30 b), the applied voltage

$$v_L = L\,di_L/dt = I_{Lm} \sin(\omega t + \pi/2) = V_{Lm} \cos \omega t$$

leads the current $i_L = I_{Lm} \sin \omega t$ (for the current we have chosen a zero initial phase) by $\pi/2$. The instantaneous power in an inductive element is given by

$$p_L = v_L i_L = V_{Lm} I_{Lm} \sin \omega t \cos \omega t = \frac{V_{Lm} I_{Lm}}{2} \sin 2\omega t = V_L I_L \sin 2\omega t$$

The instantaneous values of current, voltage and power for an inductive element are plotted in Fig. 2.30 b. The instantaneous power

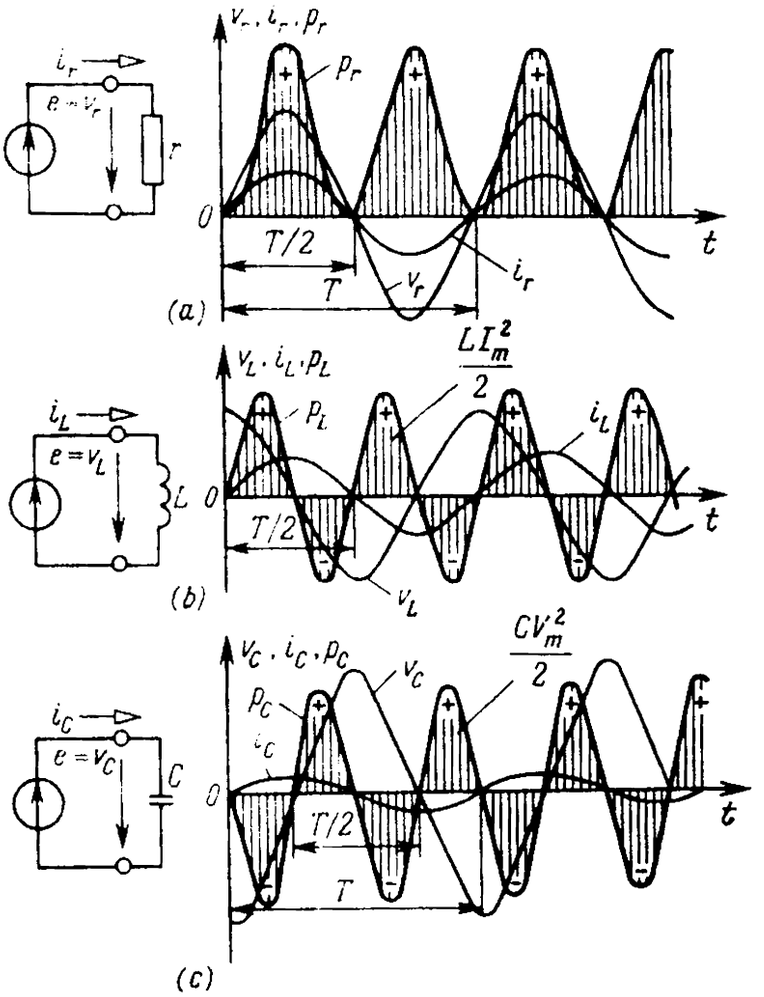


FIGURE 2.30

varies sinusoidally at twice the current frequency. The instantaneous power is positive when the current in the inductive element goes up in magnitude (irrespective of its direction); in the meantime energy is being stored in the magnetic field of the inductive element.

Let us find the energy entering an inductive element during a quarter of a period when the current and the instantaneous power are positive:

$$W_M = \int_0^{T/4} p_L \, dt = \int_0^{T/4} v_L i_L \, dt$$

On substituting the instantaneous voltage across the inductive element, $v_L = L \, di_L \, dt$, and on changing a variable, we get

$$W_M = \int_0^{I_{Lm}} L i_L \, di_L = L I_{Lm}^2/2 \qquad (2.51)$$

During the next quarter of a period, the instantaneous power p_L is negative, which means that, instead of drawing energy from a source, the inductive element supplies it to the source.

The power averaged over a period

$$P_L = \frac{1}{T} \int_0^T p_L \, dt$$

is zero in the case of an inductive element. A sinusoidal current does no work in an inductive element. Therefore, in contrast to a resistive element, the power relations for an inductive element are defined in terms of the *reactive inductive*, rather than active, *power* equal to the maximum instantaneous power

$$Q_L = V_L I_L = x_L I_L^2 = V_L^2/x_L = b_L V_L^2 = I^2/b_L \qquad (2.52)$$

Although active and reactive power has the same dimensions, reactive inductive power is measured in special units called volt-amperes reactive, or VAr for short.

In a capacitive element C (see Fig. 2.30 c), the applied voltage

$$v_C = \frac{1}{C} \int i_C \, dt = V_{Cm} \sin (\omega t - \pi/2) = -V_{Cm} \cos \omega t$$

lags behind the current

$$i_C = I_{Cm} \sin \omega t$$

by $\pi/2$. The instantaneous power in a capacitive element is

$$p_C = v_C i_C = -V_{Cm} I_{Cm} \sin \omega t \cos \omega t = -\frac{V_{Cm} I_{Cm}}{2} \sin 2\omega t$$

$$= -V_C I_C \sin 2\omega t$$

As is the case with an inductive element, the instantaneous power in a capacitive element varies sinusoidally at twice the current fre-

quency (see Fig. 2.30 c). In a capacitive element, however, the instantaneous power is positive during the interval when the applied voltage is increasing in magnitude. During these intervals, the capacitive element is being charged, and its electric field is storing energy. When the applied voltage is decreasing in magnitude, the instantaneous power is negative. Now the capacitive element is discharging, and the energy stored up in its electric field is returned to the source.

At the end of the first quarter of a period, the energy stored in the electric field

$$W_E = \int_0^{T/4} p_C \, dt = \int_0^{T/4} v_C i_C \, dt$$

is a maximum.

On substituting $i_C = C \, dv_C/dt$ and on changing a variable, we get

$$W_E = \int_{-V_{Cm}}^{0} C v_C \, dv_C = CV_{Cm}^2/2 \tag{2.53}$$

As in an inductive element, a sinusoidal current does no work in a capacitive element.

The power relations in a capacitive element are defined in terms of the *reactive capacitive power* equal to the maximum instantaneous power

$$Q_C = V_C I_C = x_C I_C^2 = I_C^2/b_C = V_C^2/x_C = b_C V_C^2 \tag{2.54}$$

If we connect an inductive element in series with a capacitive element, then at the instants when the energy stored in the magnetic field of the inductive element is increasing, the energy stored in the electric field of the capacitive element will be decreasing, and vice versa. It follows then that these elements can exchange energy not only with the associated sources, but with each other as well.

2.14 Active, Reactive and Apparent Power in a Passive One-Port

The diversity of the physical processes occurring in the elements of sinusoidal circuits complicates the formal analysis of the power relations involved.

To make the subsequent discussion more general, we will consider a voltage source connected to a passive one-port (Fig. 2.28). In the general case (see Sec. 2.12), a passive one-port may be represented by an equivalent circuit in the form of a series connection of two elements, namely an element of resistance r and an element of reactance x. The former is a resistive element of resistance r, and

the latter may be an inductive element of inductive reactance $x_L = \omega L = x$ if $x > 0$, or a capacitive element of capacitive reactance $x_C = 1/\omega C = |x|$, if $x < 0$. For both cases the applicable impedance and voltage triangles are shown in Figs. 2.27 and 2.29.

Assume that the voltage across and the current in the one-port are given by

$$v = V_m \sin(\omega t + \psi_v)$$

and

$$i = I_m \sin(\omega t + \psi_i)$$

and find the instantaneous power in the one-port, which is equal to that of the emf source:

$$p = vi = V_m I_m \sin(\omega t + \psi_v) \sin(\omega t + \psi_i)$$
$$= \frac{V_m I_m}{2} [\cos(\psi_v - \psi_i) + \cos(2\omega t + \psi_v + \psi_i)]$$

The instantaneous current, voltage and power for the inductive ($x > 0$, $\varphi = \psi_v - \psi_i > 0$) and the capacitive ($x < 0$, $\varphi = \psi_v - \psi_i < 0$) behaviour of the complex impedance of the one-port are shown in Fig. 2.31 a and b respectively.

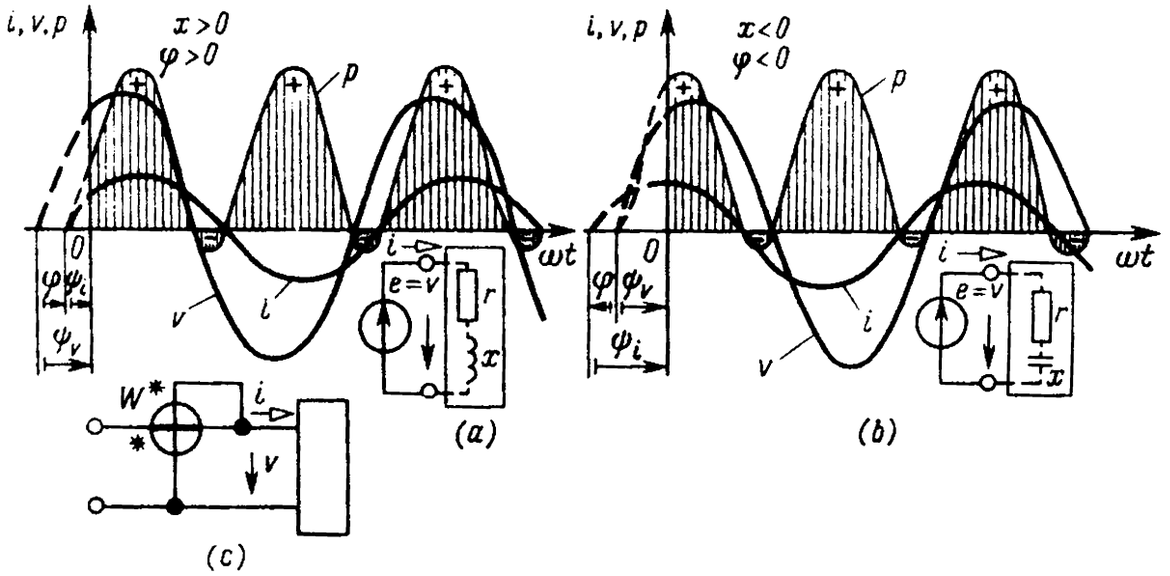


FIGURE 2.31

In each case, the energy flow is the sum of the energy flows already examined (see Sec. 2.13) for the respective ideal circuit elements. Some of the energy supplied by the source enters the one-port and is converted to other forms. The remainder of the energy is periodically exchanged between the source and the one-port.

As follows from Eq. (2.54), the average power in a passive one-port over a period (equal to the average power of the source) is given by

$$P = \frac{1}{T} \int_0^T p \, dt = VI \cos(\psi_v - \psi_i) = VI \cos \varphi \tag{2.55}$$

because the power averaged over a period of a cosine wave at twice the frequency is zero.

The phase difference between voltage and current, $\varphi = \psi_v - \psi_i$, depends on the parameters r and x of the equivalent-circuit elements (see Fig. 2.28).

Equation (2.55) defines the *active* (or *true*) *power in a one-port* and of the associated source. It is seen to be proportional to the rms values of voltage and current, and cos φ called the *power factor*. It is to be noted that the active (or true) power in a passive one-port is always positive and is independent of the sign of the phase difference φ (as will be recalled, $|\varphi| \leqslant \pi/2$). It determines the energy conditions in a passive one-port in general, that is, the average rate of the irreversible conversion of energy in all resistive elements of the passive one-port.

The active power in a passive one-port is measured with a wattmeter. A wattmeter has two measuring circuits one of which is connected in series with the one-port, so that the current in the circuit is equal to the current I in the one-port; the other circuit is connected in parallel with the one-port (across its terminals), so that the voltage across that circuit is equal to the voltage V across the one-port. For the phase difference φ between the voltage across and the current in the one-port to be properly accounted for, the measuring circuits of a wattmeter must be connected similarly with respect to the positive directions of the current and voltage. To avoid confusion, one of the leads of each measuring circuit is distinctly labelled (with an asterisk in the diagram of Fig. 2.31 c).

From the impedance triangle (see Fig. 2.27) and the voltage triangle (see Fig. 2.29) of a passive one-port it follows that the power factor is

$$\cos \varphi = r/z = V_{res}/V$$

On substituting for cos φ in (2.55), we obtain one more expression for the active (true) power in a passive one-port:

$$P = V_{res} I = rI^2 \tag{2.56}$$

The product of the rms values of the voltage between the source terminals, $V = E$, and the source current I, Eq. (2.55), defines the

apparent power of the source equal to the apparent power in the passive one-port:

$$S = VI = EI = zI^2 \tag{2.57}$$

Active (or true) power is measured in watts, and apparent power in *volt-amperes* (VA for short). In power engineering, it is usual to measure apparent power in kilovolt-amperes, kVA. Owing to this choice of units, apparent power can be readily distinguished from active (or true) power.

Apparent power defines the service capabilities of many pieces of electric equipment (generators, transformers, electric machines, and the like), and it is usually stated on their name-plates as one of the ratings: $S_{rtd} = V_{rtd}I_{rtd}$. As an example, for a generator the apparent power is equal to its maximum active (or true) power that could be developed at $\cos \varphi = 1$. For most loads, however, $\cos \varphi \neq 1$. Therefore, the capacity of a source is not fully utilized, even at the rated values of generator voltage and circuit current, because $P < S_{rtd}$.

For a proper analysis of power and energy relations in a circuit when the capacity of the source is not fully utilized, we introduce the concept of the *reactive power* of the source, equal to the reactive power of the associated passive one-port (see Fig. 2.28):

$$Q = EI \sin \varphi = VI \sin \varphi = VI \sin (\psi_v - \psi_i)$$

By introducing reactive power, we can faithfully represent all the physical processes occurring in all the inductive and capacitive elements of a passive one-port.

As follows from the impedance triangle for a passive one-port (see Fig. 2.27), $\sin \varphi = x/r$. On substituting for $\sin \varphi$ in the expression for reactive power, we obtain one more expression for the reactive power in a passive one-port:

$$Q = VI \sin \varphi = xI^2 \tag{2.58}$$

The reactive power in a passive one-port may be positive or negative, depending on the sign of the angle φ. When the input impedance of a one-port is inductive in its effect ($\varphi > 0$), the reactive power is positive; when it is capacitive ($\varphi < 0$), the reactive power is negative.

By comparing Eqs. (2.56) through (2.58), it is easy to see the following relation between the active, reactive and apparent power in a passive one-port:

$$S^2 = V^2I^2 = (VI \cos \varphi)^2 + (VI \sin \varphi)^2 = P^2 + Q^2 \tag{2.59}$$

Equation (2.59) can be conveniently interpreted geometrically on a complex plane. To this end, we multiply all sides of the impe-

dance triangle (see Fig. 2.27) of a passive one-port by I^2. The new triangle thus derived is called the *power triangle* (Fig. 2.32). From the similarity between the impedance and power triangles it follows that

$$Q/P = x/r = \tan \varphi$$

The sides of the power triangle are connected by a relation of the form

$$\underline{S} = P + jQ = VI \cos \varphi + jVI \sin \varphi = VI \exp (j\varphi) = \dot{V}\dot{I}^* \quad (2.60)$$

where $\underline{S}$ is the *complex power in a passive one-port*, and $\dot{I}^*$ is the com-

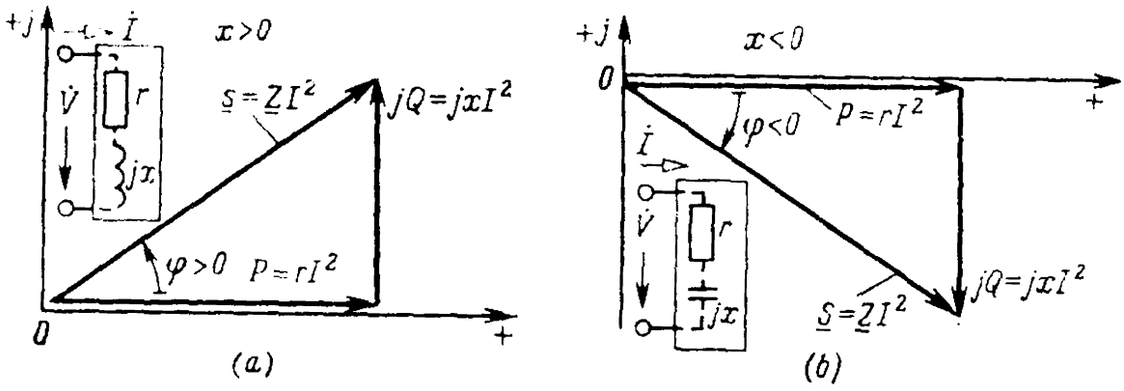


FIGURE 2.32

plex conjugate of the current $\dot{I}$. The magnitude of complex power is equal to the apparent power in a passive one-port:

$$|\underline{S}| = S = VI$$

2.15 Parallel Circuits

Figure 2.33 shows an electric circuit which is a parallel combination of a resistive element, an inductive element, and a capacitive element. Let us agree that we know in advance the conductance $g = 1/r$ of the resistive element, the complex susceptance $-jb_L = 1/j\omega L$ of the inductive element, and the complex susceptance $jb_C = j\omega C$ of the capacitive element. We also assume that the voltage across each of the elements is the same and equal to

$$\dot{V} = \dot{E} = V \angle \psi_0$$

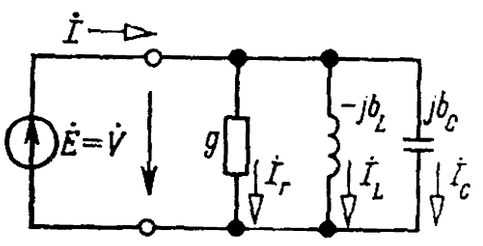


FIGURE 2.33

By Kirchhoff's current law, the complex value of the circuit current (equal to that of the emf source) is given by

$$\dot{I} = \dot{I}_r + \dot{I}_L + \dot{I}_C = \dot{V}(g - jb_L + jb_C) \quad (2.61)$$

where, in accord with Ohm's law, $\dot{I}_r = g\dot{V}$, $\dot{I}_L = -jb_L\dot{V}$, and $\dot{I}_C = jb_C V$ are the complex currents in the resistive, inductive and capacitive elements.

The sum of the complex admittances of all parallel branches in Eq. (2.61) is equal to the complex admittance of the circuit (in algebraic form);

$$\underline{Y} = 1/r - 1/j\omega L + j\omega C = g - jb_L + jb_C = g - j\,(b_L - b_C) \tag{2.62}$$

The reciprocal of complex admittance

$$1/Y = Z = z \exp(j\varphi)$$

is complex impedance. Therefore, complex admittance in exponential form is given by

$$\underline{Y} = 1/\underline{Z} = 1/z \exp(j\varphi) = y \exp(-j\varphi) \tag{2.63a}$$

and in trigonometric form by

$$\underline{Y} = y \cos\varphi - jy \sin\varphi \tag{2.63b}$$

where $y = |\underline{Y}| = \sqrt{g^2 + (b_L - b_C)^2}$ is the magnitude of the *complex admittance*, or simply *admittance*, of a circuit, and $\varphi = \arctan[(b_L - b_C)/g]$ is its argument.

In Fig. 2.34, the two components of complex admittance are shown as phasors on a complex plane for two cases: $b_L > b_C$ (Fig. 2.34 a)

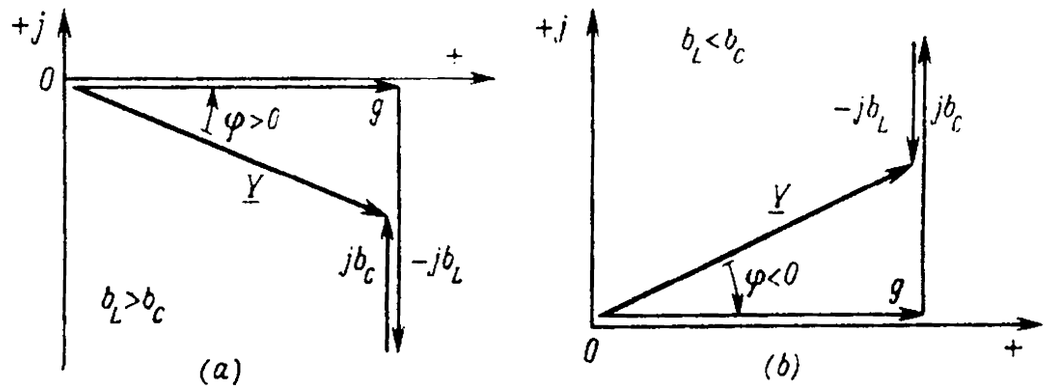


FIGURE 2.34

and $b_L < b_C$ (Fig. 2.34 b). In the former case the complex admittance of the circuit is inductive; in the latter it is capacitive.

On substituting the complex admittance in exponential form as defined by Eq. (2.63a) into Eq. (2.61), we obtain the complex current in the circuit as

$$\dot{I} = I \exp(j\psi_i) = \underline{Y}\dot{E} = \underline{Y}\dot{V} = yV \exp j\,(\psi_v - \varphi) \tag{2.64}$$

As follows from Eq. (2.64), the rms current in the common part of the circuit is

$$I = yV = \sqrt{g^2 + (b_L - b_C)^2}\,V$$

Figure 2.35 shows voltage and current phasor diagrams for the circuit in question in two cases: $b_L > b_C$ (Fig. 2.35 a), and $b_L < b_C$ (Fig. 2.35 b), given the same applied voltage $\dot{V} = V \angle \psi_v$.

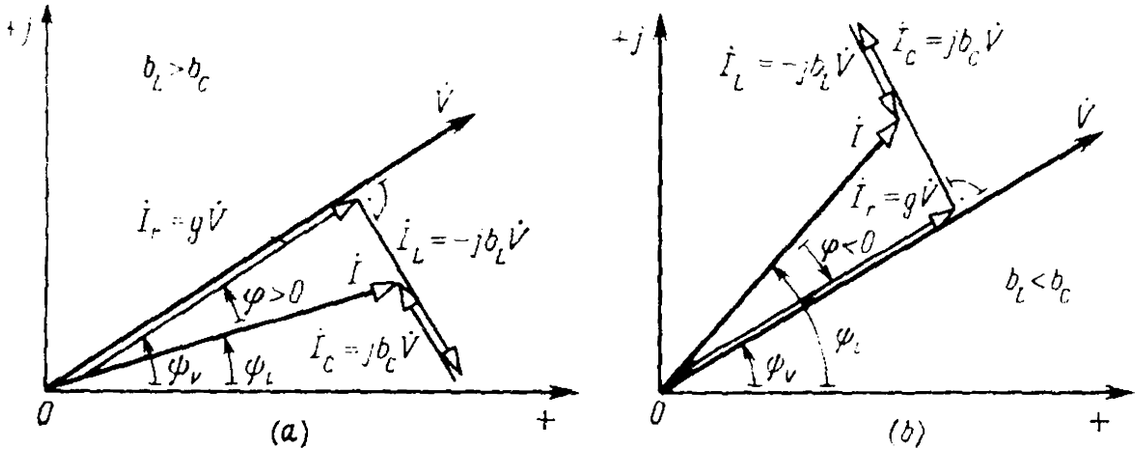


FIGURE 2.35

If the complex admittance of the circuit is inductive, the common current lags behind the voltage because $\varphi > 0$, that is, $\psi_v > \psi_i$. If the complex admittance is capacitive, the common current leads the voltage because $\varphi < 0$, that is, $\psi_v < \psi_i$. It is, to be noted that, as before, the positive values of the phase difference φ are counted counterclockwise from the complex current phasor $\dot{I}$.

The complex power in the circuit is given by

$$\underline{S} = \dot{V}\dot{I}^* = \dot{V}\,(\dot{I}_r^* + \dot{I}_L^* + \dot{I}_C^*) = gV^2 + jb_LV^2 - jb_CV^2$$
$$= \underline{Y}^*V^2 = P + j\,(Q_L - Q_C) = P + jQ$$

If a circuit contains several resistive, inductive and capacitive elements connected in parallel, its complex admittance is given by

$$\underline{Y} = y \exp(-j\varphi) = \Sigma G - j\Sigma b_L + j\Sigma b_C = \Sigma G - j\,(\Sigma b_L - \Sigma b_C)$$
$$= g - jb \qquad (2.65)$$

where $g = \Sigma G$ is the *conductance* of the circuit, and $b = \Sigma b_L - \Sigma b_C$ is its *susceptance*. As is seen, the two components of admittance have respectively the same names as their counterparts for a series circuit defined in Eq. (2.48).

The concepts of conductance and susceptance as defined above are equally applicable to any source-free two-terminal networks (one-ports).

Equation (2.65) corresponds to admittance triangles on a complex plane (Fig. 2.36). By reference to the admittance triangles and Eq. (2.65), we can readily deduce the trigonometric and exponential forms of the complex admittance of a circuit containing an arbitrary number of parallel branches with resistive, inductive and capacitive

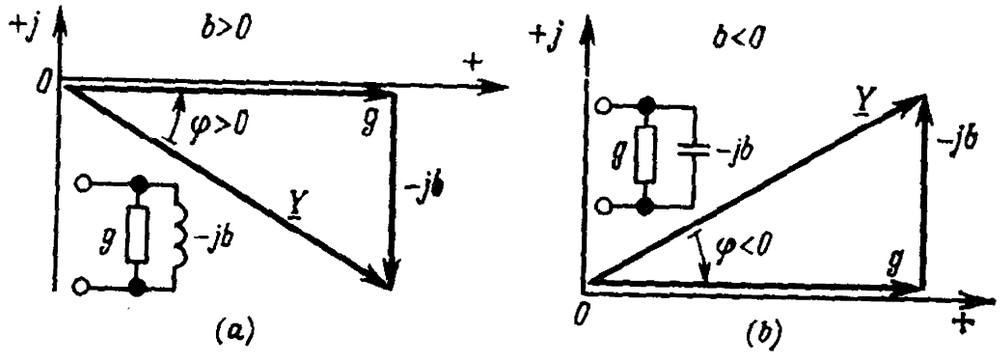


FIGURE 2.36

elements. The expressions thus derived are similar to those in (2.63), the admittance being defined as

$$y = \sqrt{g^2 + b^2} \tag{2.66a}$$

and its argument as

$$\varphi = \arctan (b/g) \tag{2.66b}$$

In the general case, the parallel branches of a circuit may contain not only resistive elements, or only inductive elements, or capacitive elements, but their parallel combinations as well. The complex admittance of a circuit with a parallel combination of n such branches is equal to the sum of the complex admittances of all the branches

$$\underline{Y} = \sum_{k=1}^{n} \underline{Y}_k = \sum_{k=1}^{n} (1/\underline{Z}_k) = 1/\underline{Z}_1 + 1/\underline{Z}_2 + \ldots + 1/\underline{Z}_k + \ldots + 1/\underline{Z}_n$$

where $\underline{Y}_k$ is the complex admittance and $\underline{Z}_k$ is the complex impedance of the kth branch.

2.16 The Conductance, Susceptance and Admittance of a Passive One-Port

In Sec. 2.12 a passive one-port was represented by its equivalent circuit consisting of a series combination of resistive and reactive elements (see Fig. 2.28). However, many problems will be simpler to solve if we replace a passive one-port with another equivalent circuit consisting of a parallel combination of elements with conductance and susceptance as shown in Fig. 2.37. The parameter of

such a passive one-port is its input complex admittance seen looking into terminals a and b

$$\underline{Y} = \dot{I}/\dot{V} = \dot{I}/\dot{E} = y \exp(-j\varphi) = y\cos\varphi - jy\sin\varphi = g - jb \tag{2.67}$$

where $\dot{V} = V\angle\psi_v = \dot{E}$ is the complex input voltage, $\dot{I} = I\angle\psi_i$ is the complex input current, and $-\varphi = \psi_i - \psi_v$ is the argument of the complex admittance. As follows from Eq. (2.67), any passive one-port may be replaced with an equivalent circuit consisting of a parallel combination of elements with conductance g and susceptance $-b$. The element with conductance is always a resistive element of conductance g, whereas the element with reactance may be an inductive element of inductive susceptance $b_L = 1/\omega L$ if $b > 0$, or a capacitive element of capacitive susceptance $b_C = \omega C = |b|$, if $b < 0$.

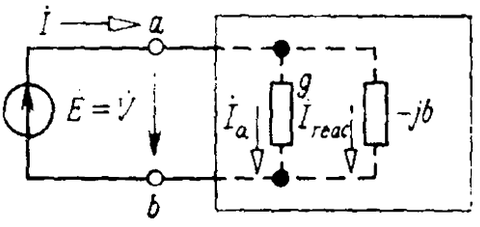


FIGURE 2.37

Depending on the sign of the susceptance b, the complex admittance of a passive one-port is inductive in its effect ($b > 0$, Fig. 2.36 a), or capacitive ($b < 0$, Fig. 2.36 b).

Now we multiply all sides of the admittance triangle in Fig. 2.36 by the complex voltage $\dot{V} = V\angle\psi_v$, and construct a current phasor diagram (Fig. 2.38) for the equivalent circuit of a passive one-port,

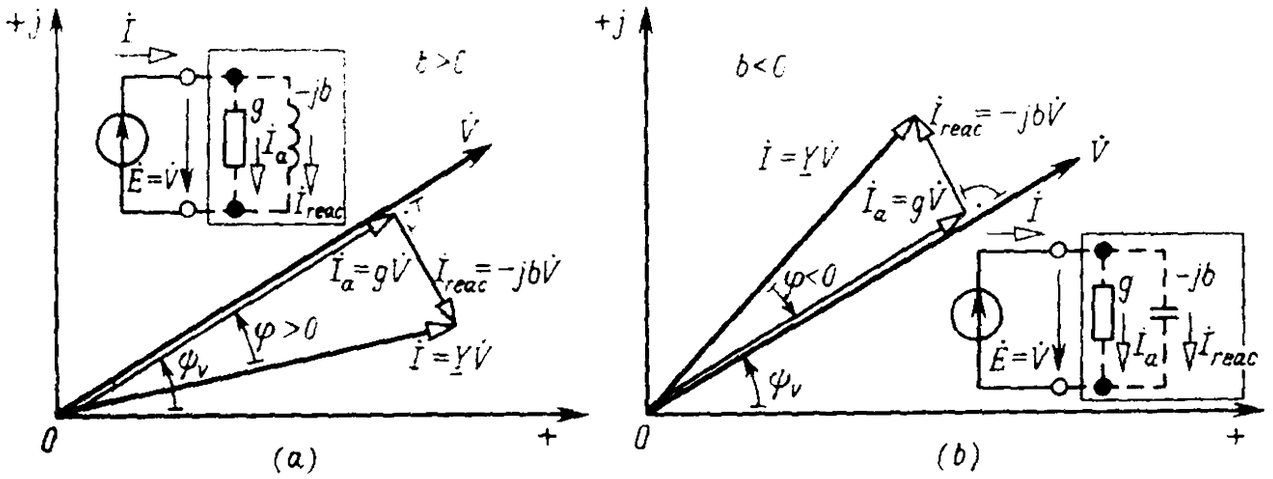


FIGURE 2.38

where $\dot{I}_a = g\dot{V}$ and $\dot{I}_{reac} = -jb\dot{V}$ are the *active and reactive components of the complex current* $\dot{I}$, respectively. The respective phasors $\dot{I}_a$, $\dot{I}_{reac}$ and $\dot{I}$ form a *current triangle* on a complex plane:

$$\dot{I} = \dot{I}_a + \dot{I}_{reac} \tag{2.68}$$

The magnitude of the active current phasor is $I_a = I \cos \varphi$; and the active current component is in phase with voltage. The magnitude of the reactive current component is $I_{reac} = I \mid \sin \varphi \mid$; the phasor $\dot{I}_{reac}$ makes an angle $\mid \pi/2 \mid$ with the voltage phasor $\dot{V}$. The *inductive reactive current* lags behind the voltage by an angle $\pi/2$ (see Fig. 2.38 a). The *capacitive reactive current* leads the voltage by the same angle $\pi/2$ (Fig. 2.38 b).

It follows from the current triangle that

$$I = \sqrt{I_a^2 + I_{reac}^2}$$

Noting Eqs. (2.67) and (2.68), we obtain several expressions for the complex power in a passive one-port (2.60)

$$\underline{S} = P + jQ = \dot{V}\dot{I}^* = \dot{V}\,(\dot{I}_a^* + \dot{I}_{reac}^*) = \underline{Y}^* V^2 = gV^2 + jbV^2 \tag{2.69}$$

2.17 Transformation of a Series into a Parallel Circuit

The analysis of electric circuits (such as an iron-cored inductor, see Chap. 7) can sometimes be markedly simplified by transforming a series connection into an equivalent parallel connection.

Suppose that we have a series combination of a resistive element of resistance r and a reactive element of impedance x (Fig. 2.39 a).

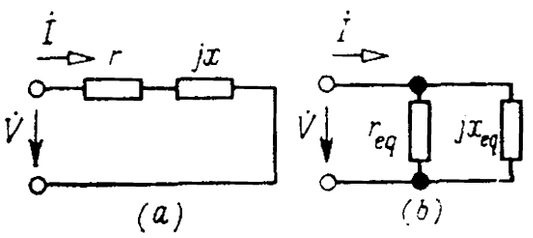


FIGURE 2.39

Its complex impedance and admittance are respectively given by

$$\underline{Z} = r + jx$$

and

$$\underline{Y} = 1/\underline{Z} = 1/(r + jx) = (r - jx)/(r^2 + x^2) = g - jb \tag{2.70}$$

The parallel connection shown in Fig. 2.39 b will be equivalent to the series connection in Fig. 2.39 a if the two have the same conductance or the same susceptance, that is

$$g_{eq} = 1/r_{eq} = g = r/(r^2 + x^2) \tag{2.71a}$$

$$-jb_{eq} = 1/jx_{eq} = -jb = -jx/(r^2 + x^2) \tag{2.71b}$$

From Eqs. (2.71) it follows that the resistances and reactances of the elements connected in parallel can be expressed in terms of the resistances and reactances of the elements connected in series in the following manner:

$$r_{eq} = (r^2 + x^2)/r \tag{2.72a}$$

$$x_{eq} = (r^2 + x^2)/x \tag{2.72b}$$

The condition for the inverse equivalent transformation can be derived by writing the impedances of the elements connected in series from Eqs. (2.72).

2.18 Series-Parallel Circuits

In some pieces of electric equipment, such as induction motors (see Chap. 14), the voltages and currents are distributed in a manner corresponding to a series-parallel connection of elements in a circuit.

The procedure leading to the overall impedance of a series-parallel connection of elements in a sinusoidal a.c. circuit is the same as is used for the same purpose in d.c. circuits (see Sec. 1.11). The first step is to find the equivalent impedance of the parallel branches.

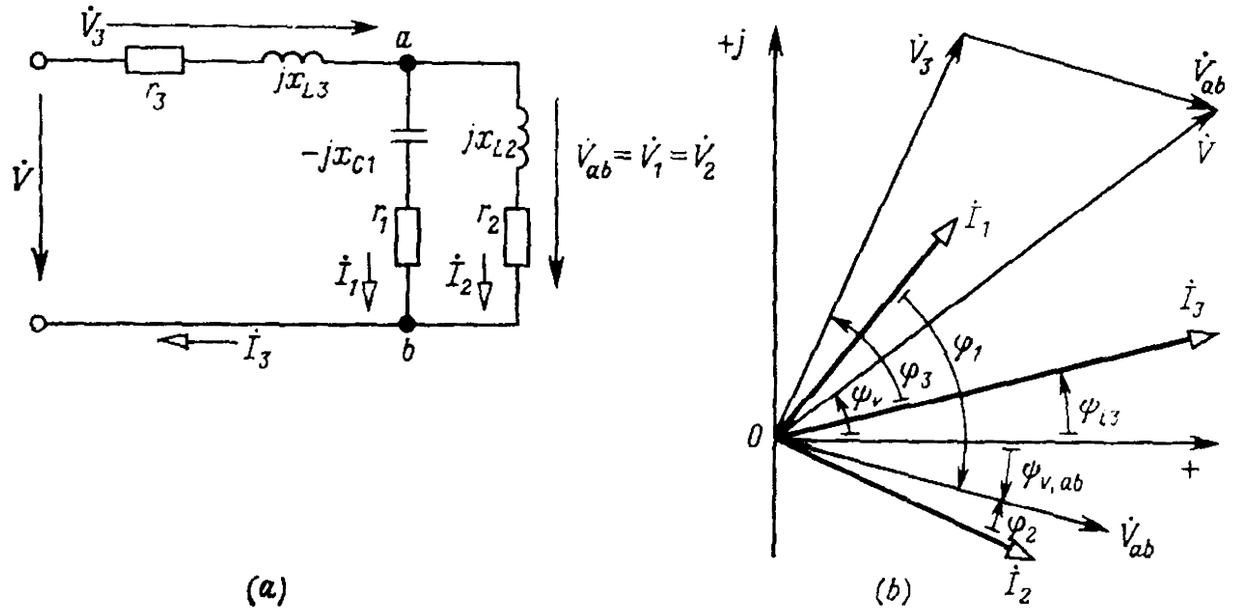


FIGURE 2.40

The second step is to replace the parallel branches with an element hdving an equivalent impedance. The third, and final, step is to eatermine the impedance of the series circuit thus derived.

Of course, for sinusoidal a.c. circuits all calculations should involve the complex impedances of the elements involved. The partial currents and voltages should likewise be found by the complex-value method.

As an example we will consider the passive one-port shown in Fig. 2.40 a. To begin with, we find the equivalent complex impedance of the two parallel branches connected between nodes *a* and *b*

$$\underline{Z}_{eq} = r_{eq} + jx_{eq} = z_{eq} \exp(j\varphi_{eq}) = \underline{Z}_1\underline{Z}_2/(\underline{Z}_1 + \underline{Z}_2)$$

where

$$\underline{Z}_1 = r_1 - jx_{C1} = z_1 \exp(j\varphi_1)$$

$$\underline{Z}_2 = r_2 + jx_{L2} = z_2 \exp(j\varphi_2)$$

are the complex impedances of the parallel branches.

The overall complex impedance of the passive one-port is

$$\underline{Z} = \underline{Z}_3 + \underline{Z}_{eq}$$

where

$$\underline{Z}_3 = r_3 + jx_{L3} = z_3 \exp(j\varphi_3)$$

The complex currents in and the complex voltages across the various parts of the one-port are given by

$$\dot{I}_3 = \dot{V}/\underline{Z} = \frac{\dot{V}}{r_3 + r_{eq} + j(x_{L3} + x_{eq})} = I_3 \exp(j\psi_{i3})$$

$$\dot{V}_3 = \underline{Z}_3 \dot{I}_3 = z_3 \exp(j\varphi_3)\, I_3 \exp(j\psi_{i3}) = V_3 \exp(j\psi_{v3})$$

$$\dot{V}_1 = \dot{V}_2 = \dot{V}_{ab} = \underline{Z}_{eq} \dot{I}_3 = z_{eq} \exp(j\varphi_{eq})\, I_3 \exp(j\psi_{i3}) = V_{ab} \exp(j\psi_{v,\,ab})$$

By applying Ohm's law, we find the complex current in each of the parallel branches

$$\dot{I}_1 = \dot{V}_{ab}/\underline{Z}_1 = \frac{V_{ab} \exp(j\psi_{v,\,ab})}{z_1 \exp(j\varphi_1)} = I_1 \exp(j\psi_{i1})$$

$$\dot{I}_2 = \dot{V}_{ab}/\underline{Z}_2 = \frac{V_{ab} \exp(j\psi_{v,\,ab})}{z_2 \exp(j\varphi_2)} = I_2 \exp(j\psi_{i2})$$

A current and voltage phasor diagram for the circuit in question is shown in Fig. 2.40 b.

The complex power in the circuit is equal to the sum of the complex powers in all the branches

$$\begin{aligned}\underline{S} &= \dot{V}\dot{I}_3^* = \underline{S}_1 + \underline{S}_2 + \underline{S}_3 = \dot{V}_1\dot{I}_1^* + \dot{V}_2\dot{I}_2^* + \dot{V}_3\dot{I}_3^* \\ &= P_1 + P_2 + P_3 + j(-Q_{C1} + Q_{L2} + Q_{L3}) \\ &= r_1 I_1^2 + r_2 I_2^2 + r_3 I_3^2 + j(-x_{C1} I_1^2 + x_{L2} I_2^2 + x_{L3} I_3^2)\end{aligned}$$

2.19 The Power Balance Around a Sinusoidal A.C. Circuit

In a sinusoidal a.c. circuit, the algebraic sum of the instantaneous powers of all the energy sources is equal to the algebraic sum of the instantaneous powers in all the loads at any instant. The same applies to the powers averaged over a period.

To begin with, we will consider energy sinks, or loads, for which the equivalent circuits contain only resistive, inductive and capacitive elements. As already noted, the power and energy relations in resistive, inductive and capacitive elements are different because the physical processes occurring in them are likewise different. In resistive elements, input energy is irreversibly converted to other forms. The mean rate of the irreversible energy conversion in a resistive element is defined as active (true or real) power P_r, Eq. (2.50). In inductive and capacitive elements, energy is periodically stored in the associated magnetic and electric fields and then returned to the external circuits. In these elements, no irreversible energy conversion takes place, so active (true or real) power P is zero. The power and energy relations for inductive and capacitive elements are defined in terms of inductive reactive power Q_L, Eq. (2.52), and capacitive reactive power Q_C, Eq. (2.53).

The power balance around a sinusoidal a.c. circuit containing an arbitrary number of energy (current and/or voltage) sources and of energy sinks or loads, such as resistive, inductive and capacitive elements, implies that, firstly, the algebraic sum of the active powers of all the energy sources is equal to the arithmetic sum of the powers in all the resistive elements

$$\Sigma V_s I_s \cos(\psi_v - \psi_i) = \Sigma r I_r^2 \tag{2.73}$$

or

$$\Sigma P_s = \Sigma P_r$$

Secondly, it implies that the algebraic sum of the reactive powers of all the energy sources is equal to the difference between the arithmetic sum of the reactive powers of all the inductive elements and the arithmetic sum of the reactive powers of all the capacitive elements

$$\Sigma V_s I_s \sin(\psi_v - \psi_i) = \Sigma x_L I_L^2 - \Sigma x_C I_C^2 \tag{2.74}$$

or

$$\Sigma Q_s = \Sigma Q_L - \Sigma Q_C$$

In the algebraic sum of active or reactive powers the term representing an emf source (see Fig. 2.41 a) takes a "+" sign if the positive direction of the current $\dot{I}$ is the same as the direction of action of the emf $\dot{E} = \dot{V}_{ab}$, that is, in the external circuit the current is flowing from terminal a which is at a higher potential than terminal b. (The positive direction of $\dot{V}_{ab} = \dot{\varphi}_a - \dot{\varphi}_b$ is from terminal a to terminal b.) Otherwise (see Fig. 2.41 b) the term associated with the emf source takes a "—" sign (now the sinusoidal emf generator is

operating as a motor). Similarly the term representing a current source $\dot{J} = \dot{I}$ (Fig. 2.41 c) takes a "+" sign; otherwise (Fig. 2.41 d) it takes a "—" sign.

The power balance around a sinusoidal a.c. circuit may be stated in complex form: The algebraic sum of the complex powers of all

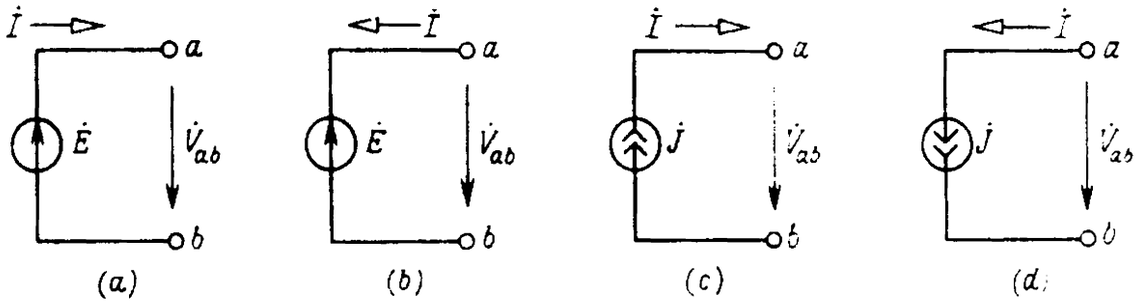


FIGURE 2.41

energy sources is equal to the algebraic sum of the complex powers in all the loads

$$\Sigma \underline{S}_s = \Sigma \dot{V}_s \dot{I}_s^* = \Sigma \underline{S}_{load} = \Sigma \dot{V}_{load} \dot{I}_{load}^* \tag{2.75}$$

The terms of the algebraic sum of the complex powers of energy sources take a "+" or a "—" sign by the same rule as applies to active and reactive powers. For loads (that is, resistive, inductive and capacitive elements) the positive directions of voltages and currents, as already noted, are chosen always the same. Thus in Eqs. (2.73) through (2.75) the terms on the right-hand sides always take a "+" sign.

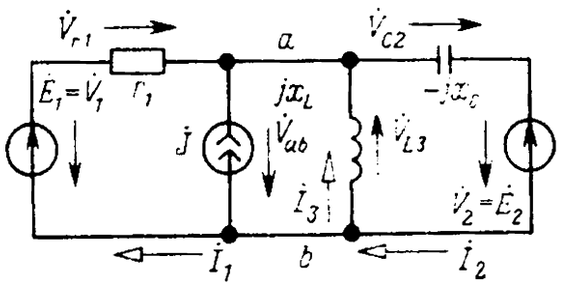


FIGURE 2.42

In the general case energy sinks (loads) may be circuit branches or one-ports rather than individual elements. As an example, we will write a power balance for the circuit of Fig. 2.42:

$$\Sigma \underline{S}_s = \dot{V}_1 \dot{I}_1^* - \dot{V}_2 \dot{I}_2^* + \dot{V}_{ab} \dot{J}^* = \dot{E}_1 \dot{I}_1^* - \dot{E}_2 \dot{I}_2^* - \dot{V}_{L3} \dot{J}^*$$
$$= \Sigma P_s + j \Sigma Q_s$$

$$\Sigma \underline{S}_{load} = \dot{V}_{r1} \dot{I}_1^* + \dot{V}_{C2} \dot{I}_2^* + \dot{V}_{L3} \dot{I}_3^* = r_1 I_1^2 - j x_C I_2^2 + j x_L I_3^2$$
$$= P_r + j (Q_L - Q_C)$$

$$\Sigma P_s = P_r$$

$$\Sigma Q_s = Q_L - Q_C$$

2.20 Power Factor Improvement

Many industrial loads (such as phase shifters, electric motors and the like) have strong magnetic fields around them. As a result, the current in such a load has a considerable reactive (inductive) component (see Fig. 2.38 a) which implies a large positive (lagging) phase difference φ between voltage and current. This impairs its power factor $\cos\varphi$ and, as a corollary, the power efficiency of the equipment or plant as a whole. A lagging power factor leads to an incomplete utilization of generators, power transmission lines and

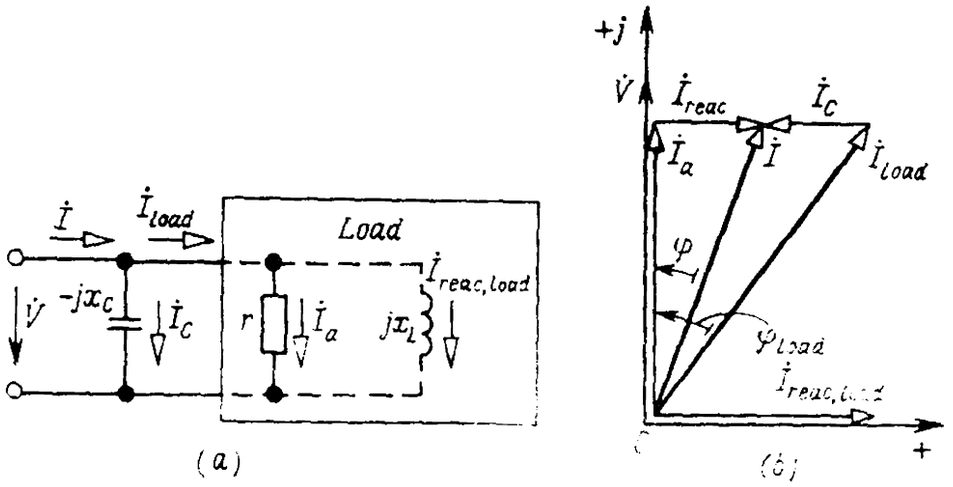


FIGURE 2.43

other facilities which are wastefully loaded by the reactive (inductive) current component. The reactive (inductive) current component is also responsible for increased power losses in all current-carrying parts (motor windings, transformers, generators, power-line conductors, and so on).

One way to improve the power factor and to relieve the plant of the reactive (inductive) current is to connect a load with a lagging power factor in parallel with a bank of capacitors. Then the reactive (capacitive) current of the capacitors will balance out the reactive (inductive) current in the load.

To illustrate this method of power factor improvement, we will represent a load with a lagging power factor as the equivalent circuit of a passive one-port (Fig. 2.43 a). The way in which the reactive (inductive) current in the load (the load current), $I_{\text{reac, load}}$, is balanced out by a bank of capacitors is demonstrated in the phasor diagram of Fig. 2.43 b. As is seen, connection of the capacitors improves the power factor: $\cos\varphi > \cos\varphi_{\text{load}}$. In most cases, an incomplete compensation of the phase difference will be sufficient. The small reactive (inductive) current, I_{reac}, at $\cos\varphi \geqslant 0.95$ does not produce any additional power loss, because $I = \sqrt{I_a^2 + I_{\text{reac}}^2} \approx \approx I_a$. Complete compensation of the inductive reactive current usu-

ally calls for the use of capacitors which is not always warranted economically.

It is usual to specify the value of the power factor that a plant should have after power-factor correction. If the load current I_{load} and the power factor cos φ of the load are known and the desired value of cos φ, that is the phase difference φ, is specified, the required capacitance can be found by reference to the current phasor diagram such as shown in Fig. 2.43 b, from which it follows that

$$I_a \tan \varphi_{load} - I_a \tan \varphi = I_C = \omega C V$$

Hence,

$$C = (P/\omega V^2)(\tan \varphi_{load} - \tan \varphi)$$

where $P = I_a V$ is the active power in the load.

The improvement of the power factor by use of capacitors is usually referred to as power factor correction in contrast to the natural power factor improvement achieved through the complete utilization of the power developed by motors and through the use of synchronous motors in which the reactive current component is very small.

2.21 Resonance in Sinusoidal A.C. Circuits

Let there be a capacitive element which is charged by a d.c. emf source to $v_C = E > 0$ (Fig. 2.44). This implies that the electric field of the capacitive element stores an energy $W_E = Cv_C^2/2$. If we disconnect the capacitive element from the source and connect it to an inductive element, a discharge current i will begin to flow around the circuit thus formed. The current will be building up gradually as its rise is opposed by the emf of self-induction, Eq. (2.2),

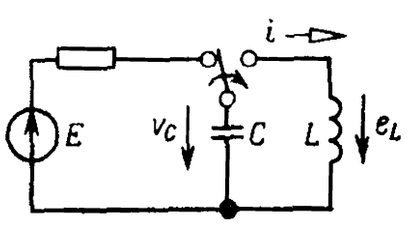


FIGURE 2.44

$$e_L = -L \, di/dt$$

As the current builds up, the magnetic field of the inductive element L stores an energy, Eq. (2.5), $W_M = Li^2/2$. The current around the circuit does not cease flowing even when the capacitive element has discharged completely ($v_C = 0$), and all of the energy stored in the electric field has been transformed into the energy stored by the magnetic field around the inductive element. The current flowing around the circuit in the same direction maintains the emf of self-induction until the energy in the magnetic field of the inductive element drops to zero, and this current charges the capacitive element in the reverse sense ($v_C < 0$). If the circuit is free from energy losses, the capacitive element will be re-charged to a voltage equal

to the original value. At the same time all of the energy stored in the magnetic field of the inductive element is transformed into the energy stored in the electric field of the capacitive element. Then the capacitive element will again begin discharging, but the current will be flowing in the reverse direction.

In an *ideal LC circuit*, that is, one free from losses, this charging and discharging of the capacitive element recurs periodically, so we observe undamped oscillations.

In a *real LC circuit* made up of an inductor and a capacitor, some energy is always lost, for example in the resistance presented by the connecting wires. This is why the respective equivalent *LC* circuit always includes a resistive element.

When an *LC* circuit is connected to an energy source (which may be a source of a sinusoidal emf or current), a resonance condition can arise in the circuit. There are two basic forms of resonance. One is series (or voltage) resonance, when the resonant circuit is connected in series with a source, and the other is parallel (or current) resonance, when the resonant circuit is connected in parallel with a source.

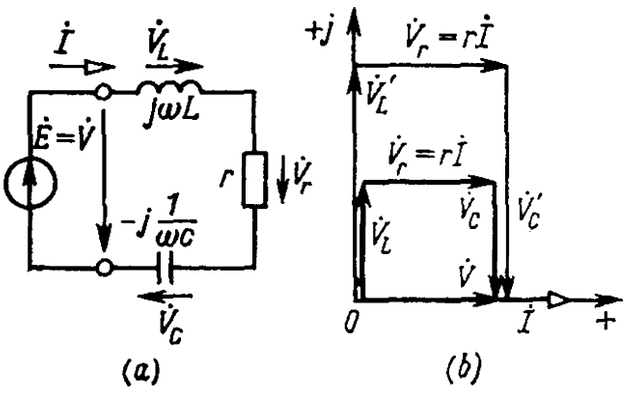


FIGURE 2.45

A. Series (voltage) resonance. This form of resonance can occur in a single-loop circuit containing an inductive element L, a capacitive element C, and a resistive element r, that is, in what is called a series resonant circuit (Fig. 2.45 a).

By Ohm's law, the complex current in the resonant circuit is

$$\dot{I} = I \exp(j\psi_i) = \dot{V}/\underline{Z} = V \exp(j\psi_v)/z \exp(j\varphi) \qquad (2.76a)$$

where

$$\underline{Z} = r + j\omega L - j1/\omega C$$

is the complex impedance of the resonant circuit,

$$z = \sqrt{r^2 + (\omega L - 1/\omega C)^2}$$

is the impedance of the resonant circuit,

$$\varphi = \psi_v - \psi_i = \arctan \frac{\omega L - 1/\omega C}{r}$$

is the phase difference between voltage and current, that is, the argument of the complex impedance, and

$$I = V/z = \frac{V}{\sqrt{r^2 + (\omega L - 1/\omega C)^2}} \tag{2.76b}$$

is the rms current.

If the angular frequency ω of the current in and of the voltage across the resonant circuit is $1/\sqrt{LC}$, the inductive reactance is equal to the capacitive reactance: $\omega L = 1/\omega C$. At the same time, the argument φ of the complex impedance is zero: $\psi_i = \psi_v$; the impedance of the resonant circuit is a minimum: $z = r$; and the rms current for a given voltage is a maximum: $I = V/r$.

The condition in a single-loop circuit containing an inductive element, a capacitive element and a resistive element, such that $\psi_i = \psi_v$, that is, when the current and the voltage are in phase is called *series* (or *voltage*) *resonance*.

At series resonance the voltages across the inductive and capacitive elements are the same in rms value and, as a consequence, in amplitude, but opposite in phase (in Fig. 2.45 b, the initial phase for the current has been chosen equal to zero). Therefore the source voltage V is equal to the voltage across the resistive element.

The angular frequency at which the above condition is observed is called the *resonant frequency* of the circuit:

$$\omega_{res} = 1/\sqrt{LC}$$

If the resistance r of the resistive element is small, the current in the circuit at resonance sharply increases as compared with its value at any other frequency. At the same time, and this is especially important, the voltages across the capacitive and inductive elements may exceed (many times) the supply voltage V. This overvoltage condition

$$V_C = V_L = (1/\omega_{res}C)\, I = V/\omega_{res}Cr > V$$

happens if

$$\omega_{res}L = 1/\omega_{res}C > r$$

On substituting ω_{res} in the above inequality, we obtain the overvoltage condition in the form

$$\sqrt{L/C} > r$$

The quantity $\rho = \sqrt{L/C} = \omega_{res}L = 1/\omega_{res}C$ has the dimensions of impedance and is called the *characteristic impedance* of a resonant circuit. The ratio of the characteristic impedance to the resistance

of the resistive element determines the resonant properties of the circuit and is called its *quality factor* or Q-factor

$$Q = \rho/r$$

The Q-factor is equal to the ratio of the reactive power (at resonance) of the inductive Q_L or the capacitive Q_C element to the active power of the resistive element. Thus,

$$Q = \omega_{res}L/r = \omega_{res}LI^2/rI^2 = Q_{L,res}/P_{res}$$

and

$$Q = 1/\omega_{res}C/r = (1/\omega_{res}C)\, I^2/rI^2 = Q_{C,res}/P_{res}$$

If, at resonance, we multiply the inductive and capacitive reactance by the same number n such that

$$x'_L = nx_{L,res} \quad \text{and} \quad x'_C = nx_{C,res}$$

the current in the circuit will remain unchanged, but the voltages across the inductive and capacitive elements will be increased n times (Fig. 2.45 b): $V'_L = nV_L$ and $V'_C = nV_C$. In consequence, given the same current, $I_{res} = V/r$, we can increase the voltages across the inductive and capacitive elements without bound.

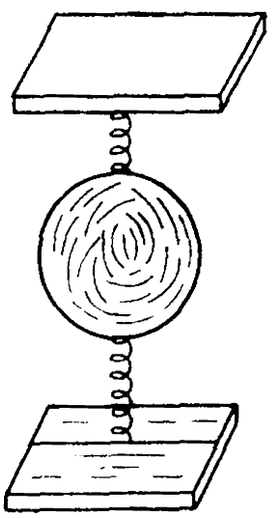


FIGURE 2.46

The physical cause of overvoltage lies in the periodic exchange of energy between the electric field of the capacitive element and the magnetic field of the inductive element. Formally, a similar interchange of energy is possible in any mechanical system possessing mass and elasticity. An example is a ball suspended on springs as shown in Fig. 2.46. In a mechanical vibrating system, energy is periodically converted from kinetic (the energy of a moving body) into potential (the energy of an elastic element) and back. If such a system is substantially free from friction, it will suffice to put in periodically a small quantity of energy from without in step with the oscillations in order to make up for the energy lost due to friction and to sustain the oscillations. In fact, the force applied from without may be a small fraction of the forces of inertia and elasticity acting within the system. In consequence, the energy applied from without to make up for losses may likewise be a small fraction of the energy of vibrations.

In an electric circuit at resonance, the small quantity of energy supplied from an external source and making up for the energy dissipated in the resistive element is sufficient for the undamped oscillations of the relatively large quantities of energy stored in the mag-

netic and electric fields of the circuit. It can be shown that at resonance the aggregate energy stored in the electric and magnetic fields remains constant at any instant

$$W_E + W_M = Cv_C^2/2 + Li^2/2 = \text{const} \tag{2.77}$$

The phase difference between the voltage across the capacitive element of a circuit and the current in the circuit is a quarter of a period. Therefore, if

$$v_C = V_{Cm} \sin \omega t$$

then

$$i = I_m \cos \omega t$$

On substituting the above expressions in Eq. (2.77), we get

$$W_E + W_M = (CV_{Cm}^2/2) \sin^2 \omega t + (LI_m^2/2) \cos^2 \omega t$$

The amplitude of the current in the circuit is

$$I_m = \omega C V_{Cm}$$

and since at resonance $\omega = \omega_{res} = 1/\sqrt{LC}$, then

$$LI_m^2 = CV_{Cm}^2$$

Hence,

$$W_E + W_M = (CV_{Cm}^2/2)\ (\sin^2 \omega_{res} t + \cos^2 \omega_{res} t)$$
$$= CV_{Cm}^2/2 = LI_m^2/2 = \text{const}$$

In telecommunications, automatic control and similar applications it is important to know the frequency dependence of the currents and voltages in circuits where a condition of resonance is likely to occur. Graphically this dependence is depicted by *resonance curves.*

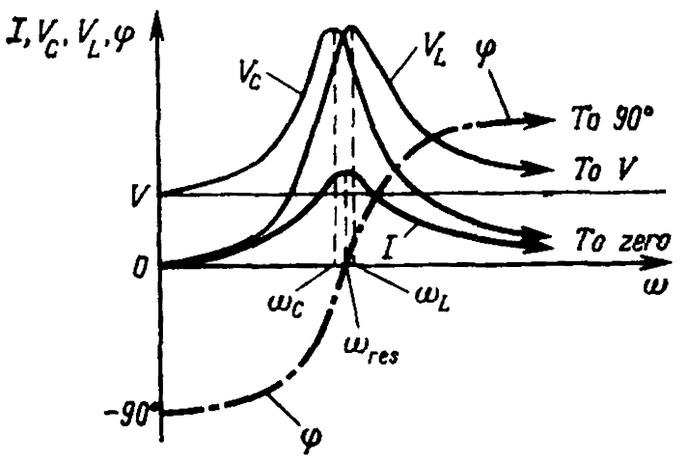


FIGURE 2.47

As is seen from Eq. (2.76b), the current in a resonant circuit depends on angular frequency, $I(\omega)$. This is because the impedance z of the resonant circuit is a function of frequency, being a maximum at resonance, that is, at $\omega = \omega_{res}$, when $\omega_{res}L = 1/\omega_{res}C$ (Fig. 2.47).

In an ideal series resonant circuit the impedance at resonance is zero, $r = 0$, and the source sees a short circuit.

The voltages across the inductive and capacitive elements of a resonant circuit reach their maximum values at frequencies other than resonant. Notably, the voltage across the capacitive element is

$$V_C = (1/\omega C)\, I = (1/\omega C)\, (V/z) = \frac{V}{\omega C \sqrt{r^2 + (\omega L - 1/\omega C)^2}}$$

$$= \frac{V}{\sqrt{\omega^2 r^2 C^2 + (\omega^2 LC - 1)^2}}$$

The maximum value of V_C (ω) is observed at an angular frequency ω_C such that the radicand in the last expression is a minimum. Therefore, in order to determine ω_C, we should equate to zero the first derivative of the radicand with respect to ω:

$$2\omega r^2 C^2 + 4\omega^3 L^2 C^2 - 4\omega LC = 0$$

Hence

$$\omega_C = \sqrt{1/LC - r^2/2L^2} = \omega_{res} \sqrt{1 - 1/2Q^2} < \omega_{res}$$

That is, the voltage across the capacitive element is a maximum at ω_C which is lower than the resonance frequency, ω_{res} (see Fig. 2.47).

Similarly we can find that the voltage across the inductive element of a resonant circuit, V_L (ω) = ωLI is a maximum at an angular frequency given by

$$\omega_L = \omega_{res} \sqrt{1/(1 - r^2 C/2L)}$$

$$= \omega_{res} \sqrt{1/(1 - 1/2Q^2)} > \omega_{res}$$

As the Q-factor of a resonant circuit is improved, the difference between ω_C or ω_L and the resonance frequency decreases, and the three resonance curves I (ω), V_C (ω) and V_L (ω) become progressively sharper.

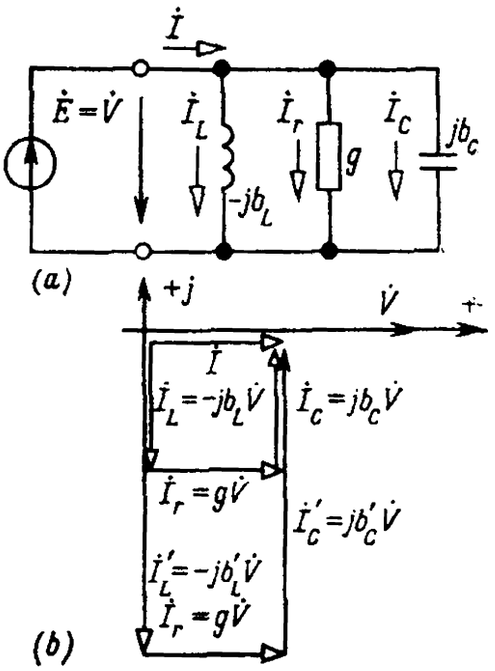


FIGURE 2.48

In most electric installations, series resonance is undesirable as it is fraught with overvoltages when the voltage is many times the operating voltage of the installation. In radio, telephony and automatic control systems, series resonance is utilized in order to tune a circuit to the desired frequency.

B. Parallel (current) resonance. This form of resonance may occur in a parallel resonant circuit, that is, one containing an inductive element, a capacitive element, and a resistive element connected in parallel (Fig. 2.48 a).

Given the supply voltage $\dot{V} = \dot{V} \angle \psi_v$, the common current is

$$\dot{I} = \underline{Y}\dot{V} = y \exp(-j\varphi)\, V \exp(j\psi_v)$$

where

$$\underline{Y} = g - jb = g - j(b_L - b_C)$$

is the complex admittance of a parallel resonant circuit;

$$y = \sqrt{g^2 + (b_L - b_C)^2}$$

is the admittance of the resonant circuit;

$$\varphi = \psi_v - \psi_i = \arctan \frac{1/\omega L - \omega C}{g}$$

is the phase difference between the voltage and the common current;

$$I = yV = \sqrt{g^2 + (b_L - b_C)^2}\, V$$

is the rms current in the resonant circuit; and $-\varphi$ is the argument of the complex admittance.

At $\omega_{res} = 1/\sqrt{LC}$, the inductive susceptance $b_L = 1/\omega L$ and the capacitive susceptance $b_C = \omega C$ of the parallel branches are the same, the argument of the complex admittance $-\varphi$ is zero (that is, $\psi_i = \psi_v$), the admittance of the resonant circuit is a minimum ($y = g$), and the common current is a minimum, too:

$$I_{res} = gV$$

The condition in a parallel resonant circuit in which the phase difference between the voltage and the common current is zero is called *parallel* (or *current*) *resonance.*

At resonance, the rms currents in the inductive and capacitive elements are the same:

$$I_L = (1/\omega_{res} L)\, V = I_C = \omega_{res} CV$$

and the phase difference between the currents is π, because the current in the inductive element lags behind the voltage across it by $\pi/2$, and the current in the capacitive element leads the voltage across it by the same angle $\pi/2$ (Fig. 2.48 b).

If, at current resonance, we multiply the inductive and capacitive susceptances by the same number n so that instead of $b_L = 1/\omega_{res} L$ and $b_C = \omega_{res} C$ we have $b'_L = nb_L$ and $b'_C = nb_C$, the currents in the respective elements, I_L and I_C, will be multiplied by the same factor n, but the common current will remain unchanged, $I_{res} = gV$. Thus we can, at least in principle, increase the currents in the inductive and capacitive elements without bound while the source current remains the same.

Resonance curves for a parallel resonant circuit are shown in Fig. 2.49. As is seen, the current in the capacitive element, $I_C = \omega CV$, is directly proportional to the angular frequency; in the inductive element the current $I_L = V/\omega L$ is inversely proportional to the angular frequency; and in the resistive element the current $I_r = V/r$ is independent of the angular frequency. The intersection of the $I_C(\omega)$ and $I_L(\omega)$ curves defines the condition of parallel (or current) resonance when $I = I_r$.

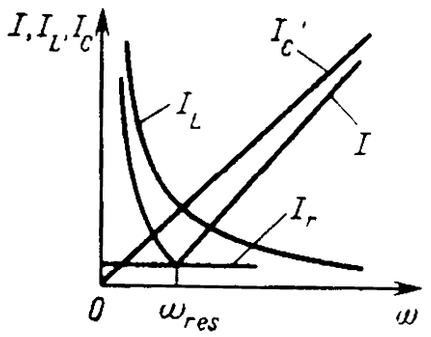


FIGURE 2.49

If the conductance g of the resistive element is zero, the admittance y of the circuit at resonance is also equal to zero, and the common current in an ideal parallel resonant circuit (the source current) vanishes, which is equivalent to an open-circuit condition.

In an equivalent circuit, to allow for energy losses in, say, the wires of a physical circuit, we may show the inductive element L connected in series with a resistive element r_L and the capacitive element C connected in series with a resistive element r_C. The condition of parallel (current) resonance in such a circuit is the equivalence of the inductive and capacitive susceptances of the respective branches: $b_L = b_C$, or, by Eq. (2.71b),

$$\frac{\omega L}{r_L^2 - (\omega L)^2} = \frac{1/\omega C}{r_C^2 + (1/\omega C)^2}$$

In this case, too, the common current at resonance is in phase with the voltage across the circuit.

It is to be noted that, in contrast to series resonance, parallel resonance does not present any hazard in electric installations. In this case, heavy currents can arise in the branches only when the branches contain large reactances, that is, when their capacitors have a high capacitance or their inductors have a low inductance. This is not surprising because the currents in the inductive and capacitive branches are independent of each other, and their values are determined (by Ohm's law) by the applied voltage.

2.22 Circuits with Inductively Coupled Elements

In Sec. 2.3 we have defined the magnetic flux Φ, the flux linkage Ψ, and the inductance L of an inductor (see Fig. 2.1).

If we have two inductors with turns w_1 and w_2 and with currents i_1 and i_2 (Fig. 2.50 a), some of the magnetic lines of force of each coil will link with the turns of the other. Therefore in addition to the self-

flux linkages Ψ_{11} and Ψ_{22} it is important to consider the additional flux linkages of the first coil with the turns of the second,

$$\Psi_{12} = \sum_{k=1}^{w_1} \Phi_{k12}$$

and of the second coil with the turns of the first,

$$\Psi_{21} = \sum_{k=1}^{w_2} \Phi_{k21}$$

where Φ_{k12} is the flux threading the kth turn of the first coil due to the current in the second coil, and Φ_{k21} is the flux threading the kth turn of the second coil due to the current in the first coil.

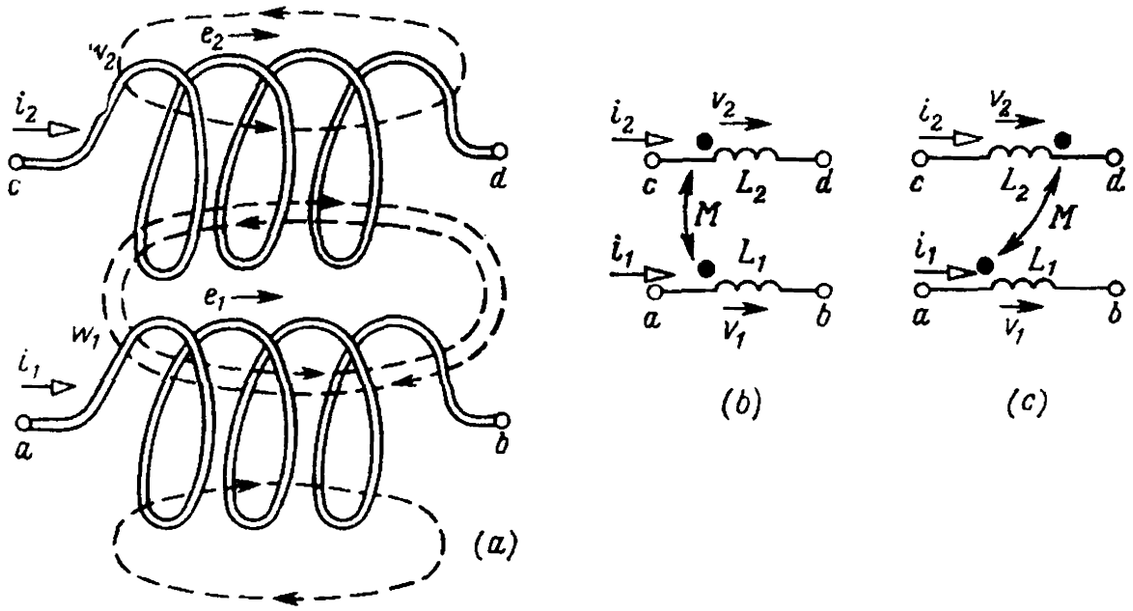


FIGURE 2.50

The ratio of the additional flux linkage Ψ_{12} of the first coil to the current i_2 in the second coil is called the *mutual inductance* of the two coils:

$$M_{12} = \Psi_{12}/i_2$$

Similarly, we can define the mutual inductance of the second and first coils

$$M_{21} = \Psi_{21}/i_1$$

Experience shows that $M_{21} = M_{12} = M$. A rigorous proof of the statement can be given by recourse to the theory of the electromagnetic field.

In linear electric circuits the mutual inductance is independent of the directions and values of the currents and is solely determined by the design and relative position of the coupled coils.

The total flux linkage Ψ of each of the two inductively coupled coils has two components which can be added together or subtracted

from each other, depending on the direction of the currents in the coils and their relative position. When the two components are added together, the coils are said to be connected *aiding*. When the two components are subtracted from each other, the two coils are said to be connected *in opposition*. In circuit diagrams the character of inductive coupling is identified by conventional signs. In equivalent circuits the "starts" of the coils are labelled with dots (see Fig. 2.50 b and c). If the currents are flowing in the same direction relative to the like terminals (see Fig. 2.50 b), the coils are connected aiding. In the circumstances, the self- and mutual-flux linkages in each coil are added together, so the total flux linkage of the first coil is

$$\Psi_1 = \Psi_{11} + \Psi_{12}$$

and the total flux linkage of the second coil is

$$\Psi_2 = \Psi_{22} + \Psi_{21}$$

If the currents are flowing in different directions relative to the like terminals (see Fig. 2.50 c), the coils are connected in opposition, and

$$\Psi_1 = \Psi_{11} - \Psi_{12}, \quad \Psi_2 = \Psi_{22} - \Psi_{21}$$

In sinusoidally excited circuits, the direction of a current refers to the assumed positive direction.

By the law of electromagnetic induction, Eq. (2.2), an emf is induced in each coil. The emf induced in the first coil is

$$e_1 = -\mathrm{d}\Psi_1/\mathrm{d}t = -\frac{\mathrm{d}(\Psi_{11} \pm \Psi_{12})}{\mathrm{d}t} = e_{1L} \pm e_{1M} \tag{2.78a}$$

and that in the second

$$e_2 = -\mathrm{d}\Psi_2/\mathrm{d}t = -\frac{\mathrm{d}(\Psi_{22} \pm \Psi_{21})}{\mathrm{d}t} = e_{2L} \pm e_{2M} \tag{2.78b}$$

where

$$e_{1L} = -\mathrm{d}\Psi_{11}/\mathrm{d}t = -L_1 \mathrm{d}i_1/\mathrm{d}t$$

$$e_{2L} = -\mathrm{d}\Psi_{22}/\mathrm{d}t = -L_2 \mathrm{d}i_2/\mathrm{d}t$$

are the emfs of self-induction in the first and second coil, respectively;

$$e_{1M} = -\mathrm{d}\Psi_{12}/\mathrm{d}t = -M\,\mathrm{d}i_2/\mathrm{d}t$$

$$e_{2M} = -\mathrm{d}\Psi_{21}/\mathrm{d}t = -M\,\mathrm{d}i_1/\mathrm{d}t$$

are the *emfs of mutual induction* in the first and second coils, respectively.

It has been noted in Sec. 2.3 that the voltage across an inductor is

$$v_L = v_{ab} = -e_L.$$

Similarly for inductively coupled coils,

$$v_1 = v_{ab} = -e_1 = -e_{1L} \mp e_{1M} = L_1 di_1/dt \pm M\, di_2/dt$$
$$= v_{1L} \pm v_{1M}$$
$$v_2 = v_{cd} = -e_2 = -e_{2L} \mp e_{2M} = L_2 di_2/dt \pm M di_1/dt \quad (2.79a)$$
$$= v_{2L} \pm v_{2M} \quad (2.79b)$$

When inductors are series-connected, the leads meeting at the common junction may have the same or the opposite polarities. In the former case the inductors are said to be connected aiding and in the latter, in opposition.

If during a time interval t_1 the currents in two inductively coupled coils vary from zero to i_1 and i_2 respectively, their common magnetic field stores an amount of energy given by

$$W_M = \int_0^{t_1} v_1 i_1\, dt + \int_0^{t_1} v_2 i_2\, dt = L_1 \int_0^{i_1} i_1\, di_1 + L_2 \int_0^{i_2} i_2\, di_2$$
$$\pm M \int_0^{i_2} i_1\, di_2 \pm M \int_0^{i_1} i_2\, di_1 = L_1 i_1^2/2 + L_2 i_2^2/2 \pm M i_1 i_2 \quad (2.80)$$

where we have used integration by parts:

$$\int_0^{i_2} i_1\, di_2 = i_1 i_2 - \int_0^{i_1} i_2\, di_1$$

Thus, as compared with the energy stored in the magnetic field due to two coils not coupled inductively, the energy in the common magnetic field due to two inductively coupled coils is increased or decreased by

$$W_{M,\ \text{mutual}} = M i_1 i_2$$

Sinusoidally excited circuits containing inductively coupled coils may be analysed by the complex-number method in about the same manner as circuits which are not coupled inductively. By analogy with Ohm's law in complex form for an inductive element, Eq. (2.32), we may write Eqs. (2.79) in complex form as

$$\dot{V}_1 = \dot{V}_{1L} \pm \dot{V}_{1M} = j\omega L_1 \dot{I}_1 \pm j\omega M \dot{I}_2 = j x_{1L} \dot{I}_1 \pm j x_M \dot{I}_2 \quad (2.81a)$$

$$\dot{V}_2 = \dot{V}_{2L} \pm \dot{V}_{2M} = j\omega L_2 \dot{I}_2 \pm j\omega M \dot{I}_1 = j x_{2L} \dot{I}_2 \pm j x_M \dot{I}_1 \quad (2.81b)$$

where $x_M = \omega M$ is the *mutual-induction impedance*, and $\dot{I}_1$ and $\dot{I}_2$ are the respective complex currents.

The respective complex emfs of self and mutual induction are given by

$$\left.\begin{aligned}\dot{E}_{1L} &= -\dot{V}_{1L} = -j\omega L_1\dot{I}_1 = -jx_{1L}\dot{I}_1\\ \dot{E}_{1M} &= -\dot{V}_{1M} = -j\omega M\dot{I}_2 = -jx_M\dot{I}_2\end{aligned}\right\} \tag{2.82a}$$

$$\left.\begin{aligned}\dot{E}_{2L} &= -\dot{V}_{2L} = -j\omega L_2\dot{I}_2 = -jx_{2L}\dot{I}_2\\ \dot{E}_{2M} &= -\dot{V}_{2M} = -j\omega M\dot{I}_1 = -jx_M\dot{I}_1\end{aligned}\right\} \tag{2.82b}$$

The complex power in each of the two inductively coupled coils is given by

$$\underline{S}_1 = \dot{V}_1\dot{I}_1^* = \dot{V}_{1L}\dot{I}_1^* \pm \dot{V}_{1M}\dot{I}_1^* = jx_{1L}I_1^2 \pm jx_M\dot{I}_1^*\dot{I}_2 = jQ_{1L} \pm \underline{S}_{12} \tag{2.83a}$$

$$\underline{S}_2 = \dot{V}_2\dot{I}_2^* = \dot{V}_{2L}\dot{I}_2^* \pm \dot{V}_{2M}\dot{I}_2^* = jx_{2L}I_2^2 \pm jx_M\dot{I}_1\dot{I}_2^* = jQ_{2L} \pm \underline{S}_{21} \tag{2.83b}$$

The terms

$$\underline{S}_{12} = jx_M\dot{I}_1^*\dot{I}_2 = jx_MI_1I_2\cos(\psi_{i2} - \psi_{i1}) - x_MI_1I_2\sin(\psi_{i2} - \psi_{i1}) = jQ_{12} - P_{12} \tag{2.84a}$$

and

$$\underline{S}_{21} = jx_M\dot{I}_1\dot{I}_2^* = jx_MI_1I_2\cos(\psi_{i1} - \psi_{i2}) - x_MI_1I_2\sin(\psi_{i1} - \psi_{i2}) = jQ_{21} - P_{21} \tag{2.84b}$$

where $Q_{12} = Q_{21}$ and $P_{12} = -P_{21}$, are, respectively, the reactive and active powers transferred from the second coil to the first and from the first coil to the second.

In the general case of an electric circuit containing n inductively coupled coils, the voltage across each kth coil is

$$\dot{V}_k = jx_{kL}\dot{I}_k \pm \sum_{p=1}^{n} jx_{Mkp}\dot{I}_p \tag{2.85}$$

where $p \neq k$.

2.23 The Potential Diagram of an Electric Circuit

The condition existing in a complex sinusoidally excited circuit can conveniently be depicted by a *potential diagram* which shows the distribution of complex potentials at various points in the circuit on a complex plane.

We will construct a potential diagram for the circuit shown in Fig. 2.51 a in which the complex branch currents $\dot{I}_1$, $\dot{I}_2$, and $\dot{I}_3$ have been found by the complex-number method. These currents along with the emfs $\dot{E}_1$ and $\dot{E}_2$ are shown in the phasor diagram of Fig. 2.51 b.

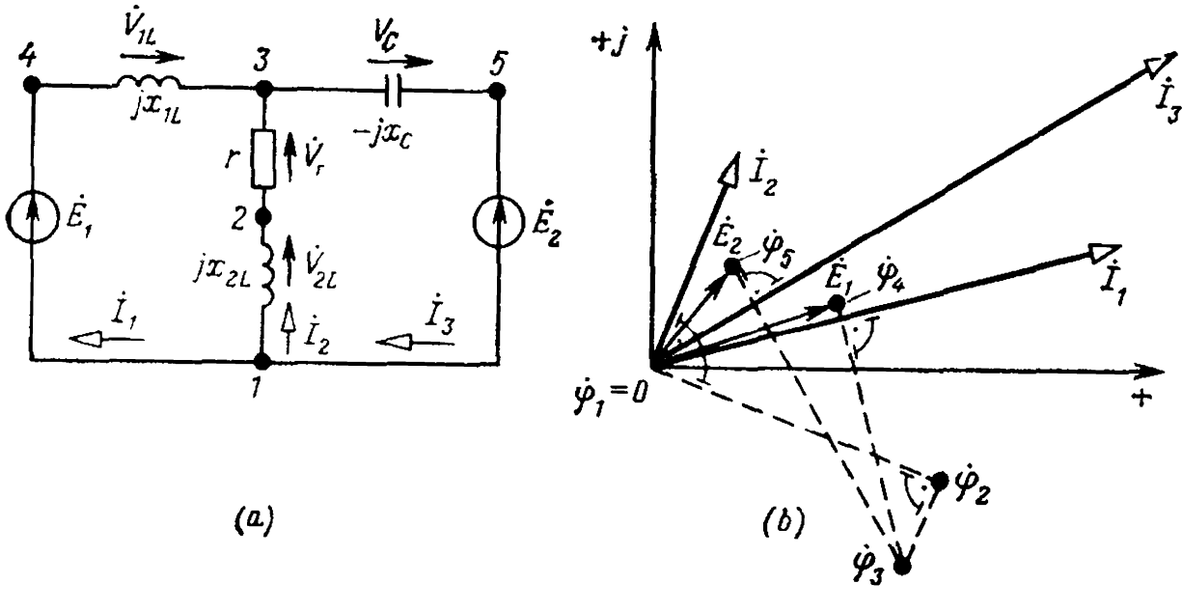


FIGURE 2.51

In constructing a potential diagram for a sinusoidal circuit, the first step, as in the case of a d.c. circuit (Fig. 1.10, point *e*), is to select the reference or datum potential point. Let it be point 1 whose potential $\dot{\varphi}_1$ is then assumed to be zero. Now we can determine the complex potentials at the other points. For the positive directions of currents indicated in Fig. 2.51 a and the specified directions of emfs,

$$\dot{\varphi}_2 = \dot{\varphi}_1 - \dot{V}_{2L} = \dot{\varphi}_1 - jx_{2L}\dot{I}_2$$

$$\dot{\varphi}_3 = \dot{\varphi}_2 - \dot{V}_r = \dot{\varphi}_2 - r\dot{I}_2$$

$$\dot{\varphi}_4 = \dot{\varphi}_3 + \dot{V}_{1L} = \dot{\varphi}_3 + jx_{1L}\dot{I}_1 = \dot{\varphi}_1 + \dot{E}_1$$

$$\dot{\varphi}_5 = \dot{\varphi}_3 - \dot{V}_C = \dot{\varphi}_3 - (-jx_C\dot{I}_3) = \dot{\varphi}_1 + \dot{E}_2$$

With the reference (datum) potential placed at the origin ($\dot{\varphi}_1 = 0$), the above relations locate the other potentials in a complex plane. For example, if we wish to locate the potential $\dot{\varphi}_2$, we turn the complex current phasor $\dot{I}_2$ through $\pi/2$ clockwise (which corresponds, in accord with Eq. (2.26), to multiplying it by $-j$), and multiply it by x_{2L}. Next we place the start of the phasor thus obtained at the origin (potential $\dot{\varphi}_1$), and note the position of its tip — it will locate the potential $\dot{\varphi}_2$ on the complex plane. Any other potential can be located on the complex plane in a similar way.

In constructing a potential diagram for a sinusoidal circuit containing inductively coupled coils, it is important to note that the voltage across an inductive element is in the general case given by Eq. (2.81).

Using a potential diagram, we can find complex voltages, $\dot{V}_{mn} = \dot{\varphi}_m - \dot{\varphi}_n$, between any two points, m and n, of a circuit, determine complex powers, $\underline{S} = \dot{V}_{mn}\dot{I}^*_{mn}$, etc.

2.24 Circle Diagrams. Phase-Shifting Networks

Circle diagrams are a convenient tool for representing and analysing circuits in which the parameters of one or several sections are varying. This is true, for example, of the phase shifters used in automatic control systems.

One of the likely arrangements of a phase shifter with three parallel branches is shown in Fig. 2.52 a. The middle branch contains two

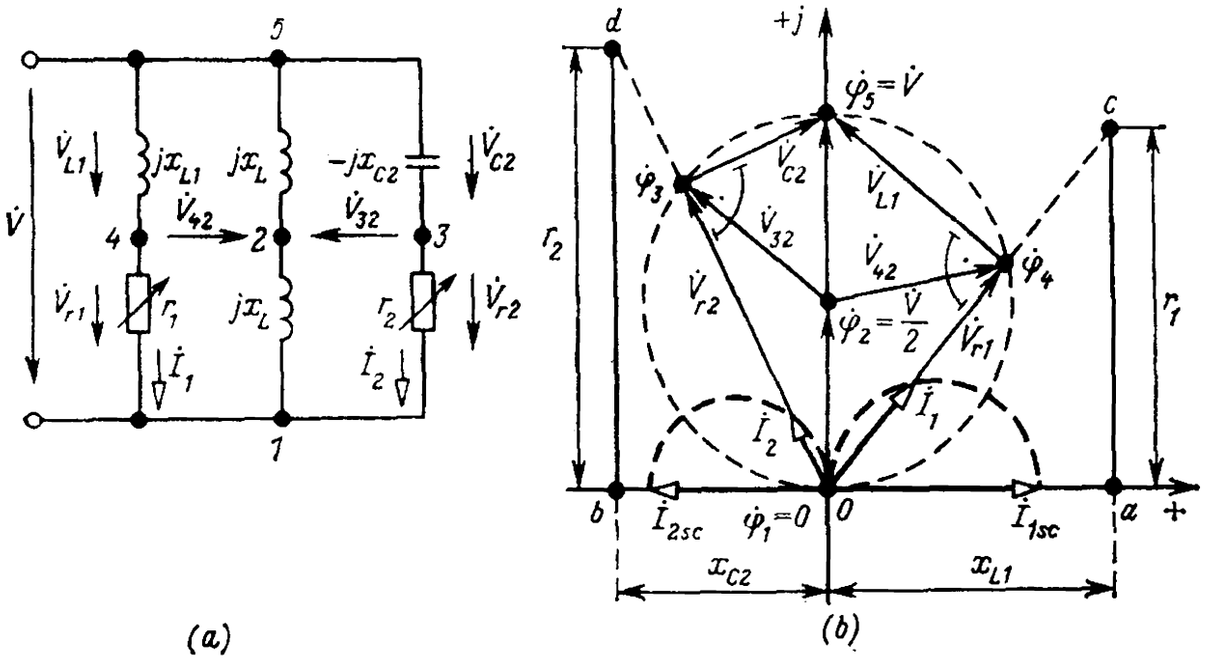


FIGURE 2.52

series-connected, identical inductive elements of constant complex reactance jx_L. The impedances of the remaining two parallel branches are varying, because they depend on the resistances r_1 and r_2 of the resistive elements.

Let the complex potential at point 1 be our reference ($\dot{\varphi}_1 = 0$). Then the complex potentials at the other points may be written as

$$\dot{\varphi}_2 = \dot{\varphi}_1 + \dot{V}/2 = \dot{V}/2 \tag{2.86a}$$

$$\dot{\varphi}_3 = \dot{\varphi}_1 + \dot{V}_{r2} = (r_2/\underline{Z}_2)\,\dot{V} \tag{2.86b}$$

$$\dot{\varphi}_4 = \dot{\varphi}_1 + \dot{V}_{r1} = (r_1/\underline{Z}_1)\,\dot{V} \tag{2.86c}$$

$$\dot{\varphi}_5 = \dot{\varphi}_2 + \dot{V}/2 = \dot{V} = \dot{\varphi}_4 + \dot{V}_{L1} = \dot{\varphi}_4 + jx_{L1}\dot{I}_1 = \dot{\varphi}_3 + \dot{V}_{C2}$$
$$= \dot{\varphi}_3 + (-jx_{C2})\dot{I}_2 \qquad (2.86d)$$

where $\underline{Z}_1 = r_1 + jx_{L1}$ and $\underline{Z}_2 = r_2 - jx_{C2}$ are the complex impedances of the respective branches.

Figure 2.52 b shows a potential diagram for the phase shifter. In the diagram, the initial phase of the supply voltage $\dot{V}$ is assumed to be $\pi/2$, that is, $\dot{V} = V \angle 90°$. It has also been noted that the voltage across the capacitive element of reactance $-jx_{C2}$ lags behind the current $\dot{I}_2$ and the voltage across the resistive element of resistance r_2 by $\pi/2$, because $\dot{V}_{r2} = r_2\dot{I}_2$ and $\dot{V}_{C2} = -jx_{C2}\dot{I}_2$, so that $\dot{V}_{r2} + \dot{V}_{C2} = \dot{V}$, which implies that the voltages $\dot{V}_{r2} = \dot{\varphi}_3$, $\dot{V}_{C2} = \dot{\varphi}_5 - \dot{\varphi}_3$ and $\dot{V} = \dot{\varphi}_5$ are the sides of a right triangle. Similarly, the voltage across the inductive element of reactance jx_{L1} leads the current $\dot{I}_1$ and the voltage across the resistive element of resistance r_1 by $\pi/2$, because $\dot{V}_{r1} = r_1\dot{I}_1$ and $\dot{V}_{L1} = jx_{L1}\dot{I}_1$, so that $\dot{V}_{r1} + \dot{V}_{L1} = \dot{V}$, which implies that $\dot{V}_{r1} = \dot{\varphi}_4$, $\dot{V}_{L1} = \dot{\varphi}_5 - \dot{\varphi}_4$ and $\dot{V} = \dot{\varphi}_5$ are the sides of a right triangle.

A change in r_1 and r_2 entails a change in $\dot{I}_1$ and $\dot{I}_2$ and, as a consequence, in $\dot{V}_{r2}$, $\dot{V}_{C2}$, $\dot{V}_{r1}$ and $\dot{V}_{L1}$, but the voltages remain to be the sides of right triangles. Therefore, the points corresponding to the complex potentials $\dot{\varphi}_3$ and $\dot{\varphi}_4$ are located on a circle whose centre is $\dot{\varphi}_2 = \dot{V}/2$ and whose radius is equal to $V/2$.

In order to locate in a complex plane the points defining the complex potentials $\dot{\varphi}_3$ and $\dot{\varphi}_4$ on the circle diagram, we lay off segments Oa and Ob equal on the adopted scale m_r to the inductive reactance x_{L1} and the capacitive reactance x_{C2}, respectively, along the real axis from the origin. Next we erect perpendiculars to the real axis from the ends a and b of the two segments and lay off on the same scale the resistances r_1 and r_2 of the resistive elements. In this way, we obtain two right triangles, Oac and Obd, similar to the respective voltage triangles, because the sides of the voltage triangles are proportional to the sides of the respective impedance triangles (the coefficients of proportionality are the currents I_1 and I_2). The hypotenuses Oc and Od of the impedance triangles cut the potential circle diagram, and the intersections define the potentials $\dot{\varphi}_4$ and $\dot{\varphi}_3$.

Thus, if we know the reactances x_{L1} and x_{C2} and the supply voltage $\dot{V}$, that is, the centre and radius of the circle diagram, and the

resistances r_1 and r_2 of the variable resistive elements, we can use the circle diagram to determine the complex potentials $\dot{\varphi}_4$ and $\dot{\varphi}_3$.

The complex current phasors

$$\dot{I}_1 = \dot{V}/(r_1 + jx_{L1}) \quad \text{and} \quad \dot{I}_2 = \dot{V}/(r_2 - jx_{C2}) \tag{2.87}$$

are directed along the hypotenuses of the impedance triangles, because $\dot{I}_1$ is in phase with $\dot{V}_{r1}$, and $\dot{I}_2$ is in phase with $\dot{V}_{r2}$.

Alternatively, the complex currents defined in (2.87) may be written as

$$\dot{V}/jx_{L1} = \dot{I}_{1,\,sc} = \dot{I}_1 - j\,(r_1/x_{L1})\,\dot{I}_1 \tag{2.88a}$$

and

$$\dot{V}/-jx_{C2} = \dot{I}_{2,\,sc} = \dot{I}_2 + j\,(r_2/x_{C2})\,\dot{I}_2 \tag{2.88b}$$

where $\dot{I}_{1,\,sc}$ and $\dot{I}_{2,\,sc}$ are the currents with the resistive elements short-circuited, that is, when $r_1 = 0$ and $r_2 = 0$. As follows from Eqs. (2.88), the current $\dot{I}_{1,\,sc}$ lags behind the supply voltage $\dot{V}$ by $\pi/2$, and the current $\dot{I}_{2,\,sc}$ leads it by the same angle (see Fig. 2.52 b).

As follows from Eqs. (2.88), each of the short-circuit currents, like the supply voltage $\dot{V}$, is equal to the sum of two components the phase difference between which is either $\pi/2$ or $-\pi/2$. This fact is represented by a phasor diagram in the form of a right triangle whose hypotenuse is the respective short-circuit current. Thus, as r_1 and r_2 are varied, the tips of the complex current phasors $\dot{I}_1$ and $\dot{I}_2$ describe each a half-circle with $\dot{I}_{1,\,sc}$ and $\dot{I}_{2,\,sc}$ as diameters (see Fig. 2.52 b).

The procedure is the same in constructing circle diagrams for other circuits with varying parameters.

2.25 The Passive Four-Terminal (Two-Port) Network

In many cases, the analysis of an electric circuit can be substantially simplified, if it is represented as a four-terminal (two-port) network (Fig. 2.53). As its name implies, this form of network has two pairs of terminals (each pair being called a port) accessible for connection to external circuits. (In Fig. 2.53, these pairs, or ports, are labelled as 1-1′ and 2-2′.) If a two-port does not contain an energy source, it is called *passive*; otherwise it is called *active*. An example of an active two-port is a vacuum-tube (or transistor) amplifier; examples of passive two-ports are a two-winding transformer, a telephone line, and a measuring bridge. Linear passive two-ports contain only linear resistive, inductive and capacitive elements.

Suppose that a source of emf $\dot{E}_1$ is connected to port 1-1′, and a load of impedance $\underline{Z}_2$ is connected to port 2-2′ (see Fig. 2.53 a). With this arrangement the two-port is said to be forward-excited; port 1-1′ is its input, and port 2-2′ is its output.

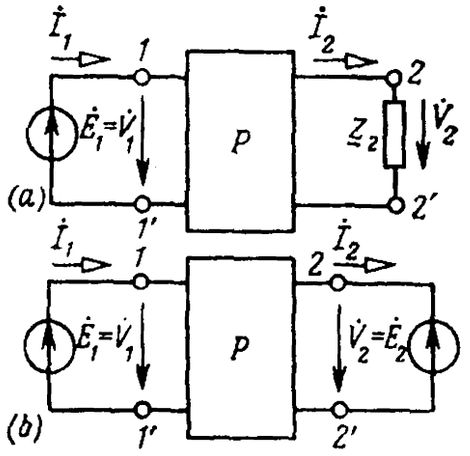


FIGURE 2.53

Let us find the relation between the input current $\dot{I}_1$ and the input voltage $\dot{V}_1 = \dot{E}_1$, on the one hand, and the output current $\dot{I}_2$ and the output voltage $\dot{V}_2 = \underline{Z}_2\dot{I}_2$ on the other, for a passive two-port. By applying the compensation principle, Eq. (1.36), we replace the load impedance $\underline{Z}_2$ with a source of emf opposing the current and equal to $\dot{E}_2 = \underline{Z}_2\dot{I}_2 = \dot{V}_2$ (see Fig. 2.53 b). The equivalent circuit thus derived contains two emf sources, so the input and output currents in the two-port can be found by use of the superposition principle, Eq. (1.35):

$$\begin{aligned} \dot{I}_1 &= \underline{Y}_{11}\dot{E}_1 + \underline{Y}_{12}\dot{E}_2 \\ \dot{I}_2 &= \underline{Y}_{21}\dot{E}_1 + \underline{Y}_{22}\dot{E}_2 \end{aligned} \tag{2.89}$$

where $\underline{Y}_{11}$ and $\underline{Y}_{22}$ are the driving-point complex admittances of the branches containing sources of emf $\dot{E}_1$ and $\dot{E}_2$, and $\underline{Y}_{12} = \underline{Y}_{21}$ are the transfer complex admittances of the respective branches (see Sec. 1.15).

Using Eqs. (2.89), we can express the input voltage $\dot{V}_1 = \dot{E}_1$ and current $\dot{I}_1$ of the two-port in terms of its output voltage $\dot{V}_2 = \dot{E}_2$ and current $\dot{I}_2$ as

$$\begin{aligned} \dot{V}_1 &= \underline{A}\dot{V}_2 + \underline{B}\dot{I}_2 \\ \dot{I}_1 &= \underline{C}\dot{V}_2 + \underline{D}\dot{I}_2 \end{aligned} \tag{2.90}$$

where

$$\underline{A} = -\underline{Y}_{22}/\underline{Y}_{21}$$
$$\underline{B} = 1/\underline{Y}_{21}$$
$$\underline{C} = \underline{Y}_{12} - \underline{Y}_{11}\underline{Y}_{22}/\underline{Y}_{21}$$
$$\underline{D} = \underline{Y}_{11}/\underline{Y}_{21}$$

are referred to as the *forward transfer parameters of the passive two-port*, but are often designated as the transmission parameters, the chain parameters, the general circuit parameters, or simply as the $ABCD$ parameters. In the general case they are complex quantities. The magnitudes of A and D are dimensionless quantities; the magnitude of B has the units of ohms, and the magnitude of C has the units of siemens. The $ABCD$ parameters are uniquely determined by the element values of the passive two-port.

In circuit analysis it is usually assumed that the $ABCD$ parameters are specified. Otherwise they must be found from the driving-point and transfer admittances which may be measured or are known.

The $ABCD$ parameters are interconnected by a simple relation of the form

$$\underline{A}\underline{D} - \underline{B}\underline{C} = 1 \tag{2.91}$$

whose validity can be readily proved by substituting their values expressed in terms of driving-point and transfer admittances.

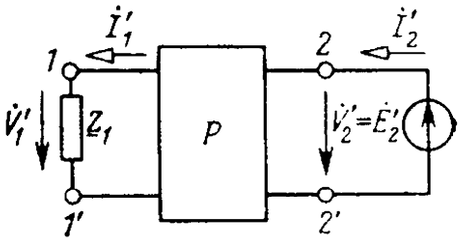


FIGURE 2.54

Equations (2.90) are referred to as the *forward transfer equations of a passive two-port.*

When a passive two-port is reverse-excited (Fig. 2.54), the source of emf is connected to port 2-2′ which is now the input, or driving point, of the two-port, and the load impedance $\underline{Z}_1$ to port 1-1′ which is now the output of the two-port.

The equations describing this two-port can be derived from its forward transfer equations (2.90) on replacing $\dot{V}_1$ with $\dot{V}_1'$, $\dot{I}_1$ with $-\dot{I}_1'$, $\dot{V}_2$ with $\dot{V}_2'$, and $\dot{I}_2$ with $-\dot{I}_2'$:

$$\begin{aligned} \dot{V}_1' &= \underline{A}\dot{V}_2' - \underline{B}\dot{I}_2' \\ -\dot{I}_1' &= \underline{C}\dot{V}_2' - \underline{D}\dot{I}_2' \end{aligned} \tag{2.92}$$

Here the primed quantities (voltages and currents) designate that the two-port is excited in the reverse direction.

On solving Eqs. (2.92) for the input quantities, $\dot{V}_2'$ and $\dot{I}_2'$, and recalling Eq. (2.91), we derive the *reverse transfer equations* of a passive two-port:

$$\begin{aligned} \dot{V}_2' &= \underline{D}\dot{V}_1' + \underline{B}\dot{I}_1' \\ \dot{I}_2' &= \underline{C}\dot{V}_1' + \underline{A}\dot{I}_1' \end{aligned} \tag{2.93}$$

In the case of a *symmetrical two-port* in which $\underline{Z}_1 = \underline{Z}_2$ and $\dot{E}_2' = \dot{E}_1$ (that is, $\dot{V}_2' = \dot{V}_1$), the input and output quantities remain the same with both forward and reverse excitation, that is, $\dot{I}_2' = \dot{I}_1$, $\dot{I}_1' = \dot{I}_2$ and $\dot{V}_1' = \dot{V}_2$. Therefore, for a symmetrical two-port the forward and reverse transfer equations must be the same, which implies that their parameters $\underline{A}$ and $\underline{D}$ must be the same.

The form of Eqs. (2.90) and (2.93) is independent of how complex the two-port itself is. Therefore, once the $ABCD$ parameters are known, we can readily find the input and output quantities of the two-port. For example, in the case of forward excitation and with the load impedance equal to $\underline{Z}_2 = \dot{V}_2/\dot{I}_2$, the input (driving-point) impedance $\underline{Z}_{1,\,\mathrm{dp}}$ can readily be found from Eqs. (2.90) as follows:

$$\underline{Z}_{1,\,\mathrm{dp}} = \dot{V}_1/\dot{I}_1 = \frac{\underline{A}\,\underline{Z}_2 + \underline{B}}{\underline{C}\,\underline{Z}_2 + \underline{D}}$$

If $\dot{V}_1$ is specified, the input current of a two-port can be found by Ohm's law

$$\dot{I}_1 = \dot{V}_1/\underline{Z}_{1,\,\mathrm{dp}}$$

Then the output voltage $\dot{V}_2$ and the output current $\dot{I}_2$ of the two-port can be found by Eq. (2.90).

Similarly, in the case of reverse excitation and with the load impedance equal to $\underline{Z}_1 = \dot{V}_1'/\dot{I}_1'$, the driving-point impedance seen looking into port 2-2′ can be found from Eqs. (2.93) as follows:

$$\underline{Z}_{2,\,\mathrm{dp}} = \dot{V}_2'/\dot{I}_2' = \frac{\underline{D}\,\underline{Z}_1 + \underline{B}}{\underline{C}\,\underline{Z}_1 + \underline{A}}$$

3 Three-Phase Circuits

3.1 Three-Phase Equipment

A very common practice in electrical engineering is to interconnect several structurally identical sinusoidal circuits operating at the same frequency and each having an energy source of its own. Each of the interconnected circuits is then called a *phase*, and the interconnection as a whole is referred to as a *polyphase system**.

Of all polyphase systems, the most commonly used ones are three-phase systems. A very important advantage offered by a three-phase system is that it uses extremely simple and inexpensive three-phase induction motors. The next most commonly used arrangements are six-phase systems, such as used in rectifiers, and two-phase systems employed in automatic control.

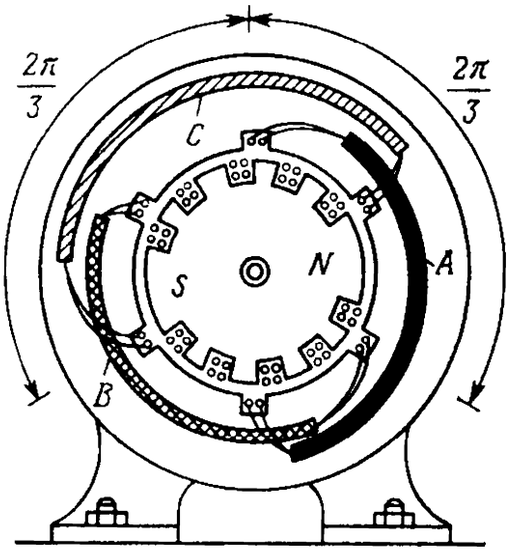


FIGURE 3.1

The energy source in a three-phase system is a three-phase generator. It differs from a single-phase sinusoidal generator (see Figs. 2.5 a and 2.8) in that, instead of one winding, its stator carries three phase windings electrically insulated from one another. In a two-pole generator, the phase windings are spaced apart by an angle $2\pi/3$ (Fig. 3.1). As the rotor spins, sinusoidal phase emfs are induced in the phase windings of the generator. Since the generator is symmetrical in design, the amplitude (peak) emfs E_m and the rms emfs E_{ph} are the same in all the phases. However, the magnetic flux set up by the rotor cuts the conductors in the phase windings at different times. For this reason, the sinusoidal emfs induced in the phase windings are displaced from one another by a third of a period or by an angle $2\pi/3$ between the winding axes.

In a generator with a multipole rotor, for each pole pair the stator carries three coils electrically insulated from one another. Col-

* As is seen, the term «phase» has two meanings in electrical engineering. Firstly, it refers to time parameter of a periodic process. Secondly, it designates a part of a polyphase system of sinusoidal circuits.

lectively, these coils form three phase windings. Arranged all the way round the periphery of the stator, the coils whose number is equal to that of pole pairs are connected within each phase winding in series or in parallel.

The phases of a three-phase generator are usually labelled with the capital letters A, B, and C. The order in which the phases are labelled is not arbitrary, but is determined by *phase sequence*, that is, the order in which the phase emfs vary in time. The designations are chosen such that the emf in phase A reaches its amplitude (peak) value one-third of a period before the emf in phase B does, and two-thirds of a period before the emf in phase C does. The order ABC

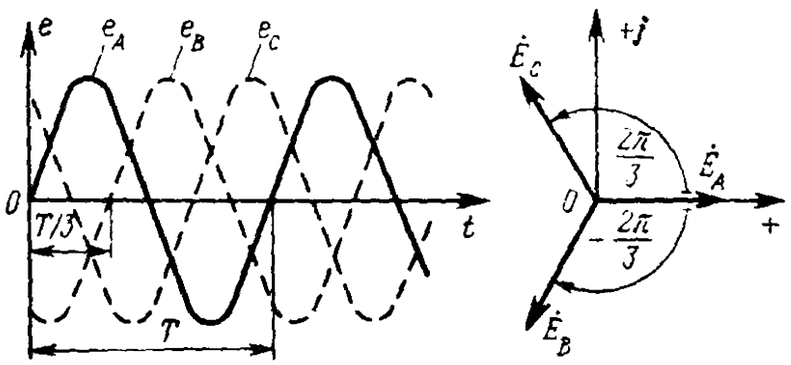


FIGURE 3.2

is taken as a *normal* or *positive phase sequence*. For the positive phase sequence the instantaneous emfs in the three phase windings of a symmetrical generator are given by

$$\begin{aligned} e_A &= E_m \sin \omega t \\ e_B &= E_m \sin (\omega t - 2\pi/3) \\ e_C &= E_m \sin (\omega t - 4\pi/3) \\ &= E_m \sin (\omega t + 2\pi/3) \end{aligned} \tag{3.1}$$

As is customary, the emf in phase A is taken to have an initial phase angle of zero.

Graphically, the instantaneous phase emfs and the three corresponding complex phasors are depicted in the plot of Fig. 3.2. The sum of the three complex emf phasors is zero. In consequence the algebraic sum of the complex phase emfs and the algebraic sum of the instantaneous phase emfs of a generator are likewise equal to zero

$$\dot{E}_A + \dot{E}_B + \dot{E}_C = 0 \tag{3.2a}$$

$$e_A + e_B + e_C = 0 \tag{3.2b}$$

The complex emfs of a symmetrical three-phase generator may be expressed in terms of the rms emf which is the same for all phases, E_{ph}, and an appropriate complex factor as follows:

$$\dot{E}_A = E_{ph} \angle 0° = E_{ph}$$

$$\dot{E}_B = E_{ph} \exp(-j2\pi/3) = E_{ph} \angle -120° = E_{ph}\left(-\frac{1}{2} - j\sqrt{3}/2\right)$$

$$\dot{E}_C = E_{ph} \exp(j2\pi/3) = E_{ph} \angle +120° = E_{ph}\left(-\frac{1}{2} + j\sqrt{3}/2\right)$$

The complex factor $\exp(j2\pi/3) = -1/2 + j\sqrt{3}/2$ is called the *phase operator* of a three-phase system and designated by the Greek letter λ (lambda). Multiplication of a complex value by λ corresponds to a rotation of the respective phasor through $2\pi/3 = 120°$ in the positive direction, that is, counterclockwise. Thus,

$$\dot{E}_A = E_{ph};\ \dot{E}_B = E_{ph}\lambda^2,\ \dot{E}_C = E_{ph}\lambda \tag{3.3}$$

To set up a three-phase system, the phases of a source must be connected in a specific manner to the phases of the associated load. Most frequently, a source and its load are interconnected in a star or in a delta.

3.2 Star-to-Star Connection of a Source and Its Load

The phase windings of a three-phase generator could be connected to three loads with three pairs of wires (as shown in Fig. 3.3), so that three noninterlinked phase circuits formed. However, non-

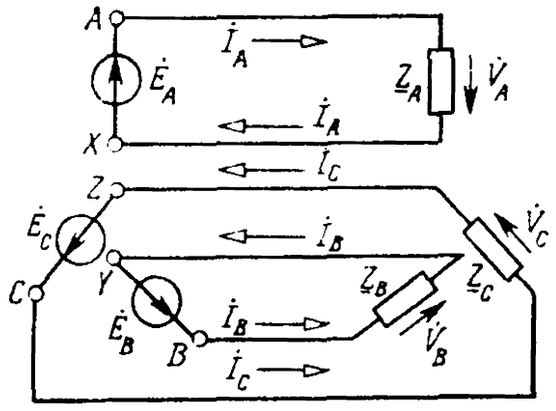


FIGURE 3.3

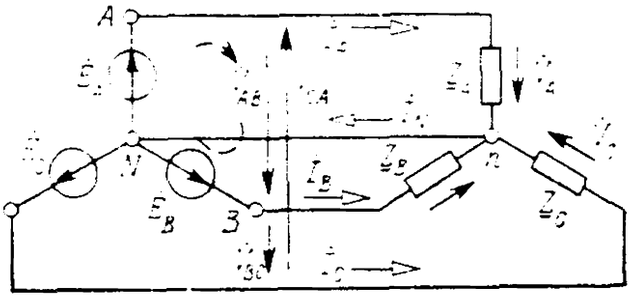


FIGURE 3.4

interlinked three-phase systems are not employed practically, although their study provides an insight into the relationships that exist when the three independent phases are combined into an interlinked three-phase system. Note the arrowheads indicating the positive directions of the phase emfs. These positive directions identify

the starts (A, B, and C) and finishes (X, Y, and Z) of the phase windings of a generator.

In a star-connected three-phase generator this form of connection is also called the wye, or Y, connection, the finishes of the phase windings meet at a common node, N (Fig. 3.4). Similarly, in a star-connected three-phase load the three phases meet at another node, n. The three return wires of the phases are combined into a common *neutral wire*. The remaining three wires interconnecting the generator and the load are called *line wires*. The node formed by the phase windings of the generator or of the load is called the *neutral point*.

Neglecting the impedances of all the wires, it is easy to determine the currents in the three phases of the generator and of the load

$$\dot{I}_A = \dot{E}_A/\underline{Z}_A, \quad \dot{I}_B = \dot{E}_B/\underline{Z}_B, \quad \dot{I}_C = \dot{E}_C/\underline{Z}_C \tag{3.4}$$

and the current in the neutral wire

$$\dot{I}_N = \dot{I}_A + \dot{I}_B + \dot{I}_C \tag{3.5}$$

A load in which the three phases have the same impedance

$$\underline{Z}_A = \underline{Z}_B = \underline{Z}_C = \underline{Z}_{ph} = z_{ph} \exp(j\varphi)$$

is called *balanced*. In the case of a balanced load, the phase currents have the same rms value I_{ph} and the same phase angle φ with respect to the corresponding phase emfs (Fig. 3.5), whereas the current in the neutral wire, Eq. (3.5), is zero. This is the reason why no neutral wire is provided in the case of a balanced load. An example of a balanced load is a three-phase motor in which the three-phase windings are connected in a star.

In a three-phase system, the voltages $\dot{V}_A$, $\dot{V}_B$ and $\dot{V}_C$ between the terminals of each phase winding in the generator or load are called the *phase voltages*.

In a *balanced three-phase system*, the rms phase voltages are the same

$$V_A = V_B = V_C = V_{ph}$$

The currents in the phase windings of the generator and in the phases of the load are called the *phase currents*. The voltages between the line wires are called the *line* (or *line-to-line*) *voltages*, and the corresponding currents are called the *line currents*.

Let us write equations for the loop (mesh) shown by a dashed arrow in Fig. 3.4, and for the other two similar loops, using Kirchhoff's voltage law and noting that

$$\dot{E}_A = \dot{V}_A, \quad \dot{E}_B = \dot{V}_B, \quad \dot{E}_C = \dot{V}_C \tag{3.6}$$

For the line voltages we get

$$\dot{V}_{AB} = \dot{E}_A - \dot{E}_B = \dot{V}_A - \dot{V}_B = V_{\text{line}} \angle 30° \tag{3.7a}$$

$$\dot{V}_{BC} = \dot{E}_B - \dot{E}_C = \dot{V}_B - \dot{V}_C = V_{\text{line}} \angle -90° \tag{3.7b}$$

$$\dot{V}_{CA} = \dot{E}_C - \dot{E}_A = \dot{V}_C - \dot{V}_A = V_{\text{line}} \angle 150° \tag{3.7c}$$

where V_{line} is the rms line voltage.

A phasor diagram showing the phase and line voltages for a source and a load connected star to star is shown in Fig. 3.5. The line voltage phasor $\dot{V}_{AB}$ is plotted on the basis of Eq. (3.7a), which

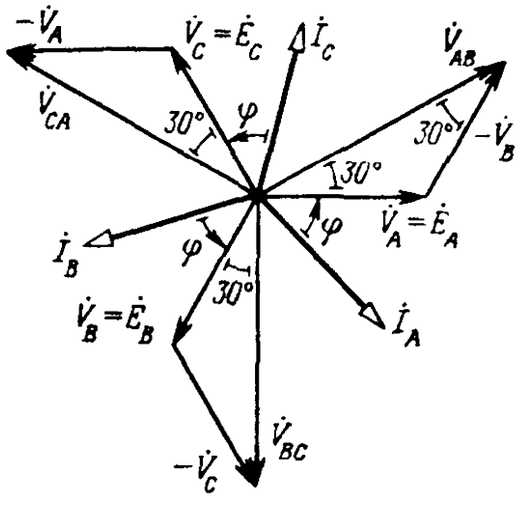


FIGURE 3.5

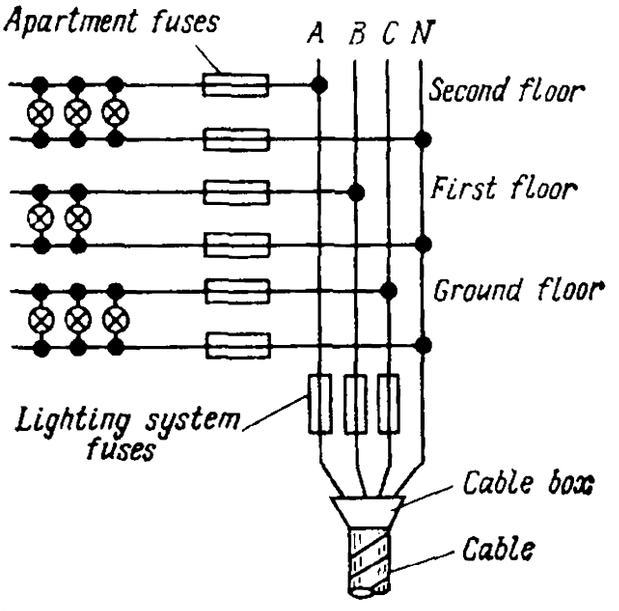


FIGURE 3.6

means that it is obtained by adding together the phasor $\dot{V}_A$ and the phasor $-\dot{V}_B$ which is equal in length to the phasor $\dot{V}_B$ but is opposite in direction. The remaining two line voltage phasors are constructed in a similar way.

If a three-phase system has a neutral wire (see Fig. 3.4), the condition defined in (3.6) is satisfied both when the load is balanced and when it is unbalanced. If there is no neutral wire, the condition is satisfied only when the load is balanced. In each case the phase and line voltage phasors form three identical isosceles triangles, with an angle of 30° at the base. From the voltage triangles it follows that the rms line and phase voltages are connected by a relation of the form

$$V_{\text{line}} = 2V_{\text{ph}} \cos 30° = \sqrt{3} V_{\text{ph}} \tag{3.8}$$

For example, if the line voltage is $V_{\text{line}} = 380$ V, the phase voltage will be $V_{\text{ph}} = 220$ V. If the line voltage is $V_{\text{line}} = 220$ V, the phase voltage will be $V_{\text{ph}} = 127$ V.

When a star-connected source is connected to a star-connected load, the line currents are equal to the respective phase currents. If the load is balanced, all the line and phase currents have the same rms value

$$I_{\text{line}} = I_{\text{ph}} \tag{3.9}$$

A three-phase motor is a balanced load. Therefore, for its connection to a source three-wire lines are used. For a lighting load (Fig. 3.6), a neutral wire is a necessity, because there is no reason to expect that this form of three-phase load will be completely balanced. No fuses or switches are allowed in the neutral wire of a four-wire lighting system, because disconnection of the neutral wire may lead to an unbalance between the phase voltages. As a result, the voltage in some phase (or phases) may be below normal, while in the remaining phases (or phase) it may be above normal, causing the lamps to blow. With this form of interlinkage, a blown fuse will only cause the lamps in one (associated) phase to go dark.

3.3 Delta-to-Delta Connection of a Source and Its Load

When a delta-connected source (this form of connection is designated with the Greek letter Δ (delta)) operates into a delta-connected load, no neutral wire is used.

We will show how a delta-to-delta three-phase system can be derived from the noninterlinked system of Fig. 3.7 a where the three

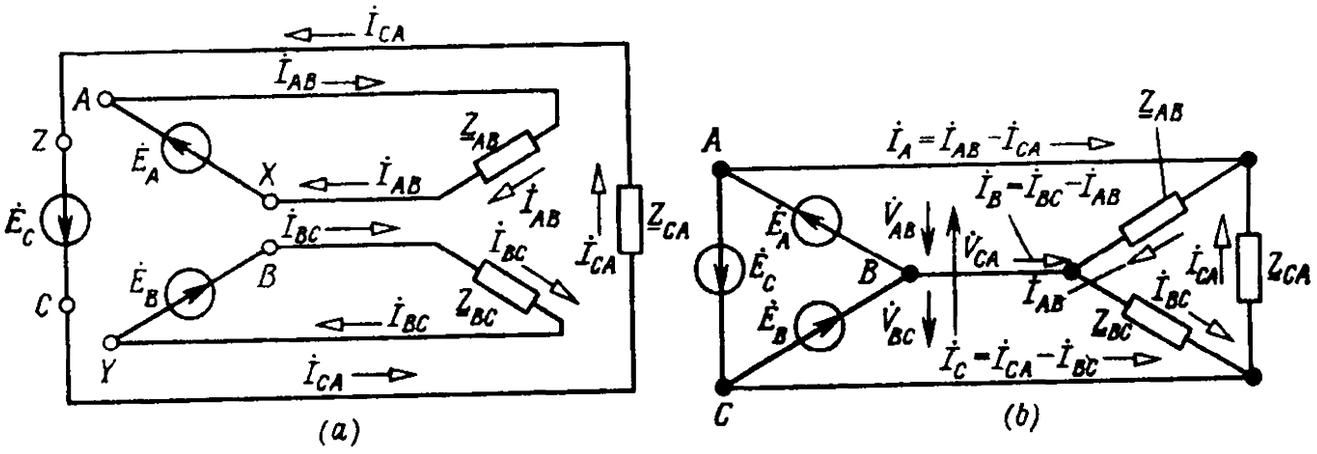


FIGURE 3.7

phase windings of a generator are connected to three loads with six wires. In order to obtain a delta connection in the generator (Fig. 3.7 b), we connect the finish X of the first winding to the start B of the second winding, the finish Y of the second winding to the start C of the third winding, and the finish Z of the third winding to the start A of the first winding. Since the algebraic sum of the sinu-

soidal phase emfs of a generator is zero, Eq. (3.2), no additional, or circulating, current will be flowing in the generator windings*. With the generator windings interconnected as explained, the voltage between the start and finish of each phase will remain unchanged, so the phase voltages are the same in the noninterlinked system of Fig. 3.7 a and in the interlinked system of Fig. 3.7 b. For this reason, the phase currents in the load, $\dot{I}_{AB}$, $\dot{I}_{BC}$ and $\dot{I}_{CA}$, connected in a delta will be the same as they are in the original system. The currents in the three combined line wires, that is, the line currents are now equal to the differences between the corresponding phase currents (by virtue of Kirchhoff's current law as applied to the load nodes):

$$\begin{aligned} \dot{I}_A &= \dot{I}_{AB} - \dot{I}_{CA} \\ \dot{I}_B &= \dot{I}_{BC} - \dot{I}_{AB} \\ \dot{I}_C &= \dot{I}_{CA} - \dot{I}_{BC} \end{aligned} \tag{3.10}$$

The line voltages are equal to the respective phase voltages, and their complex values are given by

$$\dot{V}_{AB} = \dot{E}_A, \quad \dot{V}_{BC} = \dot{E}_B, \quad \dot{V}_{CA} = \dot{E}_C \tag{3.11}$$

By Ohm's law, the complex phase currents are given by

$$\begin{aligned} \dot{I}_{AB} &= \dot{E}_A / \underline{Z}_{AB} \\ \dot{I}_{BC} &= \dot{E}_B / \underline{Z}_{BC} \\ \dot{I}_{CA} &= \dot{E}_C / \underline{Z}_{CA} \end{aligned} \tag{3.12}$$

such that for a balanced load

$$\underline{Z}_{AB} = \underline{Z}_{BC} = \underline{Z}_{CA} = \underline{Z}_{ph} = Z_{ph} \exp(j\varphi) \tag{3.13}$$

and all the phase currents have the same rms value I_{ph} and the same phase angle φ relative to the respective emfs or phase voltages.

A phasor diagram for voltages and currents is shown in Fig. 3.8. It follows from the current triangles that in a balanced three-phase system the line and phase currents are connected by a relation of the form

$$I_{line} = 2I_{ph} \cos 30° = \sqrt{3} I_{ph} \tag{3.14}$$

* If the phase emfs are other than sinusoidal, their algebraic sum may be nonzero, and there will be a circulating current flowing in the generator windings. This is one of the reasons why generator windings are not connected in a delta.

As follows from Eq. (3.11), the rms values of line and phase voltages are the same even when the load is unbalanced

$$V_{\text{line}} = V_{\text{ph}} \tag{3.15}$$

An advantage of delta-to-delta connection between a source and its load as compared with star-to-star connection is that the phase currents are independent of one another. Figure 3.9 shows a delta-

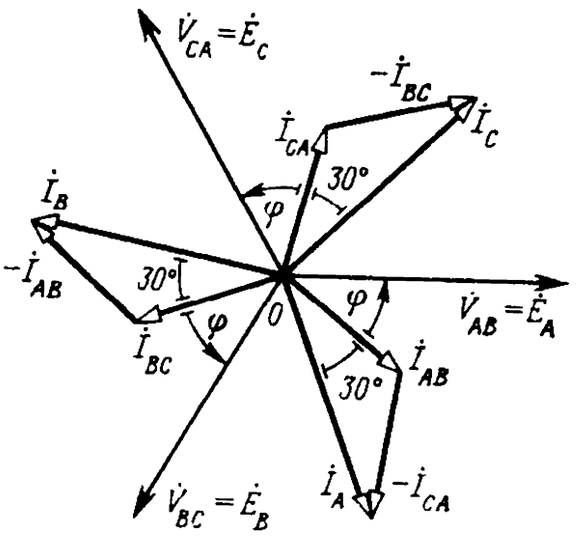


FIGURE 3.8

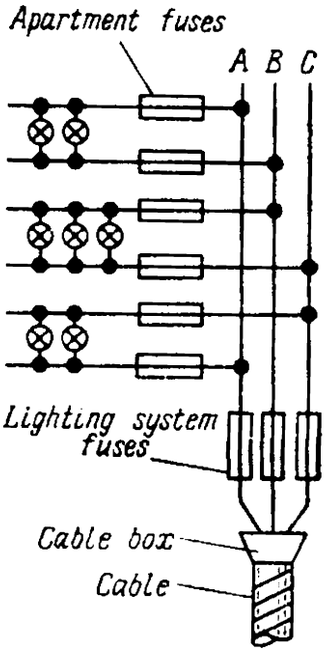


FIGURE 3.9

connected lighting load. With this form of interlinkage, if one of the fuses (say in the line wire B) blows, the lamps in the other two phases (AB and BC) will be connected in series and, given the same power, the voltage across the lamps in each of these phases will be half the line (rated) voltage; the voltage across the lamps in the third phase (CA) will remain unaffected (normal).

We have analysed star-to-star and delta-to-delta connections between a source and its load. However the techniques we have developed may be extended to include, say, star-to-delta or delta-to-star, or some other form of interconnection.

3.4 Active, Reactive and Apparent Power in a Balanced Three-Phase System

The *active* (*true* or *real*) *power of a three-phase system* refers to the sum of the active powers in all the phases of the source which is equal to the sum of the active powers in all the phases of the associated load.

In a balanced three-phase system with any form of interconnection between source and load, the active power is the same in all the

phases. Therefore, $P = 3P_{ph}$, and to each phase we may apply the expression for the active power of a sinusoidal current, Eq. (2.55):

$$P_{ph} = V_{ph} I_{ph} \cos \varphi$$

where φ is the phase angle between the *phase voltage* and the *phase current*.

On replacing the rms values of phase currents and voltages with line quantities for a star-to-star system, Eqs. (3.8) and (3.9), and for a delta-to-delta system, Eqs. (3.14) and (3.15), we obtain the same expression for the active power in a balanced three-phase system

$$P = 3V_{ph} I_{ph} \cos \varphi = \sqrt{3} V_{line} I_{line} \cos \varphi \tag{3.16}$$

Industrial loads are usually balanced or nearly so, and their power may be found by Eq. (3.16).

In the general case, the *reactive power in a three-phase system* refers to the sum of the reactive powers in all the phases of the source, equal to the sum of the reactive powers in all the phases of the associated load. By virtue of Eq. (2.58) the reactive power in a balanced three-phase system is

$$Q = 3Q_{ph} = 3V_{ph} I_{ph} \sin \varphi$$

or, on replacing the rms values of phase currents and voltages with the line quantities, we get

$$Q = \sqrt{3} V_{line} I_{line} \sin \varphi \tag{3.17}$$

The *complex power in a three-phase system* refers to the sum of the complex powers in all the phases of the source, equal to the sum of the complex powers in all the phases of the load.

The *apparent power in a balanced three-phase system* is given by

$$S = \sqrt{3} V_{line} I_{line} \tag{3.18}$$

3.5 Performance of Three-Phase Circuits Connected in a Star or in a Delta

The three phases of a load may be interlinked in a manner independent of that used in its source. It often happens that a delta-connected load is converted to a star-connected one in order, say, to reduce the starting currents of three-phase motors, to adjust the temperature in three-phase electric furnaces, etc.

Let us see how the conversion of a star connection to a delta connection can affect the rms values of currents in a balanced load of phase impedance z_{ph}. The star-to-delta conversion can be effected with, say, a simple three-pole switch as shown in Fig. 3.10.

When the load is star-connected, the rms values of its phase and line currents, Eq. (3.9), and voltages, Eq. (3.8), are connected by relations of the form

$$I_{\text{ph, Y}} = V_{\text{ph, Y}}/z_{\text{ph}} = I_{\text{line, Y}}$$

$$V_{\text{ph, Y}} = V_{\text{line}}/\sqrt{3}$$

whence it follows that

$$I_{\text{line, Y}} = V_{\text{line}}/\sqrt{3}\,z_{\text{ph}} \tag{3.19}$$

When the load is delta-connected, the rms values of its phase and line currents, Eq. (3.14), and voltages, Eq. (3.15), are connected by relations of the form

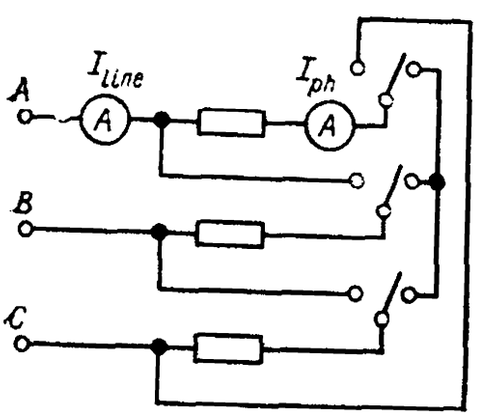


FIGURE 3.10

$$I_{\text{ph, }\Delta} = V_{\text{ph, }\Delta}/z_{\text{ph}} = I_{\text{line, }\Delta}/\sqrt{3}$$

$$V_{\text{ph, }\Delta} = V_{\text{line}}$$

whence it follows that

$$I_{\text{line, }\Delta} = \sqrt{3}V_{\text{line}}/z_{\text{ph}} \tag{3.20}$$

Given the same rms line voltage V_{line} and the same phase impedance z_{ph}, from comparison of Eq. (3.19) applicable to a star-connected load and Eq. (3.20) applicable to a delta-connected load we find that

$$I_{\text{line, }\Delta} = 3I_{\text{line, Y}}$$

and that

$$I_{\text{ph, }\Delta} = \sqrt{3}I_{\text{ph, Y}}$$

The active power in a balanced three-phase load, either star- or delta-connected, is given by Eq. (3.16) as

$$P = \sqrt{3}V_{\text{line}}I_{\text{line}}\cos\varphi$$

The conversion from a delta to a star reduces the rms value of line current, so the power in the system is decreased to one-third or, which is the same

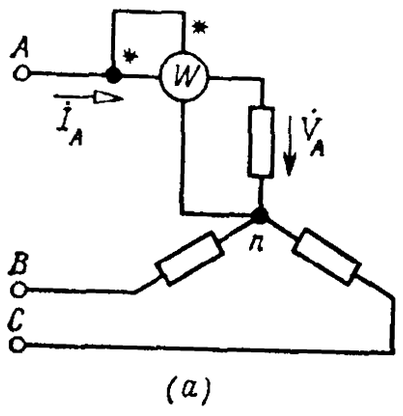
$$P_{\Delta} = 3P_{\text{Y}} \tag{3.21}$$

3.6 Measurement of Active Power in a Three-Phase System

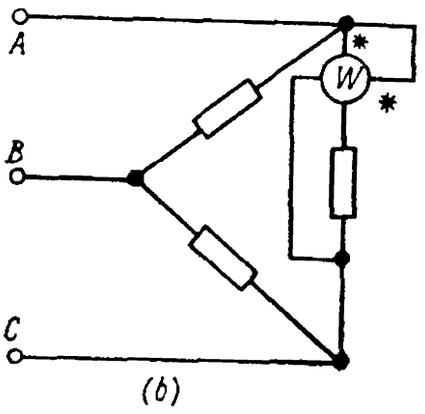
In the case of a balanced three-phase load, the active power is the same in all the phases. Therefore, if we wish to determine the total power in such a system, it will suffice to measure the active power in only one phase and to multiply it by three: $P = 3P_{\text{ph}}$.

The power in one phase, P_{ph}, is very simple to measure if the load is star-connected and has its neutral point accessible for connection. Figure 3.11 a illustrates the measurement of the active power in phase A of a load with a wattmeter. Connection of a wattmeter for measuring phase power in a delta-connected load is shown in Fig. 3.11 b.

If the phases of a balanced load are not accessible for connection, it is usual to provide an *artificial neutral point*. This neutral point is formed in a star-connected load by the voltage circuit of a watt-



(a)



(b)

FIGURE 3.11

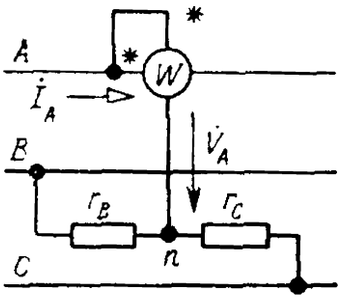


FIGURE 3.12

meter of resistance r, and two resistors $r_B = r_C = r$ (Fig. 3.12). With this arrangement, the voltage circuit of the wattmeter carries the phase voltage, and the current in the instrument is equal to the phase (line) current of the star- (or delta-) connected load. As a consequence, the wattmeter measures one-third of the load power in each case, Eq. (3.16). As a rule, wattmeters come with an artificial neutral point already arranged by their makers.

If the load in a three-phase, three-wire system is unbalanced, power is mostly measured by the *two-wattmeter method*. A distinction of the method is that even for a balanced load the two wattmeters will usually give different indications, and one of them may even be negative. In such a case, the power is equal to the *algebraic sum* of the two indications.

To prove the validity of power measurements by the two-wattmeter method, let us first express the instantaneous power in the system in terms of its instantaneous voltages and currents. The instantaneous power in any phase is equal to the product of the instantaneous voltage by the instantaneous current of that phase, and the instantaneous power in a three-phase system is equal to the sum of the instantaneous powers in the phases. Notably, for a star-connected load the instantaneous power is

$$p = v_A i_A + v_B i_B + v_C i_C$$

For a star-connected load without a neutral wire, however,

$$i_A + i_B + i_C = 0$$

and so

$$i_C = -(i_A + i_B)$$

On substituting for i_C in the equation of the instantaneous power, we get

$$p = (v_A - v_C)\, i_A + (v_B - v_C)\, i_B$$

Since the difference of phase voltages is equal to the respective line voltage, Eq. (3.7),

$$v_A - v_C = v_{AC},\ v_B - v_C = v_{BC}$$

it follows that the instantaneous power is given by

$$p = v_{AC} i_A + v_{BC} i_B \tag{3.22}$$

Thus, the instantaneous power in a three-phase system is equal to the sum of two products of line voltages by line currents. The

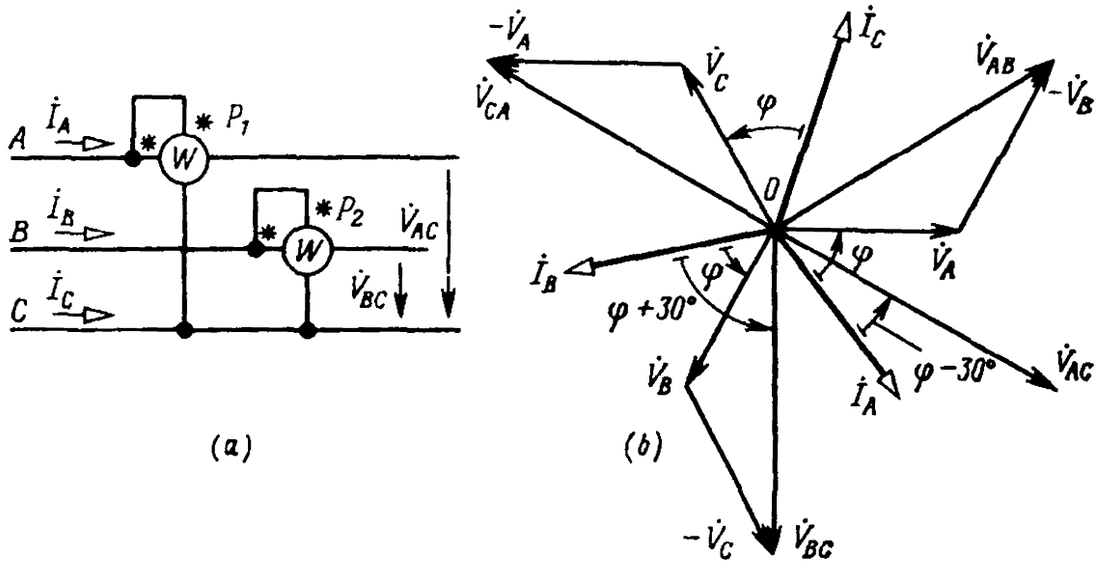


FIGURE 3.13

sum of the average values of the two products, that is, the active power in a system, can be measured with two wattmeters (Fig. 3.13a):

$$P = P_1 + P_2 = V_{AC} I_A \cos(\psi_v - \psi_i) + V_{BC} I_B \cos(\psi_v - \psi_i) \tag{3.23}$$

where $\psi_v - \psi_i$ is the phase angle between the respective line voltage and line current.

When the wattmeters are connected correctly, the positive directions of the voltages and currents are from the start to the finish of the voltage and current circuits of the wattmeters. For this, the "start"

of the first wattmeter (the terminal marked with an asterisk) should be connected to wire A, and the "finish" of the same circuit to wire C.

How the power measured in a three-phase system is distributed between the two wattmeters depends mainly on the phase angle between the line voltages and currents (see Eq. (3.23)). Let us see how the matters stand in the case of a balanced load. A phasor diagram of voltages and currents for a star-connected balanced load is shown in Fig. 3.13 b. The phase angle between the corresponding phase voltages and currents is the same and equal to the argument φ of the complex impedance of a load phase. As follows from the phasor diagram, the phase angle between the phasors $\dot{V}_{AC}$ and $\dot{I}_A$, $\dot{V}_{BC}$ and $\dot{I}_B$ in a balanced load is respectively equal to $\varphi - 30°$ and $\varphi + 30°$.

In the case of a balanced load, the rms values of line voltages and line currents are respectively the same, that is,

$$V_{BC} = V_{AC} = V_{line}$$

and

$$I_A = I_B = I_{line}$$

Thus, the sum of the two wattmeter indications is equal to the power in a balanced three-phase system

$$P = P_1 + P_2 = V_{line} I_{line} \cos(\varphi - 30°) + V_{line} I_{line} \cos(\varphi + 30°)$$

From the above equation it follows that in the case of a balanced load the wattmeter indications P_1 and P_2 will be the same only for $\varphi = 0$. If $\varphi > 60°$, the second wattmeter will give a negative indication, and an algebraic sum will be obtained.

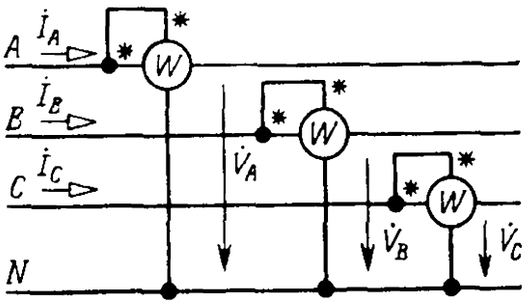


FIGURE 3.14

No proof is needed for the validity of measuring power by the two-wattmeter method when the load is delta-connected, because, with the line voltages and currents fixed in advance, the power is independent of the way in which the load phases are interlinked.

In the case of three-phase systems with a neutral wire, the simplest way to measure power is by the three-wattmeter method (Fig. 3.14). In this case, each wattmeter measures power in one phase of the load (or of the generator). The active power in the complete three-phase system is the sum of indications by the three wattmeters

$$P = P_1 + P_2 + P_3$$

A switchboard-mounted *three-phase wattmeter* is a combination of two moving systems (for a three-wire system) or of three moving systems (for a four-wire system), linked to a common shaft and operating a common pointer. Each of the moving systems is connected in the measured three-phase circuit in a manner corresponding to power measurement by the two-wattmeter or the three-wattmeter method.

3.7 A Balanced Three-Phase Circuit with Several Loads

In many cases one has a balanced (or nearly so) three-phase circuit which contains several loads such as three-phase motors.

For analysis purposes, the several loads in such a circuit should be replaced with a single equivalent one in which the phases are connected in a star. After the line current and phase voltage of the equivalent load have been found, it is an easy matter to determine the phase currents in the individual loads.

As an example, we will consider a balanced three-phase circuit (Fig. 3.15 a) containing two balanced loads in which the phases of

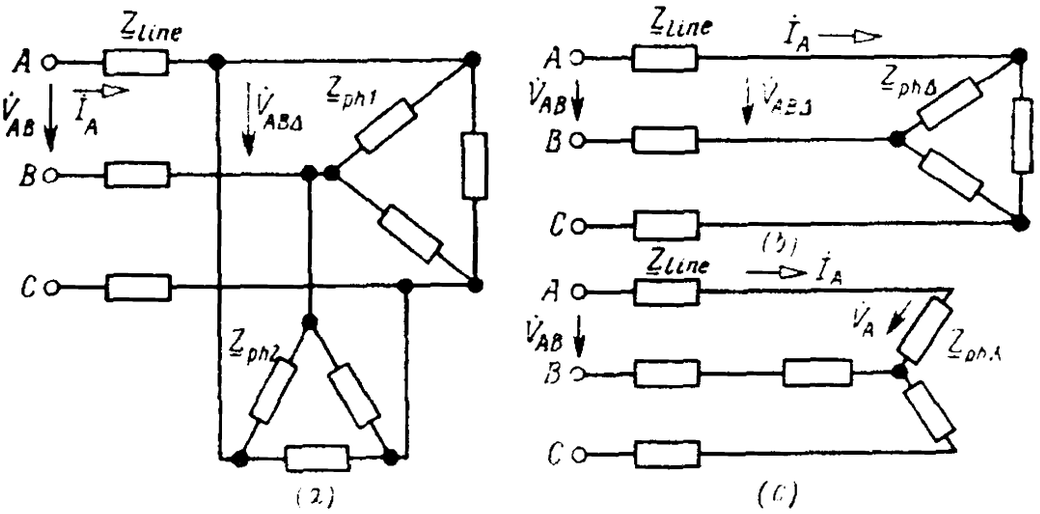


FIGURE 3.15

complex impedances $\underline{Z}_{ph1}$ and $\underline{Z}_{ph2}$ are connected in a delta. The loads are connected to a power transmission line whose conductors have a complex impedance $\underline{Z}_{line}$ and with a known line voltage, V_{line}, at the beginning of the line.

The like phases of the two balanced loads are connected in parallel. This means that the loads may be replaced with one equivalent balanced load in which the phases are connected in a delta (see Fig. 3.15 b) and have the same complex impedance

$$\underline{Z}_{ph.\,\Delta} = \underline{Z}_{ph1}\underline{Z}_{ph2}/(\underline{Z}_{ph1} + \underline{Z}_{ph2})$$

where

$$\underline{Z}_{ph1} = r_{ph1} + jx_{ph1}$$

$$\underline{Z}_{ph2} = r_{ph2} + jx_{ph2}$$

Now we transform the derived balanced delta connection into an equivalent balanced star connection (see Fig. 3.15 c) in which, on the basis of Eq. (1.23), the complex impedance of each phase is

$$\underline{Z}_{ph,\,Y} = \underline{Z}_{ph,\,\Delta}/3 = r_{ph.\,\Delta}/3 + jx_{ph\ \Delta}/3$$

If we include the complex impedance of the line conductors, the entire system may be converted into an equivalent balanced star. The complex phase impedance in the equivalent star connection is

$$\underline{Z}_{ph} = \underline{Z}_{line} + \underline{Z}_{ph,\,Y} = r_{ph} + jx_{ph}$$

and its phase impedance is

$$\underline{Z}_{ph} = \sqrt{r_{ph}^2 + x_{ph}^2}$$

The subsequent analysis does not call for the use of the complex method. It is sufficient first to determine the rms value of line current

$$I_{line} = V_{ph}/z_{ph} = V_{line}/\sqrt{3}\,z_{ph}$$

then the rms phase voltages of the loads in the equivalent star

$$V_{ph,\,Y} = z_{ph,\,Y} I_{line}$$

and, by use of Eq. (3.8), the line voltages of the loads

$$V_{line,\,\Delta} = \sqrt{3} V_{ph,\,Y}$$

The rms values of phase currents in the loads are given by

$$I_{ph1} = V_{line,\,\Delta}/z_{ph1}$$

and

$$I_{ph2} = V_{line,\,\Delta}/z_{ph2}$$

3.8 The Condition of Unbalance in a Three-Phase Circuit

Very often the condition of unbalance in a three-phase circuit arises when the phases of an unbalanced load are connected into a star without a neutral wire or with a neutral wire whose complex impedance $\underline{Z}_N$ must be considered in calculations.

Given the rms value of line voltage in the load

$$V_{AB} = V_{BC} = V_{CA} = V_{line}$$

we can expand the three-phase circuit to include an imaginary balanced star-connected three-phase emf source (Fig. 3.16) in which the rms phase emf is

$$E_A = E_B = E_C = E_{ph} = V_{line}/\sqrt{3}$$

The circuit thus derived has two neutral points. One is point N in the balanced generator, and the other is n in the unbalanced load. Together they form two nodes of the circuit, so, for circuit analysis we should use the node-pair method (see Sec. 1.13), on replacing in Eq. (1.28) the branch conductances $g = 1/r$ with complex branch admittances $\underline{Y} = 1/\underline{Z}$, and the direct emfs and currents with the complex values of the respective sinusoidal quantities. In the three-phase system considered,

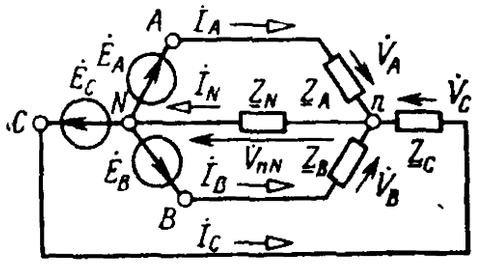


FIGURE 3.16

the complex voltage $\dot{V}_{nN}$ between the neutral point n of the load and the neutral point N of the imaginary generator is called the *neutral-point displacement voltage.* By the node-pair method,

$$\dot{V}_{nN} = \dot{\varphi}_n - \dot{\varphi}_N = \frac{\underline{Y}_A\dot{E}_A + \underline{Y}_B\dot{E}_B + \underline{Y}_C\dot{E}_C}{\underline{Y}_A + \underline{Y}_B + \underline{Y}_C + \underline{Y}_N} \tag{3.24}$$

or, subject to Eq. (3.3) and the equality

$$E_{ph} = V_{line}/\sqrt{3}$$

the complex neutral-point displacement voltage may be written as

$$\dot{V}_{nN} = \frac{V_{line}(\underline{Y}_A + \lambda^2\underline{Y}_B + \lambda\underline{Y}_C)}{\sqrt{3}(\underline{Y}_A + \underline{Y}_B + \underline{Y}_C + \underline{Y}_N)} \tag{3.25}$$

The phase voltages of the load are found by using Kirchhoff's voltage law for three loops

$$\begin{aligned} \dot{V}_A &= \dot{E}_A - \dot{V}_{nN} \\ \dot{V}_B &= \dot{E}_B - \dot{V}_{nN} \\ \dot{V}_C &= \dot{E}_C - \dot{V}_{nN} \end{aligned} \tag{3.26}$$

By Ohm's law, the phase currents and the current in the neutral wire are respectively given by

$$\dot{I}_A = \underline{Y}_A\dot{V}_A, \quad \dot{I}_B = \underline{Y}_B\dot{V}_B, \quad \dot{I}_C = \underline{Y}_C\dot{V}_C, \quad \dot{I}_N = \underline{Y}_N\dot{V}_{nN} \tag{3.27}$$

The distribution of voltages between the star-connected phases of an unbalanced load is illustrated in the potential diagram of Fig. 3.17 a.

In constructing the diagram, the neutral point N of the imaginary generator has been chosen as reference or origin, so the potential at that point is assumed to be zero. The phasors representing the three phase emfs $\dot{E}_A$, $\dot{E}_B$, and $\dot{E}_C$ of the imaginary generator are drawn from the origin. Their lengths give the complex potentials $\dot{\varphi}_A$, $\dot{\varphi}_B$ and $\dot{\varphi}_C$ of the line wires A, B and C for $\dot{\varphi}_N = 0$, and, in

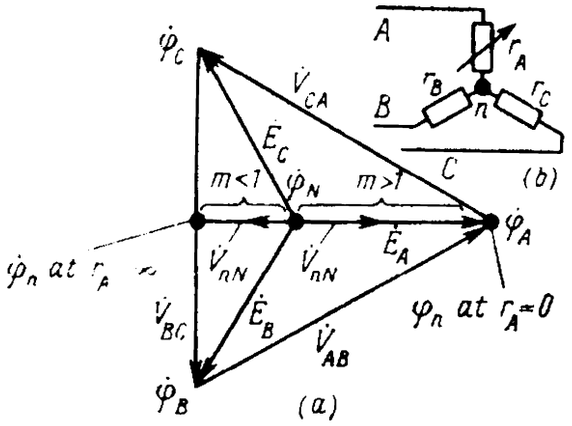


FIGURE 3.17

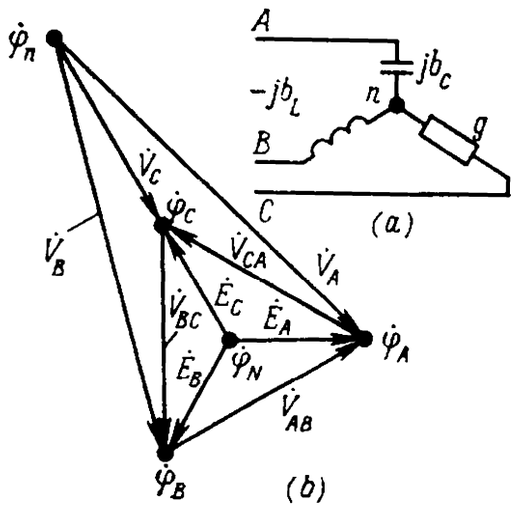


FIGURE 3.18

consequence, the complex line voltages $\dot{V}_{AB} = \dot{\varphi}_A - \dot{\varphi}_B$, $\dot{V}_{BC} = \dot{\varphi}_B - \dot{\varphi}_C$, and $\dot{V}_{CA} = \dot{\varphi}_C - \dot{\varphi}_A$. In the case of a balanced load the neutral point is not displaced, so $\dot{V}_{nN} = 0$, and the potential at the neutral point of the load is $\dot{\varphi}_n = 0$. This is why in the potential diagram the potential at the neutral point of the load, $\dot{\varphi}_n$, coincides with that of the generator, $\dot{\varphi}_N$. In the case of an unbalanced load, the neutral point is displaced and, as follows from Eq. (3.24), $\dot{V}_{nN}$ is not equal to zero. For this reason the potential at the neutral point of the load, $\dot{\varphi}_n$, is displaced relative to that at the neutral point of the generator, $\dot{\varphi}_N$, or, which is the same, from the centre of the line-voltage triangle (the neutral-point displacement).

Consider an elementary case involving a load in which the phase resistances are r_A and $r_B = r_C = r$ and there is no neutral wire (Fig. 3.17 b). Phases B and C have the same conductance $g_B = g_C = g = 1/r$, whereas the conductance of phase A, $g_A = 1/r_A$, varies from zero to infinity.

We denote the ratio $g_A/g = m$ and set out to find the neutral-point displacement voltage by Eq. (3.24) and subject to (3.3):

$$\dot{V}_{nN} = \frac{g(m+\lambda^2+\lambda)}{g(m+2)}\dot{E}_A = \frac{m-1}{m+2}\dot{E}_A$$

or

$$\dot{V}_{nN} = \frac{1-1/m}{1+2/m}\dot{E}_A$$

As g_A varies from zero to infinity, the coefficient of $\dot{E}_A$ remains real. In consequence, the neutral-point displacement voltage $\dot{V}_{nN}$ is in phase with $\dot{E}_A$ for $m > 1$, and is in anti-phase with it for $m < 1$ (Fig. 3.17 a). As special cases, when phase A is open-circuited and $g_A = 0$ or $r_A = \infty$ and $m = 0$, the neutral-point displacement is

$$\dot{V}_{nN} = -\dot{E}_A/2$$

In the circumstances, the phase voltages in the load are all the same

$$\dot{V}_A = \dot{E}_A - \dot{V}_{nN} = \left(1+\frac{1}{2}\right)V_{line}/\sqrt{3} = 1.5V_{line}/\sqrt{3}$$

$$\dot{V}_B = \dot{E}_B - \dot{V}_{nN} = \left(\lambda^2+\frac{1}{2}\right)V_{line} = -jV_{line}/2$$

$$\dot{V}_C = \dot{E}_C - \dot{V}_{nN} = \left(\lambda+\frac{1}{2}\right)V_{line} = jV_{line}/2$$

In the above expressions we recall that $V_{line} = \sqrt{3}E_A$.

For $g_A = \infty$ or $r_A = 0$, that is when point A is short-circuited to the neutral point n (see Fig. 3.17 b) it is obvious that $\dot{V}_A = 0$, $\dot{V}_B = \dot{V}_{BA} = -\dot{V}_{AB}$, and $\dot{V}_C = \dot{V}_{CA}$.

The neutral-point potential may move well beyond the limits of the line-voltage triangle, if the admittances of the load phases connected in a star without a neutral wire are different in their effect.

As an example, we will calculate the neutral-point displacement and the phase voltages for a load in which the complex phase admittances are $\underline{Y}_A = jb_C$, $\underline{Y}_B = -jb_L$, and $\underline{Y}_C = g$, such that $g = b_C = b_L$ (Fig. 3.18 a).

By virtue of Eq. (3.24), the neutral-point displacement is

$$\dot{V}_{nN} = \frac{jb_C - \lambda^2 jb_L + g\lambda}{jb_C - jb_L + g}\dot{E}_A = (-1.37 + j2.37)\dot{E}_A \qquad (3.28)$$

The load phase voltages are found in the same manner as for the load of Fig. 3.17 b. The rms voltages are found to be as follows:

$$V_A = 3.34V_{\text{line}}/\sqrt{3}, \quad V_B = 3.34V_{\text{line}}/\sqrt{3}, \quad V_C = V_{\text{line}}$$

The applicable potential diagram is shown in Fig. 3.18 b.

The circuit in Fig. 3.18 a has a very important property which is utilized in various applications. If the capacitive susceptance of phase A and the inductive susceptance of phase B are the same and constant, $b_L = b_C = b = \text{constant}$, then the current in phase

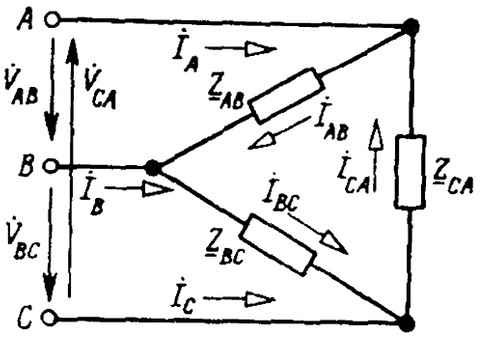


FIGURE 3.19

C is independent of the conductance $g = \text{VAr}$ of that phase. To demonstrate, it follows from the phasor diagram in Fig. 3.18 b that

$$\dot{V}_C = (\dot{E}_C - \dot{V}_{nN}) = \lambda\dot{E}_A - \left[j\frac{b}{g}(1-\lambda^2) + \lambda\right]\dot{E}_A = j\frac{b}{g}(\lambda^2 - 1)\,\dot{E}_A$$

$$\dot{I}_C = \dot{V}_C g = jb\,(\lambda^2 - 1)\,\dot{E}_A = \text{constant}$$

With the line voltages fixed in advance, the phase currents in an unbalanced delta-connected load (Fig. 3.19) are found by Ohm's law

$$\dot{I}_{AB} = \dot{V}_{AB}/\underline{Z}_{AB}, \quad \dot{I}_{BC} = \dot{V}_{BC}/\underline{Z}_{BC}, \quad \dot{I}_{CA} = \dot{V}_{CA}/\underline{Z}_{CA}$$

The line currents are found by Kirchhoff's current law

$$\dot{I}_A = \dot{I}_{AB} - \dot{I}_{CA}, \quad \dot{I}_B = \dot{I}_{BC} - \dot{I}_{AB}, \quad \dot{I}_C = \dot{I}_{CA} - \dot{I}_{BC}$$

In the analysis of a more complicated unbalanced three-phase circuit containing unbalanced loads, such as shown in Fig. 3.15 a, all the loads are transformed into a star-connected equivalent load. The necessary transformations are performed in the same order as for balanced loads (see Fig. 3.15 b and c), but the impedance of each phase in the equivalent load has to be calculated separately.

4 Periodic Nonsinusoidal Currents in Electric Circuits

4.1 General

The sine wave is the simplest form of a periodic process. In power systems, measures are taken to preserve the sine waveform of alternating currents and voltages and to avoid any variations from it. In contrast, for telecommunications, vacuum-tube and semiconductor devices variations from the sinusoidal waveform lie at the basis of their operation. Therefore, a working knowledge of an elementary theory of nonsinusoidal periodic currents and voltages is essential for a proper insight into the operating principles of automatic-control systems, electronic hardware, and a large variety of high-technology apparatus.

A *periodic nonsinusoidal function* is such that

$$f(t) = f(t + kT)$$

where T is the period of the function, or the minimum time interval after which the periodic process repeats itself, and k is an integer.

As mathematics tells us, this periodic function can be expanded into a harmonic, or Fourier, series, generally unbounded. In electric-circuit analysis, however, it is customary to limit the Fourier series to n harmonic (sinusoidal) components or simply harmonics. For example, a nonsinusoidal periodic current may be written as

$$i = I_0 + I_{1m} \sin(\omega t + \psi_{i1}) + I_{2m} \sin(2\omega t + \psi_{i2}) + \ldots + I_{nm} \sin(n\omega t + \psi_{in})$$

or

$$i = I_0 + \sum_{k=1}^{n} I_{km} \sin(k\omega t + \psi_{ik}) \tag{4.1}$$

where I_0 is the *constant* (*d.c.*) *component*; $I_{1m} \sin(\omega t + \psi_{i1})$ is the *fundamental component* (or simply the *fundamental*) whose frequency is equal to that of the nonsinusoidal periodic function, that is, the current i, the remaining terms of the expansion being called the *higher harmonics* or simply *harmonics*; and ψ_{ik} is the initial phase of the kth harmonic, dependent on the choice of the time origin ($t = 0$).

Thus a periodic nonsinusoidal current may be represented as a sum of a direct current and sinusoidal currents at frequencies which

are integral multiples of the fundamental frequency, and with different initial phases. This form of representation is often used in the analysis of circuits carrying periodic nonsinusoidal currents.

Figure 4.1 shows a plot of a periodic nonsinusoidal current i which contains only the fundamental i_1 and the second harmonic i_2.

A Fourier series expansion for a periodic nonsinusoidal voltage across any part of a circuit is written in a manner similar to Eq. (4.1)

$$v = V_0 + \sum_{k=1}^{n} V_{km} \sin (k\omega t + \psi_{vk}) \tag{4.2}$$

This is also true of source emfs

$$e = E_0 + \sum_{k=1}^{n} E_{km} \sin (k\omega t + \psi_{ek})$$

and other electric quantities.

If a circuit carrying a periodic nonsinusoidal current is linear (in which the circuit parameters r, L and C are independent of current and voltage), it can be analysed with the aid of the superposition theorem (see Sec. 1.15). This means that each harmonic and the d.c. component (if there is any) are found separately (independently of one another).

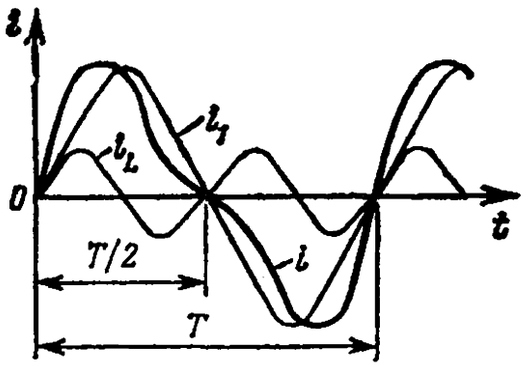


FIGURE 4.1

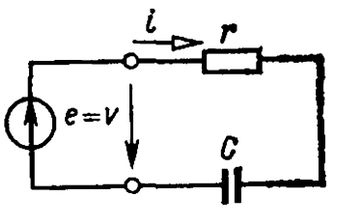


FIGURE 4.2

As a simple example, let us find the current in a circuit which is a series combination of a resistive element of resistance r and a capacitive element of capacitance C (Fig. 4.2), assuming that the applied (source) voltage is

$$v = e = V_{1m} \sin (\omega t) + V_{5m} \sin (5\omega t + \psi_{v5})$$

Accordingly, the current in the circuit is

$$i = I_{1m} \sin (\omega t - \varphi_1) + I_{5m} \sin (5\omega t + \psi_{v5} - \varphi_5)$$

By Ohm's law, the current at the fundamental frequency is

$$I_{1m} = \frac{V_{1m}}{\sqrt{r^2 + (1/\omega C)^2}}$$

and the 5th harmonic current is

$$I_{5m} = \frac{V_{5m}}{\sqrt{r^2 + (1/5\omega C)^2}}$$

By applying Eq. (2.49b),

$\varphi_1 = \text{arc tan } (-1/\omega\, Cr)$

$\varphi_5 = \text{arc tan } (-1/5\omega\, Cr)$

Of course, any other analysis techniques, including the complex-number method, used for sinusoidal circuits may be employed to find each of the harmonics in the case of a multiloop circuit.

4.2 The RMS Value of a Periodic Nonsinusoidal Current

As already noted, the instantaneous values of current and other electric quantities can be found by the superposition theorem. In practice it is as important to be able to calculate the rms values of currents, voltages, and emfs, measured by suitable instruments (ammeters or voltmeters).

In Sec. 2.6, we have defined the rms value of an alternating current, Eq. (2.17), by comparing its thermal effect with that of direct current. In fact, this definition holds for any periodic current. Therefore, we will write the rms value of a periodic nonsinusoidal current as

$$I = \sqrt{\frac{1}{T}\int_0^T i^2\,\mathrm{d}t} \tag{4.3}$$

In view of Eq. (4.1), the integral

$$\int_0^T i^2\,\mathrm{d}t = \int_0^T ii\,\mathrm{d}t$$

may be represented as a sum of four integrals, namely:

$$(1)\quad \frac{1}{T}\int_0^T I_{km}^2 \sin^2(k\omega t + \psi_{ik})\,\mathrm{d}t = \frac{I_{km}^2}{2} = I_k^2$$

because this integral is, by definition, equal to the square of the rms value, I_k, of the kth harmonic current;

$$(2)\quad \frac{1}{T}\int_0^T I_0 I_0\,\mathrm{d}t = I_0^2$$

which is the square of the d.c. component;

$$(3)\quad \frac{1}{T}\int_0^T I_0 I_{km}\sin(k\omega t+\psi_{ik})\,dt=\frac{1}{T}I_0 I_{km}\int_0^T \sin(k\omega t+\psi_{ik})\,dt=0$$

because the integral of a sinusoidal quantity over a whole number of periods is zero; and

$$(4)\quad \frac{1}{T}\int_0^T I_{km}I_{lm}\sin(k\omega t+\psi_{ik})\sin(l\omega t+\psi_{il})=0$$

where k and l are the harmonic orders, such that $k \neq l$. The integral is equal to zero because the product of two sine functions may be replaced with the difference of two cosines

$$\sin\beta\sin\gamma=\frac{1}{2}[\cos(\beta-\gamma)-\cos(\beta+\gamma)]$$

That is, the integrand in (4) is the difference of two cosines for each of which the integral over a whole number of periods is zero.

Thus the rms value of a periodic nonsinusoidal current is

$$I=\sqrt{I_0^2+I_1^2+I_2^2+I_3^2+\ldots+I_n^2}$$

or

$$I=\sqrt{I_0^2+\sum_{k=1}^{n}I_k^2} \tag{4.4}$$

That is, *the rms value of a periodic nonsinusoidal current is equal to the square root of the sum of the square of the d.c. component and the squares of the rms values of all the harmonics.* This definition also holds for the rms value of a periodic nonsinusoidal voltage

$$V=\sqrt{V_0^2+\sum_{k=1}^{n}V_k^2} \tag{4.5}$$

and any other similar periodic nonsinusoidal quantity.

4.3 The Power of a Periodic Nonsinusoidal Current

The definition of instantaneous power as

$$p=vi \tag{4.6}$$

holds for currents and voltages of any waveform.

By definition, the active power of any periodic current is equal to the instantaneous power averaged over a period

$$P=\frac{1}{T}\int_0^T p\,dt=\frac{1}{T}\int_0^T vi\,dt \tag{4.7}$$

If we substitute the expansions (4.2) and (4.1) for v and i in Eq. (4.6), the power will be represented as a sum of the same four types of integrals as in the case of the rms value of a periodic nonsinusoidal current

$$(1)\quad \frac{1}{T}\int_0^T V_{km}\sin(k\omega t+\psi_{vk})\,I_{km}\sin(k\omega t+\psi_{ik})\,dt = V_k I_k\cos\varphi_k$$

where

$$\varphi_k = \psi_{vk} - \psi_{ik}$$

(for the evaluation of the integral see Sec. 2.14);

$$(2)\quad \frac{1}{T}\int_0^T V_0 I_0\,dt = V_0 I_0$$

$$(3)\quad \frac{1}{T}\int_0^T V_0 I_{km}\sin(k\omega t+\psi_{ik})\,dt = 0$$

$$\frac{1}{T}\int_0^T I_0 V_{km}\sin(k\omega t+\psi_{vk})\,dt = 0$$

$$(4)\quad \frac{1}{T}\int_0^T V_{km}\sin(k\omega t+\psi_{vk})\,I_{lm}\sin(l\omega t+\psi_{il})\,dt = 0$$

for $k \neq l$.

Thus,

$$P = V_0 I_0 + \sum_{k=1}^{n} V_k I_k\cos\varphi_k \tag{4.8}$$

That is, *the active power of a periodic nonsinusoidal current is equal to the sum of the active powers of all the harmonics and of the d.c. components of voltage and current (the d.c. power).*

The reactive power of a periodic nonsinusoidal current is assumed to be defined as

$$Q = \sum_{k=1}^{n} V_k I_k\sin\varphi_k \tag{4.9}$$

The apparent power of a periodic nonsinusoidal current is taken to be

$$S = VI > \sqrt{P^2+Q^2} \tag{4.10}$$

4.4 Electrical Wave Filters

In a circuit carrying a periodic nonsinusoidal current, the reactance $k\omega L$ of the inductors and the reactance $1/k\omega C$ of the capacitors are different for different harmonics, and vary with the harmonic order k. Therefore, the voltage and current waveforms associated with this circuit can be varied at will through an appropriate choice of circuit parameters, L and C.

To prove the statement, we will consider two simple examples. Referring to Fig. 4.3, let the periodic nonsinusoidal voltage impressed on the inductive element have the rms value given by

$$V = \sqrt{V_1^2 + V_3^2 + V_5^2 + \ldots + V_n^2} \tag{4.11}$$

By Ohm's law, the rms value of the kth harmonic current is

$$I_k = V_k/k\omega L$$

In consequence, the rms current by Eq. (4.4) is

$$I = \sqrt{I_1^2 + I_3^2 + I_5^2 + \ldots + I_n^2}$$

$$= \sqrt{\left(\frac{V_1}{\omega L}\right)^2 + \left(\frac{V_3}{3\omega L}\right)^2 + \left(\frac{V_5}{5\omega L}\right)^2 + \ldots + \left(\frac{V_n}{n\omega L}\right)^2}$$

$$= (1/\omega L)\sqrt{\frac{V_1^2}{1} + \frac{V_3^2}{9} + \frac{V_5^2}{25} + \ldots + \frac{V_n^2}{n^2}}$$

From comparison of the expression with that for the rms value of voltage, (4.11), it is seen that the higher the order of the harmonic

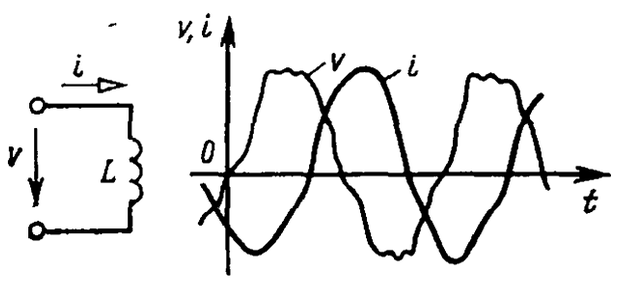


FIGURE 4.3

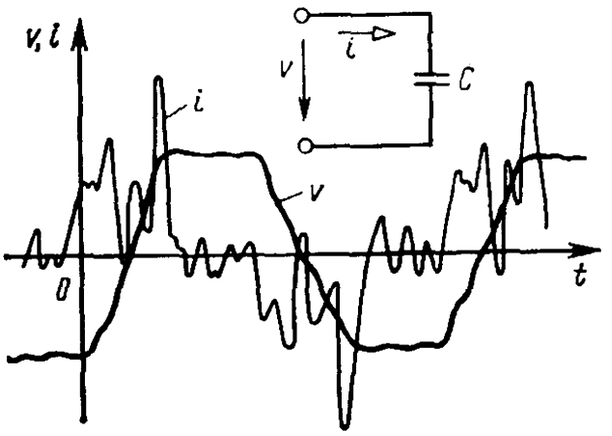


FIGURE 4.4

of the periodic nonsinusoidal current in an inductive element, the more it is suppressed. Because of this, the current waveform may differ from a sine wave substantially less than the voltage waveform (see Fig. 4.3).

Given the same rms value of voltage, Eq. (4.11), the rms value of the kth harmonic current in a capacitive element (Fig. 4.4) will be

$$I_k = V_k/(1/k\omega C) = V_k k\omega C$$

Therefore, the rms value of current, in accord with Eq. (4.4), is

$$I = \sqrt{I_1^2 + I_3^2 + I_5^2 + \ldots + I_n^2}$$
$$= \omega C \sqrt{V_1^2 + 9V_3^2 + 25V_5^2 + \ldots + n^2 V_n^2} \qquad (4.12)$$

On comparing Eqs. (4.12) and (4.11), it can be seen that the current waveform has a larger harmonic content than the voltage waveform. For this reason, the current waveform may differ from a pure sinewave more than the voltage waveform (see Fig. 4.4).

Failure to consider the nonsinusoidal waveform of voltage and current when *measuring* the parameters of capacitive and inductive elements may result in a substantial error. Suppose that, when measuring the capacitance of a capacitor, the voltage existing between its terminals is periodic, nonsinusoidal. This means that its rms value V is given by Eq. (4.11). If we assume that the voltage across and the current in the capacitor are sinusoidal, the capacitance should be deduced from the relation

$$I = V/(1/\omega C_{\text{meas}})$$

Hence the measured capacitance would be

$$C_{\text{meas}} = I/\omega V = \frac{I}{\omega \sqrt{V_1^2 + V_3^2 + V_5^2 + \ldots + V_n^2}}$$

Actually, the capacitance must be found on the basis of Eq. (4.12). Then,

$$C = \frac{I}{\omega \sqrt{V_1^2 + 9V_3^2 + 25V_5^2 + \ldots + n^2 V_n^2}}$$

On comparing the two values, it is seen that $C_{\text{meas}} > C$.

Acting similarly, we will find that $L_{\text{meas}} > L$.

Since the harmonic components of current have a greater relative value in a capacitive element and a smaller relative value in an inductive element, then, given the same voltage waveform, the error in the measured capacitance may be greater than in the measured inductance. (For example, if the error in the measured value of L is 3%, the error in the measured value of C will be 25%.)

The frequency dependence of inductance and capacitance and of the respective reactances is utilized in electrical wave filters, or simply filters. An electrical wave filter is a transducer which separates waves on the basis of their frequency. It attenuates waves at one or several frequencies and passes all others unattenuated.

An elementary tuned filter of load resistance r_{load} is shown in Fig. 4.5. It is assumed that the losses in the filter elements may be neglected. The filter parameters are such that $k\omega C = 1/k\omega L$ which means the condition of parallel resonance at the kth harmonic.

This implies that the filter impedance to the kth harmonic of a periodic nonsinusoidal currents is $z_k = \infty$. For all the other harmonics the filter impedance may be very small. This circuit is a simple band-stop filter, that is, one which attenuates the current harmonics at and near the resonance frequency.

Figure 4.6 shows another form of tuned filter. Its circuit parameters are such that $k\omega L = 1/k\omega C$, which means the condition for

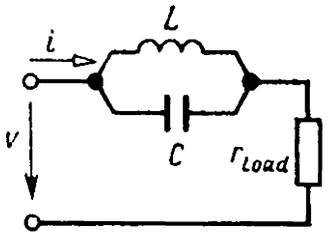


FIGURE 4.5

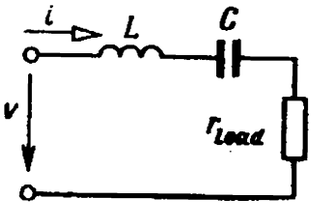


FIGURE 4.6

series resonance at the kth harmonic. This implies that the filter impedance at the kth harmonic is $z_k = 0$. At all the other harmonics the impedance may be very high. This circuit is a simple band-pass filter, that is, one which attenuates components at all frequencies except those at and in the neighbourhood of the resonance frequency.

5 Transients in Linear Electric Circuits

5.1 The Classical Method of Transient Analysis

Transients in an electric circuit occur whenever its steady state is changed. The cause of this change may be the operation of a switch, an open circuit, a short circuit, etc.

As in steady-state circuit analysis, transient analysis requires that all the circuit components be represented by their models or equivalent circuits made up of ideal resistive, capacitive and inductive elements, ideal voltage and current sources, and switches.

Physically, the occurrence of transients can be traced back to the presence of inductances and capacitances in the affected circuit. The point is that the energy stored in the magnetic field, Eq. (2.5), and in the electric field, Eq. (2.13), due to the inductive and capacitive elements cannot change instantaneously when it is switched.

The transient state of a circuit is described by differential equations, generally inhomogeneous (if the circuit contains voltage and current sources) or homogeneous (if the circuit is source-free). In the case of a linear circuit, the transient state is described by linear differential equations, and in the case of a nonlinear circuit, by nonlinear differential equations.

Consider the transient analysis of constant-parameter linear circuits which are described by linear differential equations with constant coefficients. Several methods have been developed to solve such equations and to analyse the transient response of the circuits involved. We will limit ourselves to the *classical method* which is physically instructive and is far simpler to apply to simple circuits than other forms of analysis (such as the Laplace transform, Fourier analysis, etc.).

The transient analysis of a circuit by the classical method involves several steps.

(1) To begin with, a set of differential equations is written for the circuit in question. By elimination of variables, they are reduced to a single equation for the sought current i or voltage v. For simple circuits this procedure yields a first- or second-order differential equation in which the unknown is either the current in an inductive element or the voltage across a capacitive element.

(2) The next step is to write the general solution of the differential equation thus obtained. It is the sum of a particular solution (or integral) of a nonhomogeneous differential equation and the general

solution (or the complementary function) of the corresponding homogeneous equation.

As applied to electric circuits, the particular solution is the *forced* (or *steady-state*) *response* of the circuit (if it exists), that is, direct currents and voltages if the circuit contains direct-current and direct-voltage sources, or sinusoidal voltages and currents if the circuit contains sinusoidal voltage and current sources. The respective quantities are called *steady-state currents* and *steady-state voltages*, and their symbols are i_{ss} and v_{ss}. The complementary function describes the *source-free* or *transient response* of the circuit, and the corresponding quantities are termed *transient*, their symbols being i_t and v_t. The expressions for transient quantities must contain the constants of integration, of which there must be as many as is the order of the homogeneous equation.

The free, or transient, response is due to the difference between the energy stored by the capacitive and inductive elements of the circuit at the instant immediately preceding the change in the circuit steady state and the energy stored by the same elements when the circuit has reached a new steady state. The energy stored in these elements cannot change all at once; instead the change is gradual. Hence, the existence of transients in the circuit.

(3) As the last step, the constants of integration for the general solution, $i = i_{ss} + i_t$ and $v = v_{ss} + v_t$, are found. The integration constants are determined from the *initial conditions*, that is, the conditions existing in the circuit immediately *after* it is disturbed. We will assume that the switch performing the change in the state of the circuit is ideal which implies that the switch operates instantaneously at time t. In the circumstances, the current in an inductive element and the voltage across a capacitive element immediately after switching (t_+) are the same as they are immediately before switching (t_-). These conditions are derived from what are known as *switching* or *change-of-circuit rules.*

5.2 The Switching or Change-of-Circuit Rules

By these rules, *the current in an inductive element and the voltage across a capacitive element cannot change abruptly.*

We will first prove the first rule applicable to inductive elements. Suppose that during the time interval from t_1 to t_2 the current in the inductive element changes from $i_L(t_1)$ to $i_L(t_2)$. The mean time rate of change of the energy stored in the magnetic field set up by the inductive element, Eq. (2.5), will be

$$\Delta W_M/\Delta t = (L/2)\,\frac{i_L^2(t_2) - i_L^2(t_1)}{t_2 - t_1}$$

If the time interval $\Delta t = t_2 - t_1$ tends to zero and $i_L(t_2) \neq i_L(t_1)$, then the mean time rate of change of the energy stored in the magnetic field tends to infinity.

Since circuits with an infinitely large power are nonexistent, the current in an inductive element cannot change abruptly. This is the switching or change-of-circuit rule for an inductive element; it may be written as

$$i_L(t_-) = i_L(t_+) \tag{5.1}$$

where t is the time when a change of state (a switching operation) takes place in the circuit.

The switching or change-of-circuit rule for capacitive elements can readily be deduced by analogy with the previous case. On comparing the expression for the energy stored in the magnetic field of an inductive element, $W_M = Li_L^2/2$, with that for the energy stored in the electric field of a capacitive element, $W_E = Cv_C^2/2$, Eq. (2.13), we can see that the only difference is that the former is written in terms of the current i_L and the latter in terms of the voltage v_C. Therefore, the analysis of energy relations in a capacitive element should lead to the same conclusion: The voltage across a capacitive element cannot change abruptly, or mathematically

$$v_C(t_-) = v_C(t_+) \tag{5.2}$$

where t is the time when a change of state (a switching operation) takes place in the circuit.

The current in the inductive element of a circuit, $i_L(t_-)$, and the voltage across its capacitive element, $v_C(t_-)$, that exist immediately before the state of the circuit is changed are called *initial conditions*.

If at time t_- the currents in the inductive elements of a circuit and the voltages across its capacitive elements are zero, $i_L(t_-) = 0$ and $v_C(t_-) = 0$, we have *zero initial conditions*. Otherwise, they are called *nonzero initial conditions*.

5.3 Transients in a D.C. Circuit with Only One Inductive Element

Response of a series *RL* circuit to a d.c. voltage excitation. As soon as the switch *Sw* (Fig. 5.1 a) is closed at time $t = 0$, a current i begins to flow around the circuit, and voltages appear across the resistive element, $v_r = ri$, Eq. (1.1), and across the inductive element, $v_L = L\,di/dt$, Eq. (2.3).

Using Kirchhoff's voltage law, Eq. (2.40), we write a differential equation for the circuit

$$v_L + v_r = L\,di/dt + ri = E \tag{5.3}$$

which is recognized as a first-order nonhomogeneous differential equation.

The complete solution to Eq. (5.3) is the sum of two solutions, one being the steady-state (or forced) solution and the other being the source-free (or transient) solution

$$i = i_{ss} + i_t \tag{5.4}$$

The solution holds for any instant after closure of the switch, that is, beginning at time $t = 0_+$.

The *steady-state term* in Eq. (5.4) is the direct current existing in the circuit after all transients have died out (which, theoretically, goes on for an indefinitely long time)

$$i_{ss} = E/r \tag{5.5}$$

By direct substitution it is an easy matter to verify that the above particular solution satisfies the nonhomogeneous differential equation (5.3).

The second term on the right-hand side of Eq. (5.4) is the general solution of the homogeneous differential equation of the circuit

$$L\, di_t/dt + ri_t = 0 \tag{5.6}$$

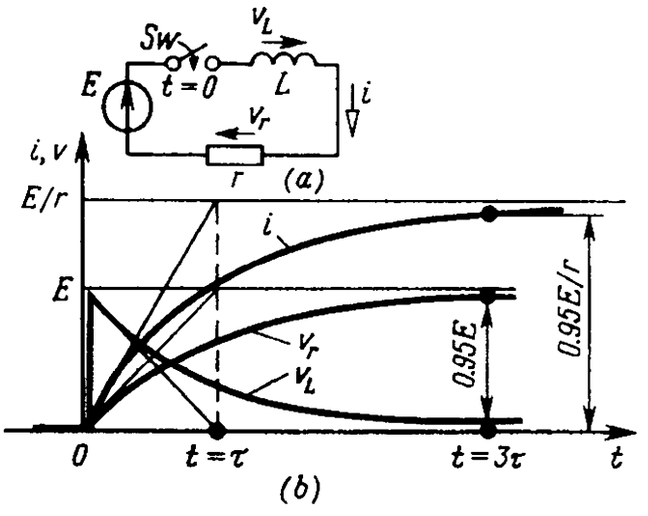


FIGURE 5.1

and is called the *free* solution because it is independent of the voltage and current sources in the circuit

$$i_t = A \exp(pt) \tag{5.7}$$

where $p = -r/L$ is the root of the characteristic equation

$$Lp + r = 0 \tag{5.8}$$

Thus, subject to Eqs. (5.5) and (5.7), the complete solution (5.4) of the nonhomogeneous differential equation (5.3) has the form

$$i = E/r + A \exp(-rt/L) \tag{5.9}$$

It remains to find the integration constant A. To this end, we refer to the switching rule for an inductive element, Eq. (5.1), at the time when the switch is closed, $t = 0$. Since the current in an inductive element cannot change abruptly, and prior to switching, that is at $t = 0_-$, it was zero, it follows that

$$i(0_-) = 0 = i(0_+) = E/r + A$$

Hence

$$A = -E/r \tag{5.10}$$

On substituting Eq. (5.10) for A in (5.9), we obtain an expression describing the rise of current in the circuit (see Fig. 5.1 b)

$$i = (E/r)\,[1 - \exp(-t/\tau)] \tag{5.11}$$

where $\tau = L/r$ has the dimension of time (H/Ω = s) and is called the *time constant* of the circuit in Fig. 5.1 a. The time constant defines the rate of current rise and is equal to the time during which the current i could reach its steady-state value, $i_{ss} = E/r$, if its rate of change remained constant and equal to the initial rate of change $\mathrm{d}i/\mathrm{d}t\,|_{t=0_+} = E/L$.

It may be assumed that all transients practically die out after a time equal to 3τ, reckoning from the instant when the switch is closed. At the end of that time interval the current will be $i\,(3\tau) = 0.95E/r$. In other words, if closure of the switch takes place at $t = 0$, this interval is $t = 3\tau$.

Now that we have found the time dependence of current, Eq. (5.11), it is an easy matter to determine the time dependence of the voltage across the resistive and inductive elements (see Fig. 5.1 b)

$$v_r = ri = E\,[1 - \exp(-t/\tau)]$$

$$v_L = L\,\mathrm{d}i/\mathrm{d}t = E \exp(-t/\tau)$$

For $0 \leqslant t < \tau$, the rate of current change may be taken approximately constant and equal to $\mathrm{d}i/\mathrm{d}t\,|_{t=0_+} = E/L$. Therefore, during the time interval defined, the voltage across the resistive element is approximately equal to

$$v_r \approx rEt/L = (r/L)\int_0^t E\,\mathrm{d}t$$

that is, directly proportional to the integral of the source emf E. This is what is usually called an *integrating network*.

When the excitation is a varying emf e, it may so happen that sometimes during the transient process $v_r \gg v_L$. For such time intervals the current in the circuit is $i \approx e/r$, and the voltage across the inductive element

$$v_L = L\,\mathrm{d}i/\mathrm{d}t \approx (L/r)\,\mathrm{d}e/\mathrm{d}t$$

is approximately proportional to the rate of change of the source emf e. Accordingly, this is what is usually called a *differentiating network*.

The source-free response of an *RL* circuit upon closure of the switch. This situation arises in the windings of electric machines

and apparatus. Since in addition to inductance, the inductor of the circuit also has a resistance r, the equivalent circuit takes the form shown in Fig. 5.2 a.

The differential equation describing the transient process in the circuit upon switch closure has the form

$$v_L + v_r = L\,di/dt + ri = 0 \tag{5.12}$$

Since Eq. (5.12) is a homogeneous differential equation (it is the same as Eq. (5.6)), its complete solution contains only a free component, Eq. (5.7)

$$i = i_t = A \exp(-t/\tau) \tag{5.13}$$

where $\tau = L/r$ is the time constant of the circuit.

It remains to find the constant A. As before we refer to the switching rule for an inductive element, Eq. (5.1). Since prior to switch closure, that is at $t = 0_-$, the inductor carried a direct current equal to $E/(r + R)$, it follows that

$$i(0_-) = E/(r + R) = i(0_+) = A$$

On substituting the above expression for A in (5.13), we obtain the current in the inductor

$$i = E/(r + R) \exp(-t/\tau) \tag{5.14}$$

Upon switch closure the current in the inductor is maintained by the energy stored in its magnetic field.

Now we can determine the time dependences of the voltages across the resistive and inductive elements (see Fig. 5.2 b)

$$v_r = ri = rE/(r + R) \exp(-t/\tau)$$

$$v_L = L\,di/dt = -rE/(r + R) \exp(-t/\tau)$$

The response of an *RL* circuit following an open-circuit. When a simple series circuit containing an inductor is opened, an arc strikes between the parting contacts. This situation occurs, for example, at the trolleys of electric-traction vehicles. Arcing can be avoided by placing a resistor in parallel with the arc-containing section of the circuit.

An applicable equivalent circuit is shown in Fig. 5.3 a. The inductor is replaced with a series combination of an inductance L and a resistance r, and the switch is replaced with a parallel combination of an ideal switch and a resistance R.

We may now apply Kirchhoff's second law and write the following differential equation for the network of Fig. 5.3 a after the switch is opened:

$$v_L + v_r + v_R = L\,di/dt + (r + R)\,i = E \tag{5.15}$$

This differential equation is the same (to within the element symbols) as Eq. (5.3). In consequence, its complete solution is the same as that for Eq. (5.3), that is, similar to Eq. (5.9)

$$i = i_{ss} + i_t = E/(r + R) + A \exp [-t (r + R)/L] \tag{5.16}$$

where $i_{ss} = E/(r + R)$ is the steady-state component or the direct current flowing in the circuit after the switch is opened.

To determine the integration constant A in Eq. (5.16), we recall the switching rule for an inductive element, Eq. (5.1). Before the

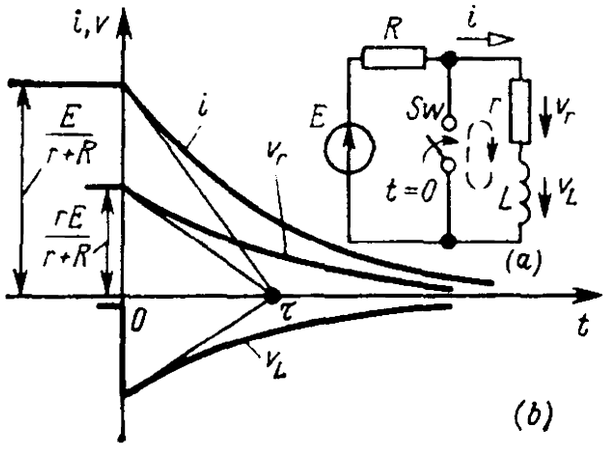


FIGURE 5.2

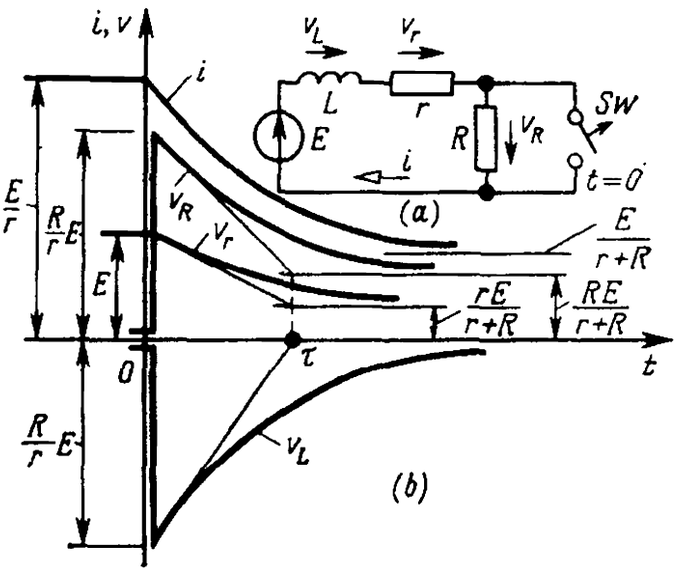


FIGURE 5.3

switch was opened, that is, at $t = 0_-$, the inductor carried the direct current E/r. Therefore, by the switching rule,

$$i\,(0_-) = E/r = i\,(0_+) = E/(r + R) + A$$

Hence,

$$A = E/r - E/(r + R) = RE/r\,(r + R)$$

On substituting the above expression for A in Eq. (5.16), we find the current in the inductor after the switch is opened (see Fig. 5.3 b).

$$i = E/(r + R)\,[1 + (R/r) \exp\,(-t/\tau)] \tag{5.17}$$

where $\tau = L/(r + R)$ is the time constant of the circuit.

Now that we know how the current varies in the circuit, Eq. (5.17), we can readily determine the time dependence of the voltages across the resistive and inductive elements (see Fig. 5.3 b)

$$v_r = ri = [rE/(r + R)]\,[1 + (R/r) \exp\,(-t/\tau)]$$

$$v_R = Ri = [RE/(r + R)]\,[1 + (R/r) \exp\,(-t/\tau)]$$

$$v_L = L\,di/dt = -(R/r)\,E \exp\,(-t/\tau)$$

Just as the switch opens ($t = 0_+$) the voltage across the resistive element R abruptly rises from zero, $v_R\ (0_-) = 0$, to $v_R\ (0_+) = ER/r$. That is why for $R \gg r$, a substantial voltage is produced between the contact tips, and it may give rise to an arc discharge.

5.4 Transients in a Direct-Current *RC* Circuit

The forced response of an *RC* network. As an example we will consider the network in Fig. 5.4 a. Initially the capacitor is uncharged and the switch Sw is open. At $t = 0$ the switch is closed, and a d.c. voltage source is inserted in the circuit. This causes the capacitor to charge.

Recalling that for a resistive element $v_r = ri$, Eq. (1.1), and for a capacitive element $i = Cdv_C/dt$, Eq. (2.11), we can use Kirchhoff's voltage law and write the following differential equation for the circuit after switch closure at time $t = 0$:

$$v_r + v_C = ri + v_C$$
$$= rC\, dv_C/dt + v_C = E \qquad (5.18)$$

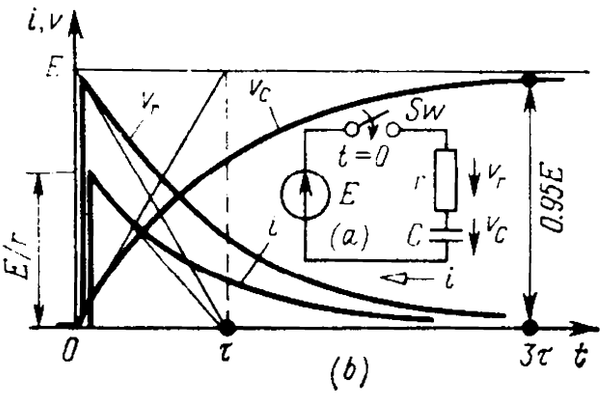


FIGURE 5.4

The complete solution of the nonhomogeneous differential equation (5.18) for the voltage across the capacitive element is the sum of the steady-state solution and the transient solution

$$v_C = v_{C,ss} + v_{C,t}$$

The steady-state term is

$$v_{C,ss} = E \qquad (5.19)$$

because the capacitive element will cease charging when v_C has reached the value of the source voltage.

Since Eq. (5.18) is a first-order differential equation, its transient (or free) term is

$$V_{C,\,t} = A \exp(pt) \qquad (5.20)$$

where $p = -1/rC$ is the root of the characteristic equation

$$rCp + 1 = 0$$

Thus, the complete solution is

$$v_C = v_{C,ss} + v_{C,t} = E + A \exp(-t/rC) \qquad (5.21)$$

In order to determine the constant A in Eq. (5.21), we invoke the switching rule for a capacitive element, Eq. (5.2). We have

assumed that before the switch was closed, that is, at $t = 0_-$, the capacitive element was uncharged. Therefore,

$$v_C(0_-) = 0 = v_C(0_+) = E + A$$

Hence, $A = -E$.

On substituting this for A in Eq. (5.21), we find the voltage across the capacitive element while it is charging (see Fig. 5.4 b)

$$v_C = E[1 - \exp(-t/\tau)] \tag{5.22}$$

where $\tau = rC$ has the dimension of time ($\Omega \times F = \Omega \times A \times s \times \times V^{-1} = s$) and is called the *time constant* of the circuit. As with the network of Fig. 5.1, it is a measure of the rate at which the voltage tends to its final value.

Now that we know the time dependence for the voltage across the capacitive element, Eq. (5.22), it is easy to establish the time dependence for the charging current and for the voltage across the resistive element (see Fig. 5.4 b)

$$i = C\,dv_C/dt = (E/r)\exp(-t/\tau)$$

$$v_r = ri = E\exp(-t/\tau)$$

It is to be noted that just as the switch is closed, or at $t = 0_+$, the current in the circuit is

$$i(0_+) = E/r$$

and the capacitive element is acting as a short circuit. Therefore at low values of r a substantial current surge may occur in the circuit.

For $0 \leqslant t < \tau$, the rate of change for the voltage across the capacitive element may approximately be taken as constant and equal to

$$dv_C/dt\,|_{t=0_+} = E/rC$$

and the voltage across the capacitive element as

$$v_C \approx (E/rC)\,t = \frac{1}{rC}\int_0^t E\,dt$$

that is, proportional to the integral of the source voltage E.

If an RC network is excited by a varying emf e, it may so happen that at some instants during the transient process when $v_r \ll v_C$, the voltage across the capacitive element will be approximately equal to the source emf, $v_C \approx e$, and the voltage across the resistive element will be proportional to the rate of change of the source voltage

$$v_r = ri = rC\,dv_C/dt \approx rC\,de/dt$$

Thus, like the series *RL* circuit examined earlier, a series *RC* network may, given certain conditions, act as an integrator or as a differentiator.

In most cases, the capacitive element of an *RC* network may be assumed to be fully charged in a time interval equal to 3τ. This time interval may be rather long (the value of τ increases with the values of r and C), and this property is utilized in time relays which are devices used to open or close a circuit (or circuits) at one or more predetermined times.

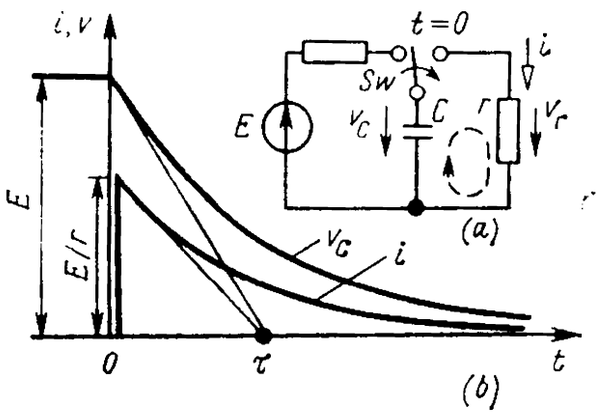


FIGURE 5.5

The discharge of a capacitor through a resistor. The electric field of a charged capacitive element stores an amount of energy, Eq. (2.13), owing to which the element can act as an energy source for some time.

As an illustration we will analyse the circuit of Fig. 5.5 a which contains a capacitive element charged to $v_C = E$. Just as the switch *Sw* is closed and the capacitive element is placed across the resistive element of resistance r, a current begins to flow around the circuit due to the change of the charge q on the capacitive element, Eq. (2.11)

$$i = -\mathrm{d}q/\mathrm{d}t = -C\,\mathrm{d}v_C/\mathrm{d}t \tag{5.23}$$

where the "—" sign indicates that the current i is the discharge current around the loop labelled by a dashed arrow in the figure; its direction is opposite to that of the voltage across the capacitor.

Noting Eq. (5.23), Kirchhoff's voltage law gives the following differential equation for the loop labelled in Fig. 5.5 a:

$$v_r - v_C = ri - v_C = rC\,\mathrm{d}v_C/\mathrm{d}t + v_C = 0 \tag{5.24}$$

Since the discharge circuit of the capacitor contains no voltage source, Eq. (5.24) is a homogeneous differential equation, and its complete solution consists solely of a free (transient) term

$$v_C = v_{C,t} = A\,\exp\,(-t/rC) \tag{5.25}$$

In order to determine the constant of integration A, we invoke the switching rule for a capacitive element, Eq. (5.2). Since before the switch was closed, that is, at $t = 0_-$, the capacitor was charged to the source voltage, it follows that

$$v_C\,(0_-) = E = v_C\,(0_+) = A$$

On substituting for A in Eq. (5.25), we derive the relation defining the manner in which the voltage across the capacitor varies on discharge (see Fig. 5.5 b)

$$v_C = E \exp(-t/\tau)$$

where $\tau = rC$ is the time constant of the circuit.

The discharge current is given by Eq. (5.23)

$$i = -C\, dv_C/dt = (E/r) \exp(-t/\tau)$$

The current rises stepwise from zero to $i(0_+) = E/r$, then decreases exponentially (see Fig. 5.5 b).

5.5 Discharge of a Capacitive Element in an *RLC* Circuit

The discharge of a capacitive element through a series combination of an inductive and a resistive element is an important process in, say, pulse generators in which a capacitor acts as an energy source.

As an illustration we will consider the circuit of Fig. 5.6. Suppose that initially, when the switch Sw is in position 1, the capacitive element C is charged by a d.c. source to a voltage equal to E. Then the switch is thrown into position 2, and the capacitor is connected to a series combination of an inductive element L and a resistive element r (in fact, this combination is an equivalent-circuit model of an inductor). Now the capacitive element begins to discharge (the discharge current is i), so its charge q and its voltage v_C diminish. In the process, the energy stored in the electric field of the capacitive element is converted to the energy stored in the magnetic field of the inductive element and is partly dissipated in the resistive element. This process may be aperiodic (damped) or oscillatory, depending on the choice of the element values (see Sec. 2.21).

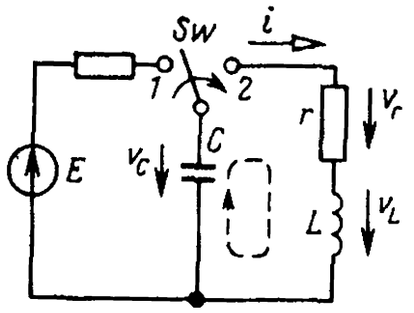


FIGURE 5.6

For the loop labelled with a dashed arrow in the figure, Kirchhoff's voltage law establishes

$$-v_C + v_r + v_L = -v_C + ri + L\, di/dt = 0 \qquad (5.26)$$

The positive direction of the current in the capacitive element is opposite to that for the voltage across that element, because i is a discharge current. Therefore for the circuit of Fig. 5.5 a

$$i = -C\, dv_C/dt \qquad (5.27)$$

On substituting (5.27) into (5.26), we obtain a second-order homogeneous differential equation

$$LC\,\mathrm{d}^2v_C/\mathrm{d}t^2 + rC\,\mathrm{d}v_C/\mathrm{d}t + v_C = 0 \tag{5.28}$$

for which the characteristic equation is

$$LCp^2 + rCp + 1 = 0 \tag{5.29}$$

The complete solution of Eq. (5.28) contains only one free (or transient) term

$$v_C = v_{C,t} = A_1 \exp(p_1 t) + A_2 \exp(p_2 t) \tag{5.30}$$

where

$$p_{1,2} = -r/2L \pm \sqrt{r^2/4L^2 - 1/LC}$$

are the roots of the characteristic equation (5.29).

For $r^2/4L^2 > 1/LC$, both roots of the characteristic equation are real, and the capacitive element discharges *aperiodically* (the *overdamped case*). For $r^2/4L^2 < 1/LC$, the roots are complex conjugate, and the capacitor produces an *oscillatory discharge* (the *underdamped case*).

Consider the case when the characteristic equation has complex conjugate roots

$$p_{1,2} = -\delta \pm j\omega_0 \tag{5.31}$$

where $\delta = r/2L$ is the *damping constant* and

$$\omega_0 = \sqrt{1/LC - \delta^2}$$

is the *natural angular frequency* of the oscillatory response.

On substituting the complex values of the roots in Eq. (5.30), we obtain the following time dependences for the voltage across the capacitive element and, by Eq. (5.27), for the discharge current in the oscillatory (underdamped) case

$$v_C = \exp(-\delta t)\,[A_1 \exp(j\omega_0 t) + A_2 \exp(-j\omega_0 t)] \tag{5.32a}$$

$$i = -C\,\mathrm{d}v_C/\mathrm{d}t = -C\exp(-\delta t)\{-\delta\,[A_1 \exp(j\omega_0 t) + A_2 \exp(-j\omega_0 t)] + j\omega_0\,[A_1 \exp(j\omega_0 t) - A_2 \exp(-j\omega_0 t)]\} \tag{5.32b}$$

As in all the previous cases, the constants of integration A_1 and A_2 can be evaluated by reference to the switching rules for an inductive element, Eq. (5.1), and a capacitive element, Eq. (5.2). Prior to and at $t = 0_-$, that is, immediately before operation of the switch, the voltage across the capacitive element was equal to the

source emf E, and no current was flowing in the inductive element. Therefore

$$v_C(0_-) = E = v_C(0_+) = A_1 + A_2$$

$$i(0_-) = 0 = i(0_+) = C[\delta(A_1 + A_2) - j\omega_0(A_1 - A_2)]$$

Hence

$$A_1 = E(\delta + j\omega_0)/2j\omega_0$$

and

$$A_2 = E(j\omega_0 - \delta)/2j\omega_0$$

Recalling that by Euler's formula, Eq. (2.25),

$$\exp(\pm j\omega_0 t) = \cos\omega_0 t \pm j\sin\omega_0 t$$

and substituting the above expressions in Eq. (5.32a), we obtain the time dependence of the voltage across the capacitive element in the form

$$v_C = (E/\omega_0)\exp(-\delta t)(\omega_0\cos\omega_0 t + \delta\sin\omega_0 t) \tag{5.33}$$

The sum of a cosine and a sine may be replaced with a single sine. To this end, we set $\omega_0/\delta = \tan\psi$, that is, we take ω_0 and δ as the sides of a right triangle (Fig. 5.7) whose hypotenuse is

$$\sqrt{\omega_0^2 + \delta^2} = \sqrt{1/LC - \delta^2 + \delta^2} = 1/\sqrt{LC}$$

On dividing and multiplying Eq. (5.33) by $1/\sqrt{LC}$, we get

$$v_C = \frac{E}{\omega_0\sqrt{LC}}\exp(-\delta t)\sin(\omega_0 t + \psi) \tag{5.34}$$

and, by Eq. (5.27), the discharge current is

$$i = -C\,dv_C/dt = (E/\omega_0 L)\exp(-\delta t)\sin\omega_0 t \tag{5.35}$$

Equations (5.34) and (5.35) show that the voltage across the capacitive element and the discharge current may be regarded as time-varying sinewaves with their amplitudes decreasing exponentially with a time constant $\tau = 1/\delta = 2L/r$.

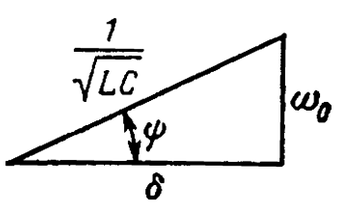


FIGURE 5.7

If we wish to construct the respective plots, we should first draw two auxiliary exponential functions, one for voltage, $\pm(E/\omega_0\sqrt{LC})\times\exp(-\delta t)$, and the other for current, $\pm(E/\omega_0 L)\exp(-\delta t)$ (Fig. 5.8). The voltage and current waveforms (see Fig. 5.8) must fall within the limits indicated by the two auxiliary exponential functions. The dotted sinusoid shown in the figure serves to find singular points on the

waveform for the voltage across the capacitive elements, such as $v_C\,(0) = E$ and $v_C\,(t) = 0$.

Now we will consider the case when the roots $p_{1,2}$ of the characteristic equation (5.29) are real negative.

If the expression $r^2/4L^2$ is greater than $1/LC$, then the roots are negative, real, and unequal such that $p_2 < p_1 < 0$. To evaluate the

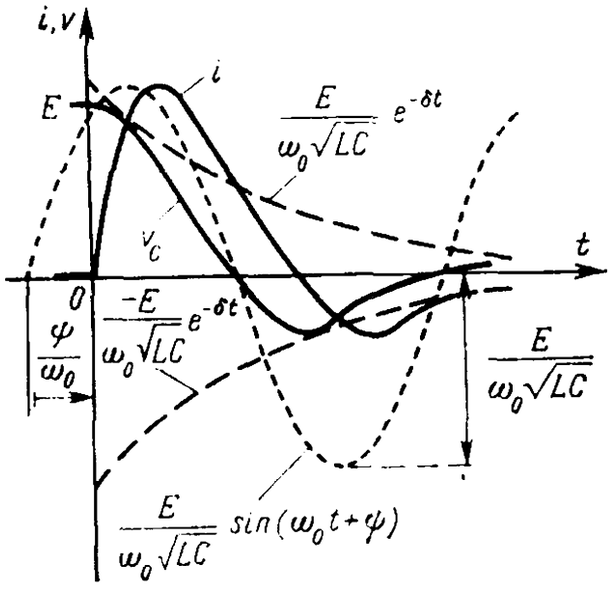


FIGURE 5.8

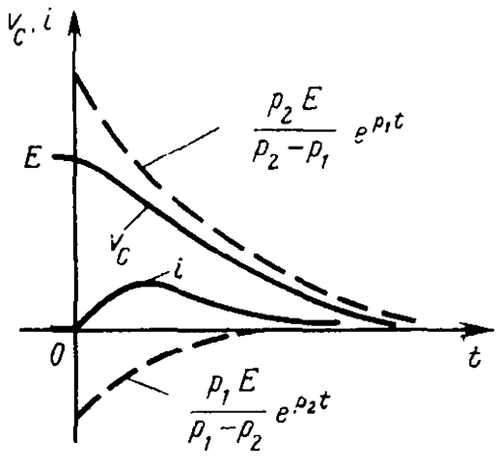


FIGURE 5.9

constants A_1 and A_2 in the complete solution (5.30), we will use, as we have done so far, the switching rules for a capacitive element and an inductive element

$$v_C\,(0_-) = E = v_C\,(0_+) = A_1 + A_2$$

$$i\,(0_-) = 0 = i\,(0_+) = C\,\mathrm{d}v_C/\mathrm{d}t\,|_{t=0_+} = C\,(p_1A_1 + p_2A_2)$$

Hence

$$A_1 = \frac{p_2E}{p_2 - p_1} > 0$$

and

$$A_2 = \frac{p_1E}{p_1 - p_2} < 0$$

On substituting the above expressions for the constants of integration in Eq. (5.30), we obtain the following time dependences for the voltage across the capacitive element

$$v_C = \frac{p_2E}{p_2 - p_1}\exp(p_1t) + \frac{p_1E}{p_1 - p_2}\exp(p_2t)$$

and for the discharge current

$$i = -C\,\mathrm{d}v_C/\mathrm{d}t = -\frac{p_1p_2EC}{p_2 - p_1}[\exp(p_1t) - \exp(p_2t)]$$

The time waveforms of voltage and current are shown in Fig. 5.9 which also includes auxiliary exponential functions plotted as dashed lines. Throughout the transient process, the voltage and current do not change sign which is another way of saying that the response is aperiodic (the overdamped case).

The limiting case of an aperiodic response is the critically damped case when $r^2/4L^2 = 1/LC$, and the characteristic equation has two roots which are negative, real, and equal: $p_1 = p_2 = p = -r/2L$. In the case of equal (or multiple) roots, the complete solution of Eq. (5.28) differs from that of Eq. (5.30) and has the form

$$v_C = (A_1 + A_2 t) \exp(pt)$$

where the constants A_1 and A_2 are evaluated on the basis of the switching rules.

The voltage across and the current in the capacitive element in the critically damped case are

$$v_C = E\,[1 + (r/2L)\,t] \exp[-(r/2L)\,t]$$

$$i = (E/L)\,t \exp[-(r/2L)\,t]$$

5.6 Response of a Series *RLC* Circuit to a D.C. Voltage Excitation

A series *RLC* circuit containing a d.c. voltage source (Fig. 5.10 a) is described by a nonhomogeneous differential equation. The complete solution of such an equation is the sum of a particular integral and the complementary function or, which is the same, the respon-

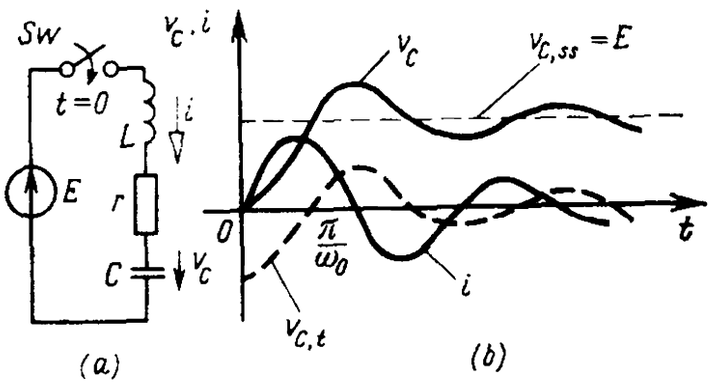


FIGURE 5.10

se of the circuit is the sum of a steady-state (or forced) solution and a transient (or free) solution. Taking the case of the voltage across the capacitive element of the circuit, we have

$$v_C = v_{C,ss} + v_{C,t}$$

where the steady-state term is $v_{C,ss} = E$ and the transient term is the same as defined in Eq. (5.30). Thus, the complete solution is

$$v_C = E + A_1 \exp(p_1 t) + A_2 \exp(p_2 t) \tag{5.36a}$$

The charging current is

$$i = C\, dv_C/dt = p_1 C A_1 \exp(p_1 t) + p_2 C A_2 \exp(p_2 t) \tag{5.36b}$$

The switch is closed at $t = 0$ and at that time the voltage across the capacitive element and the current in the circuit are zero. Therefore, by the switching rules, we can write two equations from which to evaluate the constants A_1 and A_2

$$v_C(0_-) = 0 = v_C(0_+) = E + A_1 + A_2$$

$$i(0_-) = 0 = i(0_+) = p_1 A_1 + p_2 A_2$$

Hence $A_1 = p_2 E/(p_1 - p_2)$

and

$$A_2 = p_1 E/(p_2 - p_1)$$

We will limit ourselves to the oscillatory (underdamped) case, Eq. (5.31), of the charging process. On carrying out manipulations similar to those in going over from Eq. (5.33) to Eq. (5.34), we obtain time dependences for the voltage across the capacitive element and the charging current in the circuit (see Fig. 5.10 b)

$$v_C = v_{C,ss} + v_{C,t} = E - (E/\omega_0 \sqrt{LC}) \exp(-\delta t) \sin(\omega_0 t + \psi)$$

$$i = C\, dv_C/dt = (E/\omega_0 L) \exp(-\delta t) \sin \omega_0 t$$

The voltage across the capacitive element is a maximum at $t = \pi/\omega_0$. It increases with increasing time constant $\tau = 1/\delta$ in comparison with the natural period of oscillations, $T_0 = 2\pi/\omega_0$, and in the limit it may be nearly twice the steady-state value. This overvoltage might be dangerous to the insulation of H.V. equipment. To avoid overvoltage, the transient (charging) process should be reduced to the overdamped (aperiodic) case by placing an additional resistor in series with the capacitor.

5.7 Response of Series *RL* Network to a Sinusoidal Voltage Excitation

In a circuit formed by a series combination of an inductive element L and a resistive element r (Fig. 5.11 a) excited by a sinusoidal emf $e = v = V_m \sin(\omega t + \psi_v)$, the steady-state sinusoidal current (see Sec. 2.11) is

$$i_{ss} = I_m \sin(\omega t + \psi_v - \varphi)$$

where

$$I_m = V_m / \sqrt{r^2 + (\omega L)^2}$$

is the amplitude (peak value) of the current,

$\varphi = \arctan(\omega L/r)$

is the argument of the complex impedance of the circuit, and ψ_v is the initial phase of the source emf.

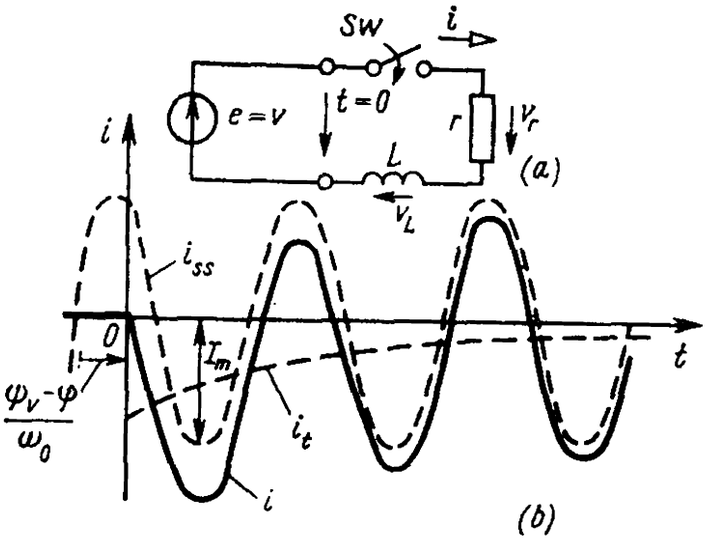


FIGURE 5.11

This circuit is described by a first-order differential equation similar to Eq. (5.3)

$$v_L + v_r = L\, di/dt + ri = e$$

Therefore the free (transient) term of current is

$$i_t = A \exp(pt)$$

and the complete solution is

$$i = i_{ss} + i_t = I_m \sin(\omega t + \psi_v - \varphi) + A \exp[-(r/L)\, t]$$

On the basis of the switching rule for an inductive element, Eq. (5.1) at $t = 0$,

$$i(0_-) = 0 = i(0_+) = I_m \sin(\psi_v - \varphi) + A$$

Hence

$$A = -I_m \sin(\psi_v - \varphi)$$

On substituting the above expression for the constant of integration in the complete solution, we obtain the time dependence of current

$$i = i_{ss} + i_t = I_m \sin(\omega t + \psi_v - \varphi) - I_m \sin(\psi_v - \varphi) \exp(-t/\tau)$$

where $\tau = L/r$ is the time constant of the circuit.

Thus during the transition from one steady state to another the current in the circuit consists of a sinusoidal (or forced) component and a free (or transient) component decreasing exponentially with time (see Fig. 5.11 b). At the end of the time interval equal to 3τ the free (transient) term may be practically neglected.

If the switching time, $t = 0$, is chosen so that the initial phase of the source voltage is $\psi_v = \varphi$, the free term of the current will be zero, and this implies that there will be no transients, and the current in the circuit reaches its new steady-state (sinusoidal) value at once.

During a transition from one steady state to another, the current in the circuit may be nearly twice the peak value, I_m, of the sinusoidal component. If the time constant of the circuit is long in comparison with the period of the sinusoidal component and the initial phase of the source voltage is $\psi_v = \varphi + \pi/2$, the maximum value of the free (transient) component may approximately be equal to I_m, and at time $t \approx T/2$ the current will be close to $2I_m$.

A similar procedure applies to the response to a sinusoidal voltage excitation for a series RC circuit and other cases. Here, too, the transient process depends on the initial phase of the voltage source: there will be no transient term in the response at $\psi_v = \varphi + \pi/2$, where $\varphi = \arctan(-x_C/r) < 0$, and it will be a maximum at $\psi_v = \varphi$, when the maximum voltage across the capacitive element may be nearly twice the peak value of the steady-state voltage. Overvoltage of this magnitude might cause a breakdown in the insulation of H.V. equipment.

5.8 The Sawtooth Voltage Generator

Transients form the basis for the operation of many electric and electronic devices. One example is the sawtooth voltage generator. A voltage wave of sawtooth form is required in the cathode-ray-tube (CRT) oscilloscope which uses an electrostatic sweep to move the electron beam horizontally (along the time axis) across the tube screen.

The circuit of a simple sawtooth voltage generator is shown in Fig. 5.12 a. When the switch Sw is closed at $t = t_0$, a d.c. emf source E is connected to a circuit formed by a series combination of a resistive element r and a capacitive element C. The branch containing a resistor R is disconnected, because the gas-discharge tube is not conducting. As a result, a transient is caused to take place in the circuit (see Fig. 5.4 b), such that the voltage across the capacitive element builds up in accord with Eq. (5.22)

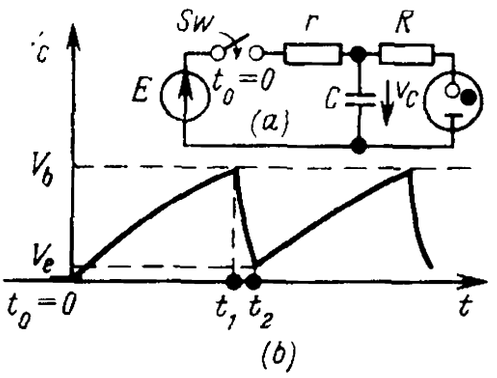


FIGURE 5.12

$$v_C = E\,[1 - \exp(-t/rC)]$$

At time $t = t_1$ the voltage across the capacitive element reaches what is known as the breakdown value, V_b, the gas-discharge tube breaks down into conduction, and the capacitive element begins to discharge (see Fig. 5.12 b).

If the charging time of the capacitive element, $\Delta t = t_1 - t_0$, is substantially shorter than the time constant $\tau = rC$, it may be assu-

med approximately that the rate of change of v_C is constant and equal to

$$dv_C/dt \mid_{t=t_0} = E/rC$$

which implies that the function v_C (t) is almost linear in the time interval Δt:

$$v_C \approx (E/rC)\ (t - t_0)$$

The element values of the discharge circuit (that is the resistance R) are chosen such that the discharge time of the capacitive element, $t_2 - t_1$, is substantially shorter than the charging time Δt. By the time t_2 the voltage across the capacitive element falls to what is known as the extinction value, V_e, and the gas-discharge tube ceases to conduct. As this happens, the capacitive element begins to charge again from the d.c. emf source, the voltage across the capacitor v_C again rises almost linearly to V_b, and the cycle just described is repeated over again. This sequence of events goes on until the switch Sw is opened.

By varying the values of r and C in the charging circuit, it is possible to adjust the linearity of v_C and the pulse duration.

6 D. C. -Excited Magnetic Circuits

6.1 Magnetic-Circuit Elements

The term *magnetic circuit* refers to the closed path followed by the lines of a magnetic flux. By extension, it also applies to the assemblage of ferromagnetic and nonferromagnetic parts of an electrical device which serve to give the magnetic flux the desired shape and intensity.

The magnetic flux can be excited by a permanent magnet or a current-carrying coil placed in a suitable part of the magnetic circuit.

One of the simplest forms of magnetic circuit is a uniformly wound torus ring (Fig. 6.1) widely used in multiwinding transformers, magnetic amplifiers, and elsewhere.

Figure 6.2 shows a more complex form of magnetic circuit in which a movable core, or plunger, is pulled inside a stationary iron-

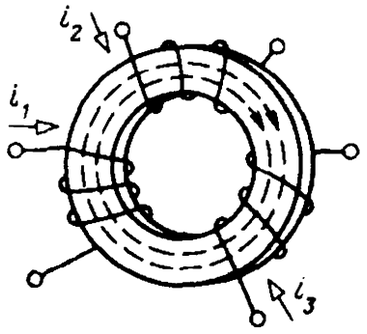


FIGURE 6.1

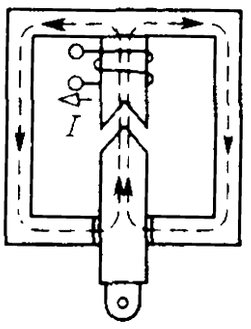


FIGURE 6.2

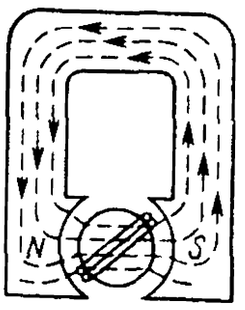


FIGURE 6.3

cored coil energized with direct or alternating current. The force of attraction, or pull, depends on the position of the plunger inside the core.

Figure 6.3 shows a magnetic circuit in which a magnetic field is set up by a permanent magnet. If we energize the moving coil mounted on a ferromagnetic cylinder with direct current, a torque will act upon the coil. Rotation of the coil does not affect the magnetic field around the magnetic circuit. This arrangement is utilized, for example, in moving-coil measuring instruments.

These and all other magnetic circuits may be classed into the *single-path type* (see Figs. 6.1 and 6.3), and the *multipath type* (see Fig. 6.2). In the former, the magnetic flux is the same at any section

of the circuit. In the latter type, the magnetic flux is different at different sections. In the general case, multipath magnetic circuits may be of a fairly complex configuration as in electric motors, generators, and similar devices.

In most cases, a magnetic circuit should be taken as nonlinear. It is only under certain assumptions and certain conditions that a magnetic circuit may be assumed to be linear.

6.2 Ampere's Law for a Constant-MMF Magnetic Circuit

What we call now Ampere's circuital law is an outcome of many experiments. It states that *the integral of the magnetic field strength over any contour (the circulation of the field intensity vector) is equal to the algebraic sum of the currents linked with that contour*

$$\oint_l \mathbf{H}\, d\mathbf{l} = \Sigma I \tag{6.1}$$

The positive direction for currents is assumed that which corresponds to a clockwise traversal of the contour (the corkscrew rule). Notably, for the contour of Fig. 6.4 the Ampere circuital law gives

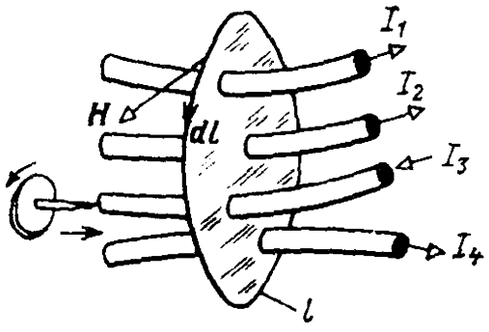


FIGURE 6.4

$$\oint \mathbf{H}\, d\mathbf{l} = I_1 + I_2 - I_3 + I_4$$

The term ΣI in Eq. (6.1) is called the *magnetomotive force* (or mmf for short).

The basic unit of mmf in the International System of Units is the *ampere* (A). The basic SI unit of magnetic field strength (or magnetic intensity) is the *ampere per metre* (A m^{-1}) for which no special name exists. Often, a multiple unit, the ampere per centimetre is used, such that 1 A cm^{-1} = 100 A m^{-1}.

In most cases, the magnetic circuit may be visualized as consisting of a series of sections within each of which the magnetic field is uniform which means that the magnetic field strength is the same over the cross section and equal to the magnetic intensity H_k along the mean path of a section of length l_k. For such magnetic circuits, the integration in Eq. (6.1) may be replaced with summation.

If the magnetic field is set up by a coil carrying a current I and having w turns, then for the magnetic-circuit contour linked with

the turns and made up of n sections we may, instead of Eq. (6.1), write

$$\sum_{k=1}^{n} H_k l_k = Iw \tag{6.2a}$$

If the contour links with the turns of m current-carrying coils, then

$$\sum_{k=1}^{n} H_k l_k = \sum_{p=1}^{m} I_p w_p = \sum_{p=1}^{m} F_p \tag{6.2b}$$

where $F_p = I_p w_p$ is the mmf.

Thus, by Ampere's circuital law, the mmf F is equal to the sum of the products of the magnetic intensities by the lengths of the respective sections of a magnetic-circuit contour. The product $H_k l_k = U_{Mk}$ is sometimes called the *magnetic difference of potential*, mdp, of a part of a magnetic circuit.

6.3 Properties of Ferromagnetic Materials

The magnetic state of a point in an isotropic medium (that is, a medium whose properties are the same, in whatever direction they are measured) is well defined by the *magnetic intensity vector* **H** and the *magnetic induction vector* **B**, both of which point in the same direction.

The basic SI unit of magnetic induction is the *tesla* (T), such that $1\ \mathrm{T} = 1\ \mathrm{Wb\ m^{-2}} = 1\ \mathrm{V\ s\ m^{-2}}$. It is defined as the magnetic induction of a uniform magnetic field in which the magnetic flux Φ (see Sec. 2.3) through an area of 1 $\mathrm{m^2}$ perpendicular to the magnetic lines of the field is equal to one weber (Wb).

In vacuum, the magnetic induction and the magnetic intensity are connected by a simple relation

$$B = \mu_0 H$$

where $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\ m^{-1}}$ is the *magnetic constant*. For ferromagnetic materials, the magnetic induction is, in the general case, a nonlinear function of the magnetic intensity.

For an experimental study of the magnetic properties of a ferromagnetic material, all measurements should be made on a specimen in which the magnetic field is uniform. One such specimen may be a torus ring fabricated from the ferromagnetic material of interest (Fig. 6.5), in which the length of the magnetic lines is many times its thickness (a thin-walled torus ring). The coil wound on the torus ring should be uniform and have a number w of turns.

For a torus ring made of an isotropic ferromagnetic material, with the coil turns wound touching, it may be taken that all magnet-

ic lines are circles, and the magnetic intensity and induction vectors are tangent to a circle. In Fig. 6.5, the dashed circle shows the mean magnetic line and the vectors **H** and **B** tangent to it at one of the points.

When calculating the magnetic intensity and induction for a thin-walled torus ring, it may be assumed that all magnetic lines have the same length equal to the mean line length, that is, $2\pi r$.

Suppose that the ferromagnetic material of a thin-walled torus ring is completely demagnetized and there is no current, I, flowing in the coil ($B = 0$ and $H = 0$). If, now, we gradually raise the d.c. current I in the coil, a magnetic field will be set up in the ferromagnetic material. By virtue of the Ampere circuital law, Eq. (6.1), the intensity of this field is given by

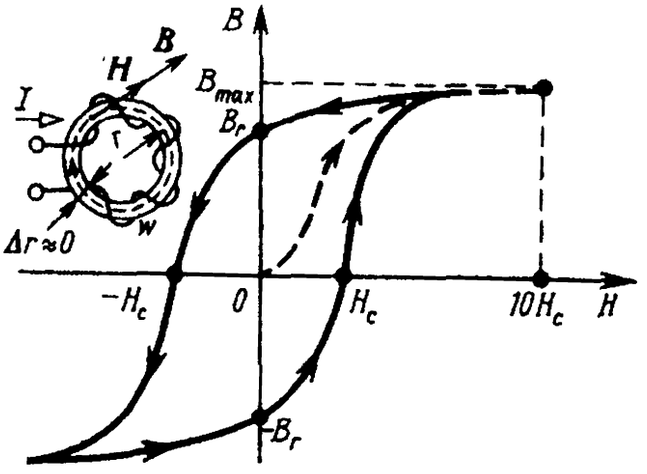


FIGURE 6.5

$$H = Iw/2\pi r \qquad (6.3)$$

For each value of H, the ferromagnetic material of the thin-walled torus ring will be in a particular state of magnetization in turn corresponding to a particular value of magnetic induction B.

If the initial magnetic state of the torus ring material is such that $H = 0$ and $B = 0$, a gradual rise in magnetizing current will yield a nonlinear relation B (H) depicted graphically by what is known as the *initial magnetization curve* (the dashed curve in Fig. 6.5). Beginning at certain values of H, the magnetic induction in a thin-walled torus ring practically ceases to increase, but remains constant and equal to B_{max}. This condition is referred to as *saturation.*

If, upon saturation, the direct current in the coil is gradually reduced, thereby reducing the magnetic intensity, Eq. (6.3), the magnetic induction in the specimen will go down, too. In going down, however, the B (H) curve does not retrace the path it followed when going up. On reversing the direction of current flow in the coil and raising the value of current, we obtain a new B (H) curve. At some maximum negative value of H, the ferromagnetic material again reaches the condition of saturation. If, now, we repeatedly reduce the reverse saturation current, then raise the forward current to saturation and so on, then, after several cycles of reversible magnetization, we obtain a symmetrical loop (the full line in Fig. 6.5). This closed B (H) cycle is called the *limiting static hysteresis loop* (or the static hysteresis limit cycle) of the ferromagnetic material. If, during

a symmetric cyclic magnetization the B (H) curve fails to reach saturation, the resultant symmetric B (H) curve is called a *minor symmetric hysteresis loop* of the material.

The static hysteresis limit cycle of a ferromagnetic material si specified (see Fig. 6.5) in terms of the *coercive force* H_c which is the negative magnetic intensity required to reduce B to zero, the *residual magnetic induction* or *remanence* B_r, and the *squareness ratio* $k_{\square} = B_r/B_{H=10H_c}$.

According to the value of H_c associated with the static hysteresis limit cycle, all ferromagnetic materials may be classed into two broad groups, *magnetically soft* and *magnetically hard*. The former are

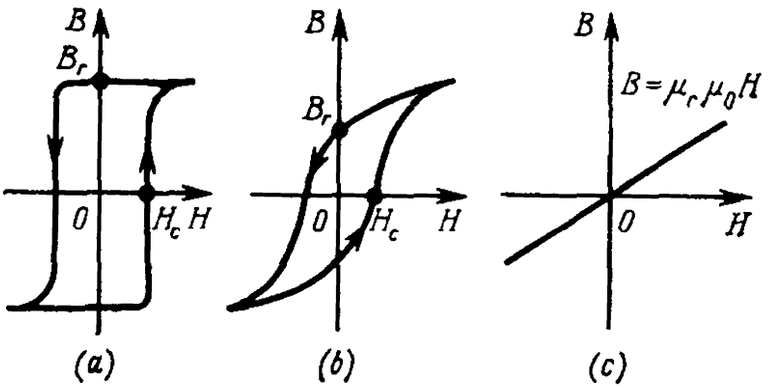


FIGURE 6.6

those for which $H_c < 0.01$-0.05 A · m^{-1}. The latter are those for which $H_c > 20$-30 kA m^{-1}. Magnetically hard materials go to make permanent magnets. Magnetically soft materials are used in the fabrication of cores for electric devices operating with a cyclically varying magnetic field.

In turn, magnetically soft materials are classed into those having a nearly rectangular B/H loop, for which the squareness ratio is $k_{\square} > 0.95$ (see Fig. 6.6 a); those having a rounded B/H loop for which $0.4 < k_{\square} < 0.7$ (see Fig. 6.6 b); and those for which the B (H) relation is a linear one: $B = \mu_r\mu_0 H$ (see Fig. 6.6 c), where μ_r is the *relative permeability*.

The desired form of the B/H loop can be obtained, using specimens made of either various ferromagnetic alloys, or ferrites which are composed of mixed oxides and generally have the form of polycrystalline ceramics. In contrast to ferromagnetic alloys, ferrites have a very high resistivity.

Rectangular-B/H-loop materials are used in large quantities as cores in memory devices for computers, in magnetic amplifiers, and other automatic-control components. Round-B/H-loop materials are used as cores for electric machines and apparatus. They operate under cyclic magnetization along minor loops. In designing such cores, the symmetrical minor loops are usually replaced with the

normal magnetization curve which is the locus of the minor-loop tips (or cusps) obtained for a thin-walled torus ring magnetized with a low-frequency sinewave current in the coil (Fig. 6.7).

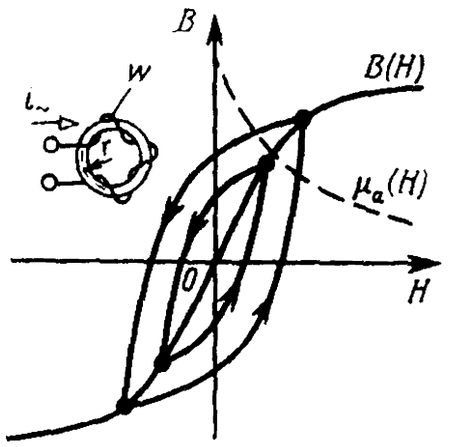


FIGURE 6.7

Using the normal magnetization curve of a ferromagnetic material, we can deduce the *absolute permeability*

$$\mu_a = \mu_r \mu_0 = B/H \qquad (6.4)$$

as a function of the magnetic intensity H. The respective curve is shown dashed in Fig. 6.7.

Figure 6.8 shows the normal magnetization curves for electrical-sheet steel (used in the fabrication of cores for electric machines, transformers and some other devices), cast iron, and permalloy.

Linear B/H-loop ferromagnetic materials are used to make cores for the inductors used in high-Q tuned circuits. Such circuits are

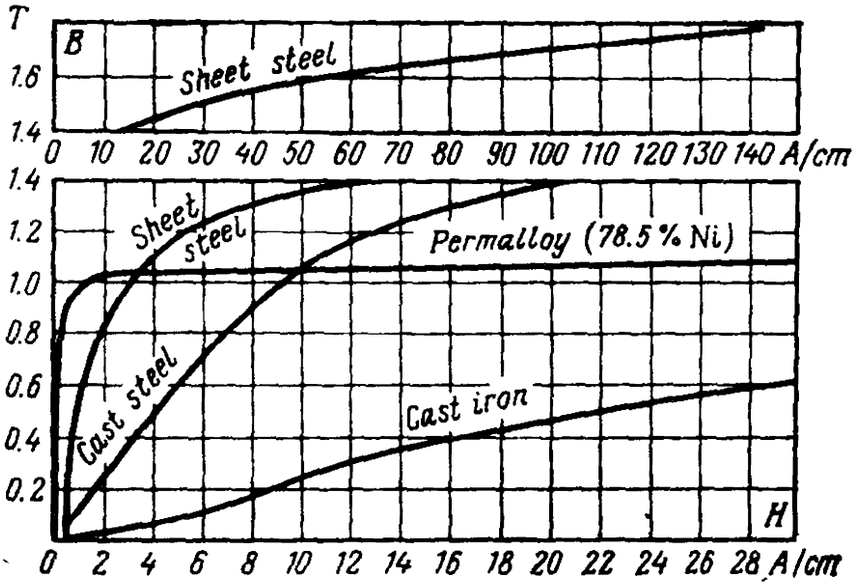


FIGURE 6.8

employed in receivers, transmitters, miniature communication antennas, etc.

If a part of a magnetic circuit with a cross-sectional area S cannot be regarded as thin-walled, its magnetic field will not be uniform, and the design calculations must be based on the average magnetic induction $B_{av} = \Phi/S$ (that is, the magnetic induction averaged over the cross-sectional area), and the average magnetic intensity H_{av} (the value of H taken along the mean magnetic line

of force). As an example, for a torus ring rectangular in cross section, with an inside radius r_1, an outside radius r_2 and a height h, fabricated from a linear B/H material, that is, such that $B = \mu_r \mu_0 H$ (see Fig. 6.6 c),

$$H_{av} = wI/2\pi r_{av} = wI/\pi (r_1 + r_2)$$

$$B_{av} = \Phi/S = \frac{\mu_r \mu_0}{r_2 - r_1} \int_{r_1}^{r_2} (wI/2\pi r)\, dr = \frac{\mu_r \mu_0 wI}{(r_2 - r_1)\, 2\pi} \ln (r_2/r_1)$$

where

$$\Phi = h \int_{r_1}^{r_2} B\, dr = \mu_r \mu_0 h \int_{r_1}^{r_2} H\, dr$$

H is given by Eq. (6.3), and

$$S = h (r_2 - r_1)$$

From the expressions derived above it follows that

$$B_{av} = \frac{\mu_r \mu_0 (r_1 + r_2)}{2 (r_2 - r_1)} \ln (r_2/r_1)\, H_{av}$$

In our subsequent discussion we will simplify calculations by neglecting the variations in the magnetic field over the cross-section of a magnetic circuit and by assuming that at any section the magnetic intensity and the magnetic induction are the same as they are along the mean magnetic line.

6.4 The Single-Path Magnetic Circuit

The problem most commonly encountered in the design of a single-path (undivided) magnetic circuit is to find the mmf $F = Iw$ required to set up some specified magnetic flux or magnetic flux density (induction) within some portion of the magnetic circuit (most often within the air gap).

An example of a *single-path (undivided) magnetic circuit* is shown in Fig. 6.9. As is seen, it consists of a ferromagnetic core of a constant cross-sectional area S_1 and of length l_1, and an air gap of the same cross section and of length l_2. The magnetic properties of the core are defined by the B/H curve in Fig. 6.10 and, therefore, in accord with Eq. (6.4), by the relation $\mu_a (H)$.

By the Ampere circuital law, Eq. (6.2),

$$\sum_{k=1}^{2} H_k l_k = H_1 l_1 + H_2 l_2 = Iw = F \qquad (6.5)$$

where H_1 is the magnetic intensity in the core, and H_2 is the magnetic intensity in the air gap.

In the air gap the magnetic induction B_2 and the magnetic intensity H_2 are connected by a simple relation of the form

$$B_2 = \mu_0 H_2$$

In the core, this relation is

$$B_1 = \mu_{a1} H_1$$

Also, in a single-path (undivided) magnetic circuit the magnetic flux is the same at any section

$$\Phi = B_1 S_1 = B_2 S_2 \tag{6.6}$$

where S_1 is the cross-sectional area of the core and S_2 is that of the air gap.

If we know the magnetic flux Φ, then we can use Eq. (6.6) to find the magnetic inductions B_1 and B_2. The magnetic field inten-

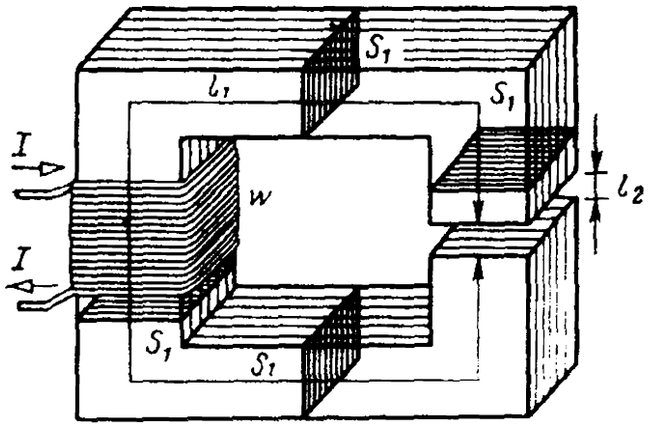


FIGURE 6.9

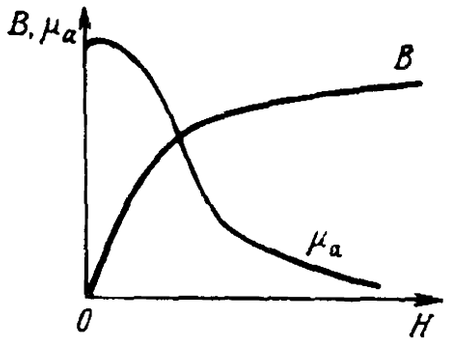


FIGURE 6.10

sity in the core, H_1, can be found from the normal magnetization curve in Fig. 6.10, and the magnetic field intensity in the air gap is $H_2 = B_2/\mu_0$. Finally, using Eq. (6.5), we find the sought mmf.

It takes more effort to solve the inverse problem: given the mmf F, one is to find the magnetic flux. Now the procedure is as follows.

On replacing the magnetic field intensities in Eq. (6.5) with the respective magnetic inductions, we get

$$(B_1/\mu_{a1})\, l_1 + (B_2/\mu_0)\, l_2 = Iw = F$$

or, subject to Eq. (6.6),

$$\Phi\,(l_1/S_1\mu_{a1} + l_2/S_2\mu_0) = \Phi\,(r_{M_1} + r_{M_2}) = \Phi \sum_{k=1}^{2} r_{Mk} = Iw = F \tag{6.7}$$

where

$$r_{Mk} = l_k/S_k\mu_{ak}$$

is the reluctance of the kth part of the magnetic circuit, being nonlinear if the B/H relation for the part is nonlinear (see Fig. 6.10), that is, $\mu_{ak} \neq \text{const}$.

For a nonlinear reluctance, $r_{\mathbf{M}}$, we can construct a *weber-ampere characteristic*, that is, a plot of the magnetic flux as a function of the magnetic potential difference $U_{\mathbf{M}}$ for the respective part of the magnetic circuit, by reference to the normal magnetization curve, or the B/H graph, of the ferromagnetic material. This is done by multiplying the ordinates and abscissae of all points on the normal magnetization curve by the cross-sectional area S of that part and its mean length l, respectively.

Figure 6.11 shows weber-ampere characteristics for the ferromagnetic part, $\Phi\,(U_{\mathbf{M}1})$, which has a nonlinear reluctance $r_{\mathbf{M}1}$,

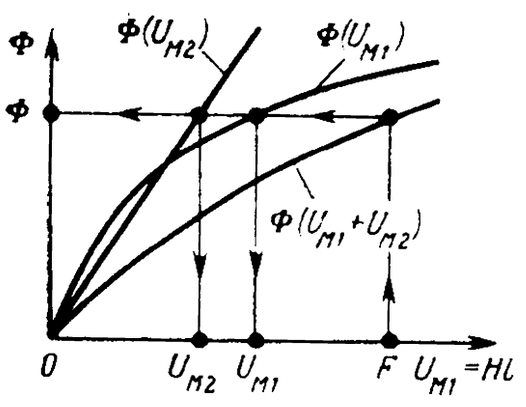


FIGURE 6.11

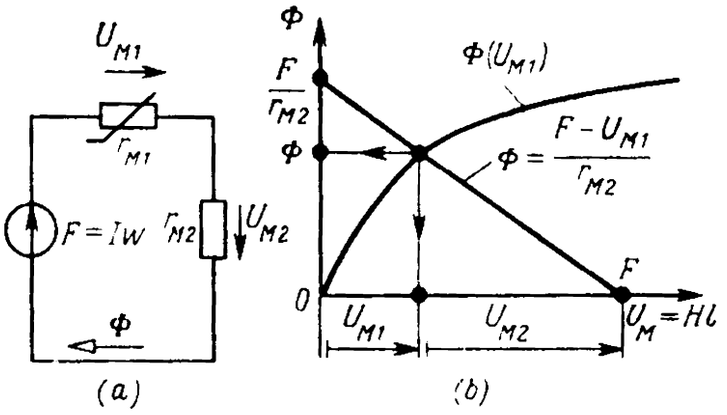


FIGURE 6.12

and for the air gap, $\Phi\,(U_{\mathbf{M}2})$, which has a linear reluctance $r_{\mathbf{M}2} = l_2/S_2\,\mu_0$, of the magnetic circuit in Fig. 6.9.

There is a fair analogy between the analysis of nonlinear d.c. electric circuits and constant-mmf magnetic circuits. Indeed, as follows from Eq. (6.7), the magnetic potential difference across a part of a magnetic circuit is equal to the product of the reluctance of the part by the magnetic flux in that part, $U_{\mathbf{M}} = r_{\mathbf{M}}\,\Phi$. This expression resembles Ohm's law, $V = rI$, Eq. (1.1), for a resistive element in a d.c. electric circuit, and is therefore sometimes called the magnetic Ohm's law. The sum of the magnetic potential differences around a magnetic circuit is equal to the sum of the mmfs of the circuit, $\Sigma U_{\mathbf{M}} = \Sigma F$, Eq. (6.2b), which resembles Kirchhoff's voltage law for d.c. electric circuits, $\Sigma V = \Sigma E$, Eq. (1.8).

By extending the analogy a bit further, we can replace the single-path (undivided) magnetic circuit in Fig. 6.9 with an equivalent circuit such as shown in Fig. 6.12 a. This equivalent circuit is analogous to that of a nonlinear electric circuit with its elements connected in series (see Fig. 1.31), to within the element symbols. This implies that single-path (and, for that matter, multipath) constant-mmf

magnetic circuits may be analysed using all the graphical and analytical methods developed for the analysis of nonlinear d.c. electric circuits (see Sec. 1.21).

As an illustration, we will analyse the undivided magnetic circuit of Fig. 6.12 a graphically, namely by the addition of weber-ampere characteristics (Fig. 6.11) and by the load-line method (Fig. 6.12 b).

In the case of the first method, the weber-ampere characteristic of the entire magnetic circuit, $\Phi\ (U_{M1} + U_{M2})$, is obtained by adding together the weber-ampere characteristics of its two parts, the core and the air gap, in terms of magnetic potential difference. With the mmf $F = Iw$ specified in advance we refer to the weber-ampere characteristic of the entire magnetic circuit and find the magnetic flux Φ. Once the magnetic flux is determined, we can refer to the weber-ampere characteristics of the two parts of the circuit and find the respective magnetic potential differences.

In the case of the second method, the load line, Eq. (1.39), for the air gap (which is the second or linear part of the magnetic circuit), defined as

$$\Phi = U_{M2}/r_{M2} = (F - U_{M1})/r_{M2}$$

is a straight line passing through the point F on the axis of abscissae, and the point F/r_{M2} on the axis of ordinates. The intersection of the load line with the weber-ampere characteristic of the ferromagnetic part of the circuit, $\Phi\ (U_{M1})$, defines the magnetic flux Φ in the circuit and the magnetic potential differences across the ferromagnetic part, U_{M1}, and across the air gap, U_{M2}. The magnetic induction in the air gap is $B_2 = \Phi/S_2$.

6.5 The Permanent-Magnet Undivided Magnetic Circuit

As an example, we will consider the simplest permanent-magnet undivided (single-path) magnetic circuit, which is a torus ring (Fig. 6.13 a). Suppose that the torus ring of length l and of cross-sectional area S is made of a magnetically hard material with part of its B/H relation shown in Fig. 6.13 b. The material of the torus ring has been magnetized so that its remanence is B_r.

Removal of a part $l_{air} \ll l$ from the torus ring (Fig. 6.13 c, where subscript a stands for air) leaves an air gap in which the remaining part of the ring sets up a magnetic field. If we neglect variations in the magnetic field in the air gap, the magnetic field everywhere in the gap can be specified by a magnetic intensity H_{air} and by a magnetic induction $B_{air} = \mu_0 H_{air}$. Since the magnetic lines of force buckle outwards in the air gap, the cross-sectional area of the gap, S_{air}, is greater than that of the permanent magnet, $S_M = S$.

By Ampere's circuital law, Eq. (6.5), for the contour following the centre line of the magnetic circuit

$$H_M l_M = H_{air} l_{air} = 0 \tag{6.8}$$

where H_M is the magnetic intensity in the permanent magnet and l_M is its mean path length.

From Eq. (6.8) it follows that

$$H_M = -(l_{air}/l_M)\, H_{air} = -\,(l_{air}/l_M)\,(B_{air}/\mu_0) \tag{6.9}$$

Also, since the magnetic flux Φ in an undivided magnetic circuit is the same at any section, then

$$S_{air} B_{air} = S_M B_M \tag{6.10}$$

On substituting for B_{air} from (6.10) into (6.9), we obtain an equation for the line passing through the origin (see Fig. 6.13 b)

$$H_M = -\,(S_M l_{air}/S_{air} l_M)\,(B_M/\mu_0) = -\,N_M B_M \tag{6.11}$$

where

$$N_M = S_M l_{air}/S_{air} l_M \mu_0$$

is the demagnetization factor for the permanent magnet.

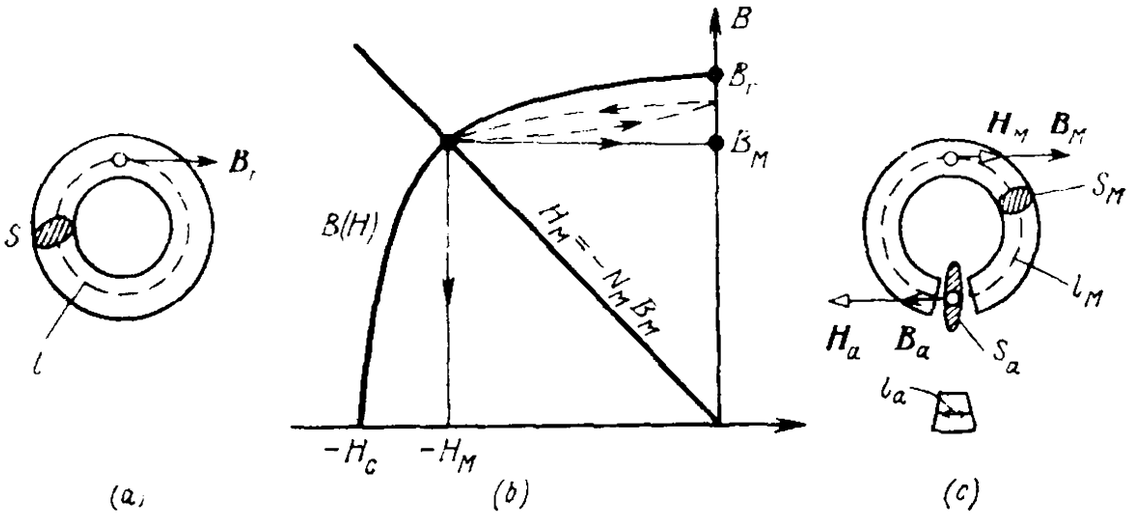


FIGURE 6.13

The intersection of the line $H_M = -N_M B_M$ and the B/H curve defines the magnetic induction in the magnet, $B = B_M$, and, in consequence, in the air gap, Eq. (6.10).

If we introduce a ferromagnetic piece of low reluctance into the air gap slowly, the magnetic induction in the magnetic circuit will retrace the minor hysteresis loop shown dashed in Fig. 6.13 b. If we move the ferromagnetic piece repeatedly into and out of the air gap, variations in the magnetic induction of the magnet will follow some steady-state minor loop.

For large values of magnetic induction in the air gap, permanent magnets are fabricated from magnetically hard materials, that is, those for which the coercive force H_c is high.

6.6 D.C.-Excited Electromagnetic Devices

For their operation many d.c. electromagnetic devices, such as measuring instruments, electromechanical relays, and electromagnets, depend on the electromechanical effect of the magnetic field. In analysing or designing such devices, it is often required to express the forces acting on the various parts of the magnetic circuit in terms of variations in the energy stored in the magnetic field.

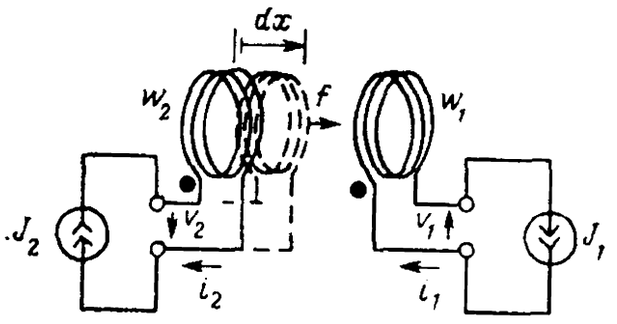


FIGURE 6.14

As an example, we will consider a system made up of two inductors, one stationary with w_1 turns, and the other movable with w_2 turns, connected each to a d.c. current source, J_1 and J_2, respectively (Fig. 6.14).

Suppose that the force of attraction f causes coil w_2 to move a distance dx along the x-axis in a time dt. Over this time interval the system will have received the amount of energy given by

$$p_1 dt + p_2 dt = v_1 J_1 dt + v_2 J_2 dt$$

where p_1 and p_2 are the instantaneous powers supplied by the two sources, whereas v_1 and v_2 are the voltages between the terminals of the two inductors, respectively.

For simplicity we neglect the losses in the coil wire. Then the energy supplied by the current sources can be assumed to be expended to do mechanical work and to augment the energy stored in the magnetic field of the system

$$v_1 J_1 dt + v_2 J_2 dt = f dx + dW_M \tag{6.12}$$

The voltages v_1 and v_2 between the terminals of the two coils arise owing to variations in the overall flux linkages in each of the inductors (see Sec. 2.22)

$$\Psi_1 = \Psi_{11} + \Psi_{12} = L_1 J_1 + M J_2 \tag{6.13}$$
$$\Psi_2 = \Psi_{22} + \Psi_{21} = L_2 J_2 + M J_1$$

Since in our system the currents in the coils, J_1 and J_2, and the inductances of the coils, L_1 and L_2, remain constant, variations in the overall flux linkages, Ψ_1 and Ψ_2, can only be caused by variations

in the mutual inductance M. (In the general case the inductances of the coils may vary too owing to changes in their physical dimensions.) By the law of electromagnetic induction, Eq. (2.78), the voltages between the terminals of the coils are

$$v_1 = d\Psi_1/dt$$

and

$$v_2 = d\Psi_2/dt \tag{6.14}$$

By virtue of Eqs. (6.13) and (6.14), we may re-write Eq. (6.12) as

$$J_1 d\Psi_1 + J_2 d\Psi_2 = d\,(L_1 J_1^2 + L_2 J_2^2 + 2MJ_1J_2) = f\,dx + dW_M \tag{6.15}$$

In the above equation, the term in the parentheses is equal, on the basis of Eq. (2.80), to twice the energy stored in the magnetic field of the system, $2W_M$. Hence,

$$dW_M = f\,dx$$

Thus, the electromechanical force which causes coil w_2 to move can be found from the corresponding change in the energy of the magnetic field

$$f = dW_M/dx \tag{6.16}$$

The derivative is positive, which implies that the electromechanical force f tends to displace the movable coil so that the energy stored in the magnetic field is increased.

For some devices it is legitimate to think that small displacements of the moving element of the system do not practically entail variations in the flux linkages, that is, in Eq. (6.15) $d\Psi_1 = 0$ and $d\Psi_2 = 0$. Then the system receives no energy from the sources and, in consequence,

$$f = -dW_M/dx \tag{6.17}$$

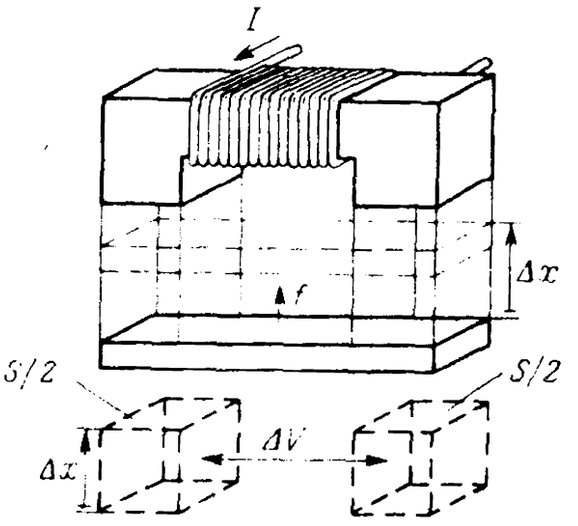


FIGURE 6.15

Or, in words, the element moves in the direction of action of the force at the cost of a decrease in the energy of the magnetic field, for example, following a decrease in the volume occupied by the magnetic field while its intensity remains unaffected.

As an example, we will apply the condition defined in Eq. (6.17) to an approximate determination of the pull developed by a lifting magnet in which the magnetic field is excited by the direct current traversing the magnet coil (Fig. 6.15).

Before proceeding any further, we will allow ourselves a small digression. As physics tells us, the d.c. magnetic field in a linear ferromagnetic medium for which $\mu_r = \text{const}$, or in a medium free from a ferromagnetic material for which $\mu_r = 1$, stores so much energy that

$$w_M = BH/2 = B^2/2\mu_r\mu_0 \tag{6.18}$$

which is called the *specific energy* of a magnetic field. The validity of Eq. (6.18) can be proved by using Eq. (2.5) for a coil wound on a thin-walled torus ring of cross-sectional area S and mean path length l, made of a linear ferromagnetic material for which $B = \mu_r\mu_0 H$. Then

$$w_M = \Psi I/2V = BH/2$$

where $\Psi = wSB$, $I = Hl/w$, and $V = lS$ is the volume of the core.

Now we may go back to finding the pull of a lifting magnet. If we assume that the magnetic induction B in the air gap between the core and the armature of the electromagnet remains constant as the armature travels a distance Δx, then the specific energy of the magnetic field in the air gap will not change, either. In consequence, the displacement of the armature through a distance Δx causes the energy of the magnetic field to change by

$$\Delta W_M = -(B^2/2\mu_0)\,\Delta V = -(B^2/2\mu_0)\,S\Delta x < 0$$

Since we have assumed that the magnetic induction remains unchanged as the armature moves, then, on the basis of Eq. (6.15), we get

$$f = -\Delta W_M/\Delta x = SB^2/2\mu_0 > 0 \tag{6.19}$$

Equation (6.19) gives an approximate value of the pull of a lifting electromagnet of any type in which the magnetic field is excited by a d.c.-energized coil. If more accurate results are desired, the distinction in design of each type must be included into consideration.

The energy stored in the magnetic field of the system generally depends not only on the relative position of its parts. Therefore, in defining the forces, arising in the magnetic field, one should apply the concept of partial derivative of the magnetic field energy with respect to the coordinate of the movable element, as it will be done in what follows.

7 The Iron-Cored Inductor in an A.C. Circuit

7.1 An Alternating Magnetic Flux in an Iron-Cored Inductor

The design of and the functions served by the magnetic circuits of various a.c. devices (machines, transformers, and so on) are many and diverse. This chapter will be only concerned with inductors (coils) wound on single-path (undivided) cores made of a ferromagnetic material.

When an iron-cored coil is connected to a source of sinusoidal voltage $v = V_m \sin \omega t$ (Fig. 7.1 a), the alternating current i flowing in the coil sets up in the core an alternating magnetic flux Φ which induces in the coil an emf of self-induction $e_L = -w\, d\Phi/dt$. If we neglect the coil resistance and assume that all the magnetic lines of the coil field have their path completed only through the core, then by Kirchhoff's voltage law for the loop shown dashed in Fig. 7.1 a we may write the following equation:

$$v = -e_L \tag{7.1a}$$

or

$$v = V_m \sin \omega t = w\, d\Phi/dt \tag{7.1b}$$

Equation (7.1) can be used to determine how the magnetic flux varies with time. Since

$$d\Phi = (V_m/w) \sin \omega t\, dt$$

then

$$\Phi = (V_m/w) \int \sin \omega t\, dt = -(V_m/\omega w) \cos \omega t + A$$

$$= \frac{V\sqrt{2}}{2\pi f w} \sin (\omega t - \pi/2) + A$$

The constant of integration A is equal to some constant magnetic flux which is nonexistent in the core of an a.c. magnetic device in a steady state. In consequence, $A = 0$, and the magnetic flux is

$$\Phi = \Phi_m \sin (\omega t - \pi/2) \tag{7.2a}$$

where

$$\Phi_m = V/4.44\, fw \tag{7.2b}$$

That is, *when the voltage existing between the terminals of an inductor is sinusoidal, the magnetic flux in its core is likewise sinusoidal.*

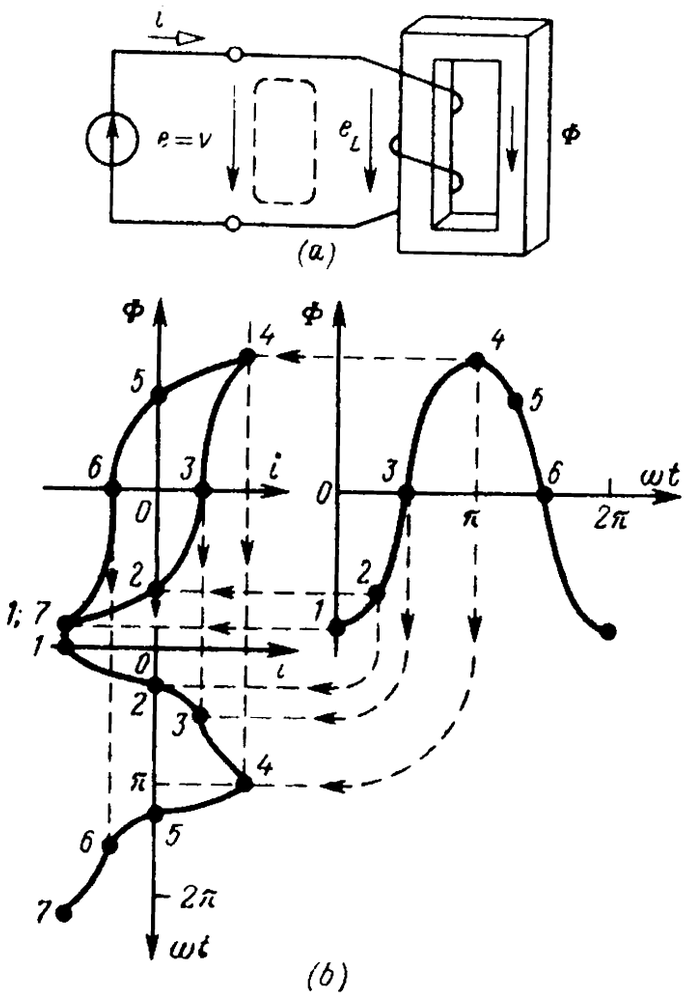


FIGURE 7.1

Since the rms voltage V between the coil terminals and the emf of self-induction E_L are the same, Eq. (7.1a), it follows from Eq. (7.2b) that

$$E_L = 4.44\, fw\Phi_m \qquad (7.2c)$$

Equation (7.2c) is used to calculate the emfs induced in transformer windings, for which reason it is often called the *equation of transformer emf.*

When finding the magnetic flux, we neglected the coil resistance and the leakage flux, that is, that part of the flux that has its path completed outside the core. So long as we are interested in time variations of the magnetic flux in a magnetic circuit, these two factors are of no material significance. In other cases, however, the resistance of the coil and the leakage flux have to be included in consideration. Therefore, Eqs. (7.2) may only be regarded as approximate relations between the rms voltage V across the coil terminals and the emf of self-induction E_L, on the one hand, and the magnetic flux of peak value Φ_m in the core and the number w of coil turns, on the other.

7.2 Magnetization of the Magnetic Circuit

Under the assumptions made above, when a sinusoidal voltage is impressed on an iron-cored coil, the magnetic flux in the core is likewise sinusoidal and independent of the properties of the core material. We set out to demonstrate that in the circumstances the magnetizing current in the coil is nonsinusoidal.

Referring to the applicable B/H curve for the core material, for example in Fig. 6.6 b, we begin by plotting the weber-ampere characteristic Φ (i) for the coil in question (see Fig. 7.1 a). To do so, we multiply the ordinates of the B/H loop by the cross-sectional

area S of the core ($\Phi = BS$), and its abscissae by the core length l, and divide the product by the number of turns in the coil (by Ampere's circuital law, $i = Hl/w$). The graph thus constructed is shown in Fig. 7.1 b. The same figure shows a sinusoidal magnetic flux plotted on the basis of Eq. (7.2a) and the time relation of the coil current. As is seen, the nonlinearity of the Φ (i) relation causes the current to be nonsinusoidal, although the magnetic flux in the core is sinusoidal. As the saturation of the core is increased, the current departs progressively more from a sine waveform.

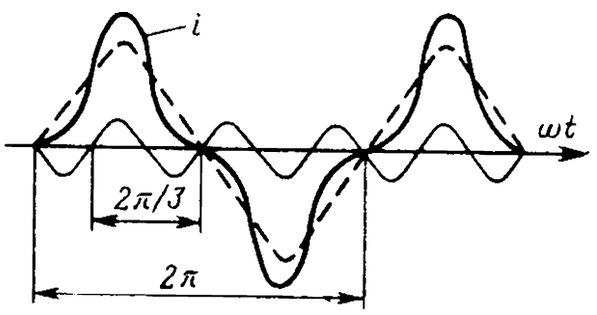


FIGURE 7.2

By comparing the plot of variations in the magnetizing current with the graph obtained by combining two sinewaves for one of which the frequency is three times the frequency of the other (Fig. 7.2), we can see that at saturation the magnetizing current contains a predominant proportion of the third harmonic. The difference between the plots of magnetizing current in Fig. 7.1 b and Fig. 7.2 is that in the former case the current contains, in addition to the fundamental and the third harmonic, also some higher harmonics.

7.3 The Ideal Iron-Cored Coil

Qualitatively, the magnetic-flux pattern in an iron-cored coil is shown by dashed lines in Fig. 7.3. The greater proportion of the magnetic flux is confined within the core; this is the main magnetic

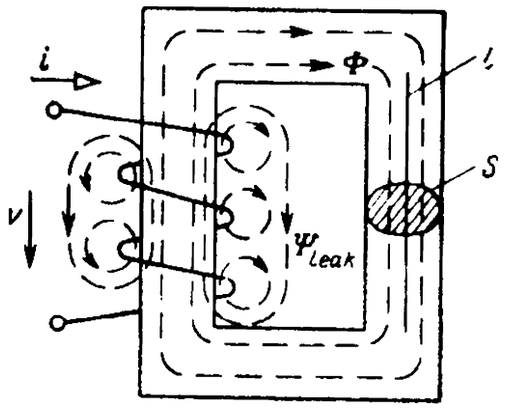


FIGURE 7.3

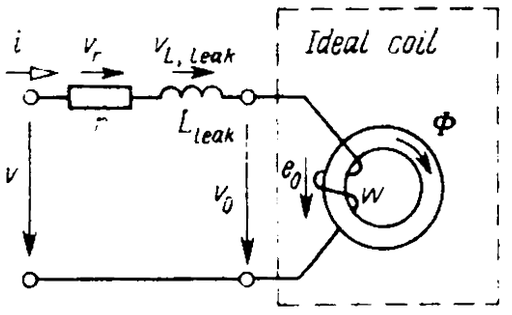


FIGURE 7.4

flux Φ. The remaining part links some of the turns and groups of turns and has its path partly through the core and partly through the air. This part of the magnetic flux is difficult to analyse quanti-

tatively and is usually defined in terms of an integral quantity called the *leakage flux linkage*, Ψ_{leak}.

The leakage flux linkage largely depends on the coil design, notably, the relative position of its turns, the wire cross-sectional area, and the like, and is less dependent on the magnetic properties of the core. In view of this, we may deem that the leakage flux linkage is proportional to current

$$\Psi_{leak} = L_{leak} i$$

where L_{leak} is the *leakage inductance* of the coil (it is constant for a given coil design).

The total flux linkage with the coil turns is

$$\Psi = w\Phi + \Psi_{leak} \tag{7.3}$$

If we consider the resistance r of the coil and the leakage flux linkage, then instead of Eq. (7.1b) the voltage between the coil terminals will be given by

$$v = ri + \frac{d\Psi}{dt} = ri + \frac{d\Psi_{leak}}{dt}\frac{di}{dt} + w\frac{d\Phi}{dt}$$

$$= ri + L_{leak}\frac{di}{dt} + w\frac{d\Phi}{dt} = v_r + v_{L,\,leak} + v_0 \tag{7.4}$$

As follows from Eq. (7.4), the equivalent circuit for a real iron-cored coil is a series combination of a resistive element of resistance r, an inductive element of inductance L_{leak}, and an ideal coil (Fig. 7.4).

The ideal coil has neither leakage inductance nor a resistance. The behaviour of the ideal coil is solely dependent on the core characteristics, and the voltage between its terminals is determined by the emf of self-induction alone

$$v_0 = -e_0 = w\,d\Phi/dt$$

7.4 Phasor Diagrams and Equivalent Circuits for an Iron-Cored Coil

Different simplifying assumptions may be made in trying to analyse a real iron-cored coil, depending on the core characteristics and the form of magnetization.

To begin with, we will analyse an iron-cored coil solely on the basis of the static magnetic properties of the core (see Fig. 6.6).

1. The core is fabricated from a ferromagnetic material which has a practically linear B/H relation: $B = \mu_r\mu_0 H$ (see Fig. 6.6 c).

The flux linkage with an ideal coil (see Fig. 7.4) is always equal to $w\Phi$, Eq. (7.3), because its leakage flux linkage $\Psi_{leak} = 0$. In the uniform, closed undivided core of an ideal coil of cross-sectional area S, the magnetic field may be deemed to be uniform, that is,

$\Phi = BS$, where B is the magnetic induction at the centre line of the core, as determined by the magnetic intensity at the centre line, $H = iw/l_{av}$.

Since in our case the B/H relation for the core is linear, it follows that

$$\Phi = BS = \mu_r \mu_0 HS = \frac{\mu_r \mu_0 S w}{l_{av}} i \tag{7.5}$$

On substituting for Φ from Eq. (7.5) into Eq. (7.4), we get the voltage between the terminals of a real coil

$$v = ri + L_{leak}\, di/dt + L\, di/dt \tag{7.6}$$

where

$$L = \mu_r \mu_0 S w^2 / l_{av}$$

is the inductance of the ideal coil.

When the excitation is a sinusoidal current, the equivalent circuit corresponding to Eq. (7.6) is the one drawn up for a real

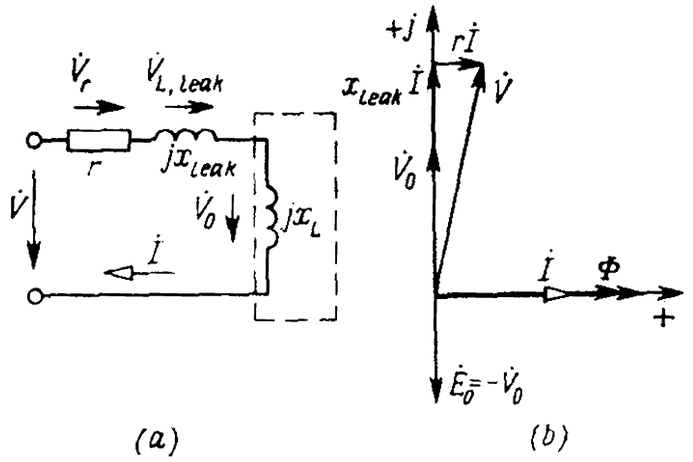


FIGURE 7.5

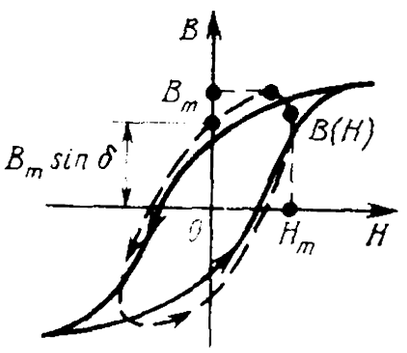


FIGURE 7.6

coil (Fig. 7.5 a) whose core is made of a linear magnetic material. The equivalent circuit for the ideal coil, which is a linear inductive element, is enclosed in the dashed box in Fig. 7.5 a.

Since all the elements in the equivalent circuit of a real coil are linear, it may be analysed by resort to the complex-number method; the results of such an analysis, subject to Eq. (2.33), appear in the phasor diagram of Fig. 7.5 b.

2. The core is fabricated from a round B/H loop ferromagnetic material (see Fig. 6.6 b).

Graphically, the magnetostatic properties of the core are represented by the B/H relation in Fig. 7.6, where $B = \Phi/S$ is the average magnetic induction over the cross-sectional area S, and $H = iw/l_{av}$ is the average magnetic intensity, that is, one at the centre line of

the core. For purposes of an approximate analysis of the ideal coil (see Fig. 7.4) we may replace the B/H loop with an equivalent ellipse. The equivalent ellipse with its centre at the origin must be so shaped, positioned and traversed that its B/H equation can sufficiently accurately describe the magnetization of the core along the static B/H loop. As a rule, the total area of the equivalent ellipse and hysteresis loop should be not less than 80-90% of the area of each of them taken separately.

Given a sinusoidal excitation voltage v, the equation of the equivalent ellipse in parametric form is

$$B = B_m \sin \omega t$$
$$H = H_m \sin (\omega t + \delta) \tag{7.7}$$

where B_m is the peak value of magnetic induction, H_m is the peak value of magnetic intensity, δ is the phase difference between the induction and the intensity, ω is the angular frequency of cyclic magnetization, and t is time.

Since, with the hysteresis loop replaced by an equivalent ellipse, the magnetic induction and intensity in the core vary sinusoidally, it is legitimate to apply the complex-number method to the analysis of the ideal coil. To this end, we represent the magnetic intensity and the magnetic induction in the core by their respective complex equivalents, Eq. (2.21)

$$\dot{H} = H_m \exp (j\delta)/\sqrt{2}$$
$$\dot{B} = B_m/\sqrt{2}$$

Similarly the complex current $\dot{I}$ in the ideal coil can be written by Eq. (6.2a), the complex voltage between its terminals $\dot{V}_0$ and its complex emf $\dot{E}_0$ by Eqs. (2.33) and (7.1) as follows:

$$\dot{I} = l_{av} H/w = I \exp (j\delta) \tag{7.8a}$$
$$\dot{V}_0 = -\dot{E}_0 = j\omega w \dot{B} S = jV_0 \tag{7.8b}$$

where

$$I = l_{av} H_m/\sqrt{2} w$$

and

$$V_0 = E_0 = \omega w S B_m/\sqrt{2}$$

are the rms values of current, voltage and emf of self-induction for the ideal coil.

By Ohm's law in complex form, Eq. (2.47), and subject to Eqs. (2.23) and (7.8), the complex impedance of a sinewave-excited ideal coil is

$$\underline{Z} = \dot{V}_0/\dot{I} = j\,(V_0/I)\exp(-j\delta) = (V_0/I)\exp j\,(90^\circ - \delta)$$
$$= (V_0/I)\sin\delta + j\,(V_0/I)\cos\delta = r_h + jx_L = r_h + j\omega L \qquad (7.9)$$

where

$$r_h = \omega w^2\,(B_m S/H_m\, l_{av})\sin\delta$$

is the resistance accounting for the hysteresis losses in the ideal coil, and

$$x_L = \omega w^2\,(B_m S/H_m l_{av})\cos\delta$$

is the inductive reactance of the ideal coil.

On replacing the ideal coil with a series combination of a resistive element, r_h, and an inductive element, x_L, we obtain an equivalent

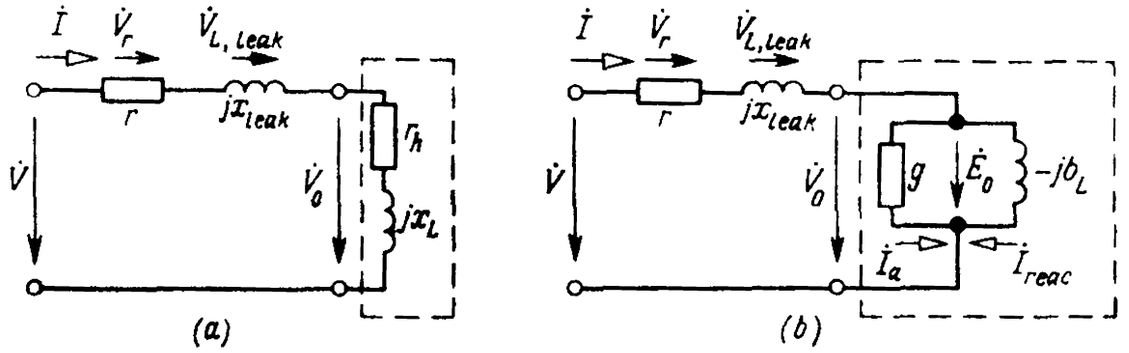


FIGURE 7.7

circuit for the real coil in the case at hand (Fig. 7.7 a). As is seen from Eqs. (7.8b) and (7.9), the emf of self-induction $\dot{E}_0$ of the ideal coil is represented in the equivalent circuit by a branch which contains the resistive element to account for the losses that might take place in the core.

As often as not the equivalent circuit for a real coil has the form shown in Fig. 7.7 b, which is derived from the equivalent circuit in Fig. 7.7 a on replacing the series combination of a resistive element and an inductive element with a parallel combination of a conductance and an inductive susceptance in accord with Eq. (2.71)

$$g = r_h/[r_h^2 + (\omega L)^2],\quad b_L = \omega L/[r_h^2 + (\omega L)^2]$$

where g is the conductance and b_L is the inductive susceptance of the ideal coil.

The phasor diagram applicable to the equivalent circuit of Fig. 7.7 b appears in Fig. 7.8. As follows from Eq. (7.7), the phasor

$\dot{\Phi} = \dot{B}S$ is drawn with a zero initial phase. In accord with Eqs. (7.7) and (7.8a), the current phasor $\dot{I}$ leads the magnetic flux phasor by an angle δ, called the *loss angle* of the ideal coil. The current phasor $\dot{I}$ is depicted as the sum of the active current phasor $\dot{I}_a$ and the reactive current phasor $\dot{I}_{reac}$, such that $\dot{I}_a$ is in phase with the voltage phasor $\dot{V}_0$, and $\dot{I}_{reac}$ lags $\dot{V}_0$ by an angle $\pi/2$.

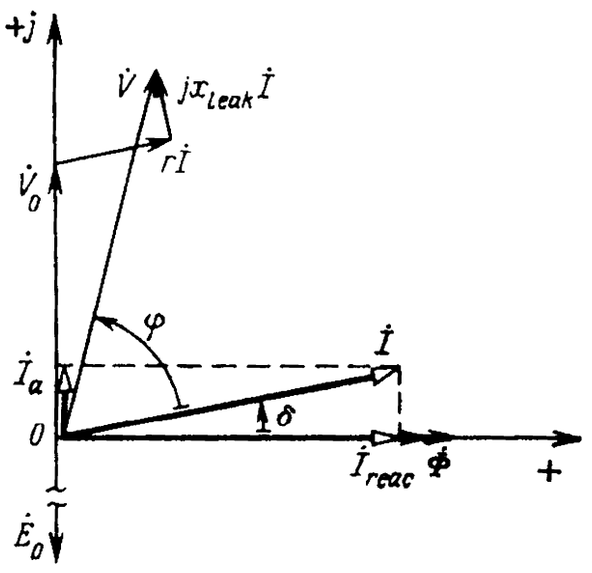


FIGURE 7.8

In order to find the terminal voltage $\dot{V}$ of the ideal coil, we should add the voltage drops across the resistance of the coil $\dot{V}_r = r\dot{I}$ and across its inductive leakage reactance, $\dot{V}_{L,leak} = jx_{leak}\dot{I}$. The emf of self-induction phasor $\dot{E}_0$ lags behind the magnetic flux phasor $\dot{\Phi}$ by an angle $\pi/2$, Eq. (7.8b).

In the general case, the dependence of the average magnetic induction on the magnetic intensity at the core centre line (the average magnetic intensity) is found from the dynamic (incremental) rather than the static hysteresis loop (see Sec. 7.5). Therefore, the equivalent ellipse defining the parameters for the equivalent circuit of the ideal coil in an a.c. circuit should in the general case correspond to the dynamic hysteresis loop.

7.5 Core Losses

Hysteresis is responsible for the loss of some energy in the core, To demonstrate, at any instant the power lost in an ideal coil (see Fig. 7.7) is given by

$$p = v_0 i = (w\, \mathrm{d}\Phi/\mathrm{d}t)\ (Hl_{av}/w) = Sw\ (\mathrm{d}B/\mathrm{d}t)\ (l_{av}/w)\ H$$
$$= Sl_{av}H\ \mathrm{d}B/\mathrm{d}t$$

where v_0 is the terminal voltage of the ideal coil, and i is the current in that coil.

For a periodic current the average power loss, that is the active power in the ideal coil over a period

$$P = \frac{1}{T}\int_0^T v_0 i\, \mathrm{d}t = Sl_{av}\oint H\, \mathrm{d}B \qquad (7.10)$$

is proportional to the area of the hysteresis loop multiplied by the core volume $V = Sl_{av}$.

As has been indicated, the area of the dynamic hysteresis loop differs, in the general case, from that of the static hysteresis loop. The point is that the time-varying magnetic flux induces in the core eddy currents i_e (Fig. 7.9) which are functions of the frequency of the magnetic flux, the resistivity of the core material, and the core design. These eddy currents cause an additional loss of energy and heat dissipation in the core. Also, they have a demagnetizing effect on the core. This is the reason why in order to make up for the effect of eddy currents and to produce the same magnetic flux as would

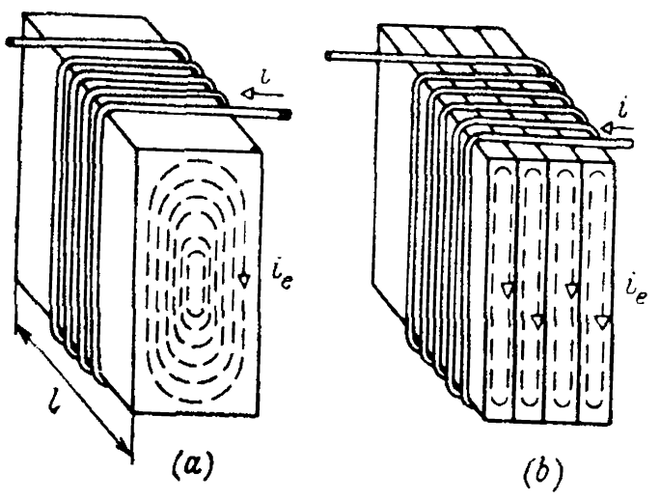


FIGURE 7.9

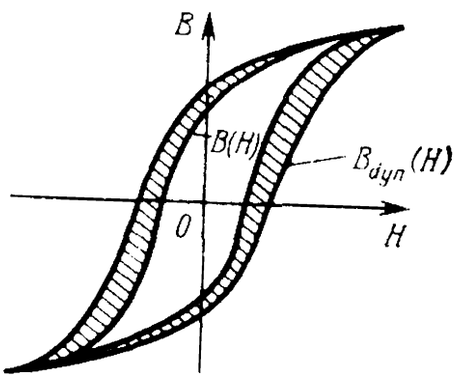


FIGURE 7.10

exist in their absence we should apply a higher magnetizing current and, as a consequence, a higher magnetic intensity.

This also explains why the area of the *dynamic hysteresis loop*, $B_{dyn}(H)$, for a core with eddy currents is larger than that of the static hysteresis loop, $B(H)$ (Fig. 7.10). If the static hysteresis loop lies within the dynamic hysteresis loop, the core losses may be resolved into two components.

One component is the *hysteresis loss*. It is proportional to the area of the static hysteresis loop shown unshaded in Fig. 7.10. The other component is the *eddy-current loss* which is proportional to the area shown shaded in Fig. 7.10 and equal to the difference in area between the static and dynamic hysteresis loops.

Eddy currents in a core can be minimized in two ways. The first way is to reduce the area of the loops traversed by eddy currents; the second way is to bring up the resistivity of the core material. The area of eddy-current loops at frequencies up to 20 kHz is usually reduced by building up a core from electrical-sheet steel laminations insulated from one another with a varnish (Fig. 7.9 b). For cores intended for operation at power frequency (50-60 Hz), the laminations are usually 0.35-0.5 mm thick. At higher frequencies, the thick-

ness is reduced to 0.02-0.05 mm. The material of punchings is a silicon steel (with as much as 0.5-4.5% Si added). The addition of silicon substantially increases the resistivity of the material, but has a negligible effect on its magnetic properties.

For engineering purposes, the hysteresis loss can be found by the formula

$$P_h = \sigma_h f B_m^n G \tag{7.11}$$

where σ_h is the hysteresis constant depending on the core material and found by experiment, f is the frequency, G is the mass of the core, B_m is the maximum induction reached. The exponent n is $n = 1.6$ for $B_m < 1$ T, and $n = 2$ for $B > 1$ T.

The eddy-current loss is given by

$$P_e = \sigma_e f^2 \gamma G B_m^2 \tag{7.12}$$

where σ_e is the eddy-current constant dependent on the core material and design, f is the frequency, G is the mass of the core, γ is the resistivity of the core material, and B_m is the maximum induction reached.

At values of magnetic induction in excess of 1 T, it may be assumed that the total core losses are proportional to B_m^2 and, in consequence, to Φ_m^2. Therefore the core losses $P = P_h + P_e$ are proportional to the square of the peak value of magnetic flux in much the same way as the copper losses (those in the coil wires) are proportional to the square of the peak value of current.

In practical cases, core losses are found from tables which usually give the total core losses as a function of the amplitude of magnetic induction (Table 7.1).

TABLE 7.1 Specific Losses in Electrical-Sheet Steel for Several Values of Magnetic Induction

B_m, T	Losses, W kg^{-1}		B_m, T	Losses, W kg^{-1}	
	laminations 0.35 mm thick	laminations, 0.5 mm thick		laminations 0.35 mm thick	laminations, 0.5 mm thick
0.6	0.585	0.685	1.1	1.61	1.92
0.7	0.685	0.890	1.2	1.94	2.24
0.8	0.855	1.13	1.3	2.31	2.58
0.9	1.05	1.38	1.4	2.66	2 95
1.0	1.30	1.64	1.5	3.0	3.40

7.6 The Voltage-Current Characteristic of an Iron-Cored Coil

If the amplitude of the current in a coil is so high that the magnetic induction in the core reaches saturation, the coil may no longer be represented by a linear equivalent circuit. As a circuit element, such a coil is specified by giving its nonlinear voltage-current characteristic, that is, the dependence of the rms a.c. voltage between the coil terminals on the rms a.c. current in the coil.

To simplify the analysis of a nonlinear iron-cored coil, we neglect its leakage inductance and resistance, and also hysteresis and eddy currents. We also assume that the nonlinear relation between the average induction $B = \Phi/S$ and the average magnetic intensity $H = iw/l_{av}$ is approximately given by $B = aH - cH^3$, where B and H are the instantaneous magnetic induction and magnetic intensity, while a and c are coefficients dependent on the properties of the ferromagnetic material used and the core design (say, the air-gap length). This B/H relation is not unlike the normal magnetization curve (see Fig. 6.7).

Suppose that the coil is energized with the sinusoidal current

$$i = I_m \sin \omega t$$

Then the magnetic flux in the core will be

$$\Phi = BS = aSH - cSH^3 = \frac{awS}{l_{av}} I_m \sin \omega t - \frac{cSw^3}{l_{av}^3} I_m^3 \sin^3 \omega t \qquad (7.13)$$

and the voltage between the coil terminals, in accord with Eq. (7.1b), will be

$$v = w\, d\Phi/dt = \frac{aS\omega w^2}{l_{av}} I_m \left[\cos \omega t - \frac{3cS\omega w^4}{l_{av}^3} I_m^2 \sin^2 \omega t \cos \omega t\right] \qquad (7.14)$$

Recalling that

$$\sin^2 \omega t \cos \omega t = \sin \omega t \sin 2\omega t/2 = (\cos \omega t - \cos 3\omega t)/4$$

the voltage in Eq. (7.14) may be written as

$$v = \left(\frac{aS\omega w^2}{l_{av}} I_m - \frac{3cS\omega w^4}{4l_{av}^3} I_m^3\right) \cos \omega t + \frac{3cS\omega w^4}{4l_{av}^3} I_m^3 \cos 3\omega t$$

$$= V_{1m} \cos \omega t + V_{3m} \cos 3\omega t \qquad (7.15)$$

As is seen, the voltage between the terminals of a nonlinear coil excited by a sinusoidal current contains the fundamental and the third harmonic. The amplitude of the third harmonic, V_{3m}, is often a small fraction of the amplitude of the fundamental, V_{1m}. If so, the third harmonic may be neglected in finding the rms voltage. The ratio of the rms or peak values or of the amplitudes of voltage

to current gives the inductive reactance of the coil

$$x_L = \frac{V}{I} = \frac{V_{1m}}{I_{1m}} = \omega\left(\frac{aSw^2}{l_{av}} - \frac{3cSw^4}{2l_{av}^3} I^2\right) = \omega L(I) \tag{7.16}$$

where

$$L(I) = \frac{aSw^2}{l_{av}} - \frac{3cSw^4}{2l_{av}^3} I^2 \tag{7.17}$$

is the nonlinear inductance of the ideal coil, neglecting the harmonics. Since under the assumption made the current in and the voltage across the coil vary sinusoidally, the circuit analysis involved may be carried out by the complex-number method.

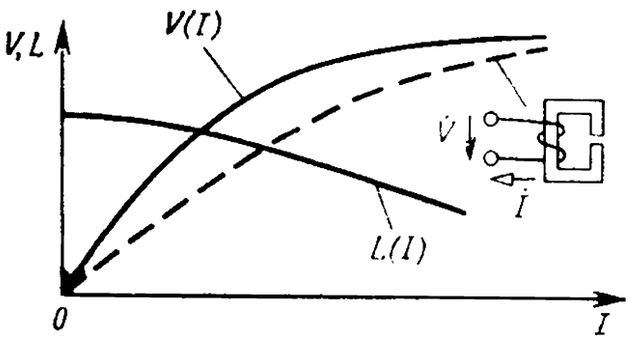


FIGURE 7.11

It is seen from Eq. (7.17) that the inductive reactance of a nonlinear iron-cored coil decreases with increasing rms current.

The voltage-current characteristic $V(I)$ and the inductance-current characteristic $L(I)$ of the ideal iron-cored coil in question are shown in Fig. 7.11 by full lines. If the core has an air gap, the nonlinearity of the $V(I)$ curve is reduced, because the weber-ampere characteristic of the core (the dashed line in Fig. 7.11) is reduced too.

If we include in consideration hysteresis, eddy currents, coil resistance, and harmonics, the $V(I)$ characteristic of a real iron-cored coil will differ somewhat from that of the ideal coil.

7.7 Ferroresonance

The condition of resonance occurring in a tuned circuit containing a nonlinear iron-cored inductor is called *ferroresonance*. There are two basic forms of ferroresonance, series (or voltage) ferroresonance and parallel (or current) ferroresonance.

Series ferroresonance, as its name implies, may take place in a circuit (Fig. 7.12 a) made up of a series combination of a nonlinear iron-cored inductor and a capacitor connected to a sinusoidal voltage source. The core must be closed, because an air gap would strongly reduce the nonlinearity of the voltage-current characteristic of the coil (see Fig. 7.11).

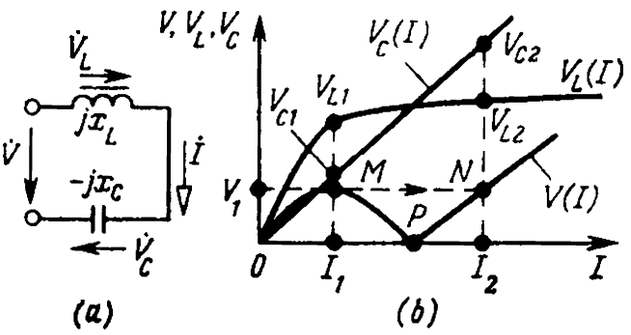


FIGURE 7.12

In analysing series ferroresonance, let us neglect all forms of losses in the circuit and voltage and current harmonics. Under such an assumption, we may apply the complex-number method of circuit analysis. The supply voltage applied to the circuit terminals (see Fig. 7.12 a) is

$$\dot{V} = \dot{V}_L + \dot{V}_C = j\omega L\,(I)\,\dot{I} + \dot{I}/j\omega C, \tag{7.18}$$

where $\dot{V}_L$, the voltage across the inductive element, is in antiphase with $\dot{V}_C$, the voltage across the capacitive element (see Fig. 2.45). In consequence, the rms supply voltage is equal to the magnitude of the difference between the rms voltages across the inductive and capacitive elements

$$V = |V_L - V_C|. \tag{7.19}$$

In order to determine the magnitude of the difference, we construct, subject to Eq. (7.17), voltage-current characteristics in a common system of coordinates (Fig. 7.12 b) for the nonlinear inductive element

$$V_L\,(I) = \omega L\,(I)I = x_L\,(I)\,I$$

and for the linear capacitive element

$$V_C\,(I) = (1/\omega C)\,I = x_C I$$

The same figure shows the voltage-current characteristic of the circuit in Fig. 7.12 a, which relates the rms supply voltage to the current in the circuit, $V\,(I)$.

At low values of $\dot{V}$, the current in the circuit $\dot{I}$ is in time-quadrature lagging with the voltage, because the circuit impedance is inductive in its effect, $V_L > V_C$. As the supply voltage is gradually raised, the rms current I increases, and so do V_L and V_C. This goes on until the supply voltage approaches a critical value, V_1, located by the operating point M, that is, the point where the line parallel to the axis of abscissae touches the voltage-current curve of the circuit. Any further increase in V, however small, causes the current to change abruptly from I_1 to I_2 (the operating point N). At the same time, an abrupt change occurs in the voltage across the inductive element from V_{L_1} to V_{L_2} (this is an insignificant change due to the saturation of the core), and in the voltage across the capacitive element from V_{C_1} to V_{C_2} (this is a far more noticeable change). At further increase in the supply voltage, $V > V_1$, following the abrupt

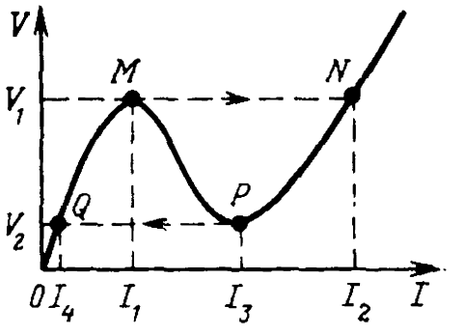


FIGURE 7.13

changes in voltage and current, we observe a gradual rise in the circuit current and in the voltages V_L and V_C.

If now we reduce the supply voltage V gradually (theoretically to zero), we could, it might seem, to bring about a condition of series resonance for which $V_C = V_L$, corresponding to the operating point P on the voltage-current characteristic of the circuit. Actually, however, losses of energy in the coil cause the voltage-current characteristic (Fig. 7.13) to differ from that plotted in Fig. 7.12 b. The voltage corresponding to the operating point P is $V_2 > 0$, and not $V = 0$. With any further decrease in the supply voltage V, however small, the current abruptly changes from I_3 to I_4 (the operating point Q). In the circumstances, the reduction of the supply voltage to zero causes the current to fall down to zero (the origin) as well.

The voltage V_L across a saturable-core coil changes very little as the supply voltage V is varied (see Fig. 7.12 b). This behaviour of the voltage-current characteristic is widely utilized in *saturable-reactor voltage regulators* (or *stabilizers*). In them, the voltage across a saturable-core coil is utilized as a stabilized (regulated) output voltage.

Parallel (or *current*) *ferroresonance* occurs in a tuned circuit in which a nonlinear saturable-core coil is connected in parallel with a capacitor, and both are excited by a source of sinusoidal voltage. The condition of parallel ferroresonance can be analysed in much the same way as series ferroresonance. In this case, however, no abrupt changes occur in the currents.

8 Transformers

8.1 General

A *transformer* is a static electric device (that is, one free from rotating or moving parts) most often used to convert one alternating voltage to another alternating voltage (or other alternating voltages) at the same frequency by mutual induction.

A transformer has at least two windings linked by a common magnetic flux but electrically insulated from each other (except in an autotransformer). In most transformers the windings are put on a core assembled from electrical-sheet steel laminations. The purpose

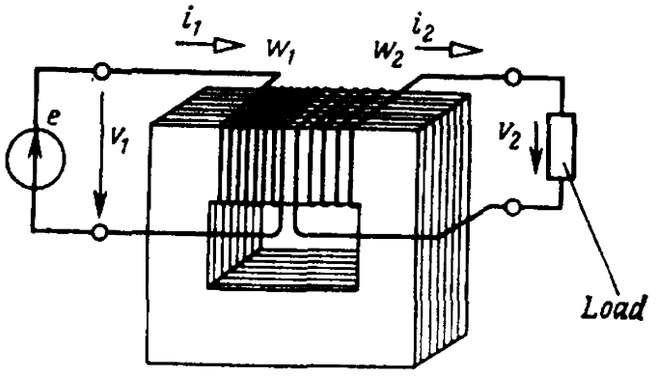


FIGURE 8.1

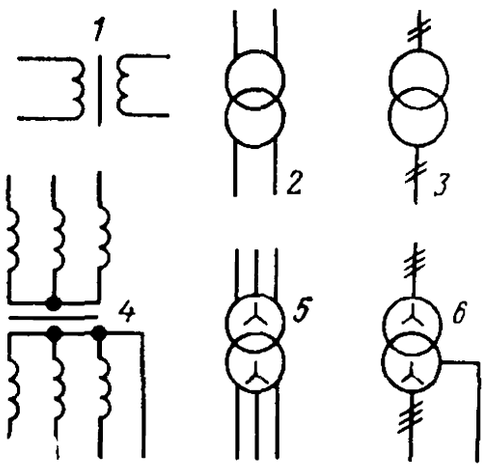


FIGURE 8.2

of the core is to enhance the inductive coupling between the windings, and the objective of using a laminated core is to reduce eddy currents (Fig. 8.1). No cores are used in air-cored transformers operating at frequencies above around 20 kHz, since the core is not magnetized all the same because of a marked increase in eddy currents.

The transformer winding connected across a supply source (which may be a power line, a generator, etc.) is called *primary*. All the quantities associated with this winding are also called primary (the primary turns, the primary voltage, the primary current, etc.), and their symbols are indexed "1", for example, w_1, v_1, i_1 (see Fig. 8.1). The transformer winding connected across a load is called *secondary*, and all the quantities associated with it are referred to as secondary and indexed "2".

Transformers may be single-phase (for single-phase circuits) and three-phase (for three-phase circuits). In a three-phase transformer,

the *primary* or *secondary* winding refers to the three phase windings carrying the respective (primary or secondary) voltage. The graphical symbols most widely used in the USSR for single-phase transformers (1, 2, 3) and for three-phase transformers (4, 5, 6) are shown in Fig. 8.2.

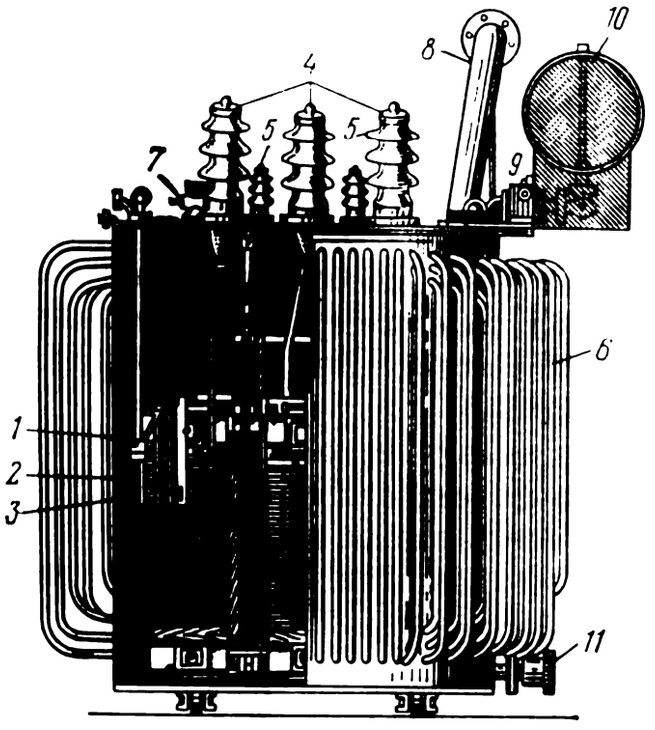


FIGURE 8.3

Each transformer has a nameplate which bears all of its ratings. These include the voltage on the *high-voltage* (*H.V.*) *side*, the voltage on the *low-voltage* (*L.V.*) *side*; its apparent power (in VA or kVA); its current (in A) at the rated apparent power; the frequency; the number of phases; the phase interconnection; its duty (long-term or short-term); and the method of cooling.

According to the manner in which transformers are cooled, they are classed into dry and oil-immersed. In the latter case, the core-and-coils unit is immersed in an oil-filled steel tank. A cut-away view of an oil-immersed transformer with a tubular tank is shown in Fig. 8.3 where 1 is the core, 2 is the L.V. winding, 3 is the H.V. winding, 4 are the H.V. terminal bushings, 5 are the L.V. terminal bushings, 6 is the tubular tank, 7 is the oil-filling cock, 8 is the explosion stack, 9 is the Buchholtz gas relay, 10 is the oil conservator, and 11 is the oil drain cock.

If the primary voltage V_1 is lower than the secondary voltage V_2, we have a *step-up transformer*; otherwise ($V_1 > V_2$) we have a *step-down transformer*.

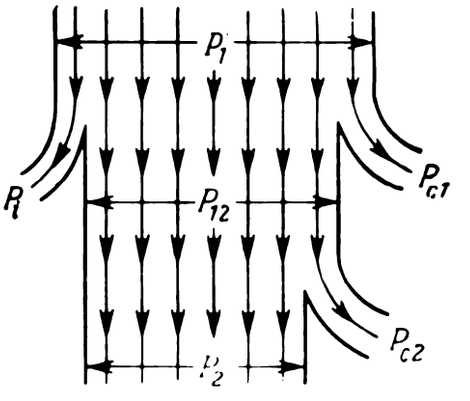


FIGURE 8.4

Figure 8.4 shows the distribution of power in a transformer. Here P_1 is the power in the primary winding; $P_{c.1}$ is the copper loss in the primary winding; P_1 is the iron (or core) losses which include both hysteresis losses and eddy-current losses; $P_1 - P_{c.1} - P_1 = P_{12}$ is the power in the secondary winding which includes $P_{c.2}$, or the copper losses in the secondary, with the remainder P_2 being the power supplied by the transformer to the associated load, such that $P_2 = P_{12} - P_{c.2} = P_1 - P_{c.1} - P_1 - P_{c.2}$. In operation under

rated conditions, the difference $P_1 - P_2$, or the total losses in the transformer, accounts for as little as 1-2% of the rated power in the primary winding. In the diagram, the loss fraction is shown exaggerated for better presentation.

A single-phase transformer operates in practically the same way as one phase of a three-phase transformer. Therefore, in order to simplify the presentation, we will first analyse the operation of a two-winding, single-phase transformer, and the findings will then be extended to include what concerns three-phase transformers only.

8.2 Operation of a Single-Phase Transformer

In sketch form, the arrangement of a single-phase transformer is shown in Fig. 8.5 a. On the secondary side, where the winding has w_2 turns, the transformer acts as a supply source for a load

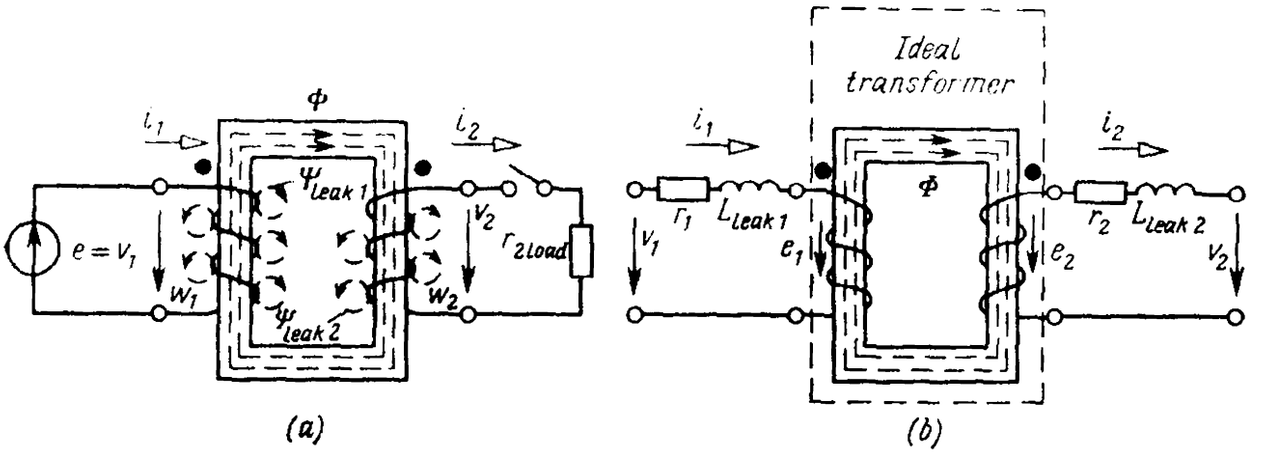


FIGURE 8.5

whose resistance is $r_{2,\,load}$. On the primary side, where the winding has w_1 turns, it acts as a load with respect to its own power supply.

To begin with, suppose that the secondary circuit of the transformer is open and that the supply voltage $v_1 = e$ causes a current i_1 to flow in its primary winding. The mmf $i_1 w_1$ induces in the core a magnetic flux whose positive direction is determined by the corkscrew rule (see Fig. 2.1 a). This flux induces an emf of self-induction, e_{L1}, in the primary and an emf of mutual induction, e_{M2}, in the secondary. (Neither of the emfs is shown in the diagram.) When the secondary circuit is completed, the emf of mutual induction e_{M2} gives rise to a current, i_2, in the load resistance $r_{2,\,load}$.

For the polarity of the primary and secondary windings indicated in Fig. 8.5 and the assumed positive directions for the currents i_1 and i_2, the magnetic flux induced in the core by the mmf $i_2 w_2$ opposes the magnetic flux due to the mmf $i_1 w_1$. In consequence, the primary and secondary windings are connected in opposition, which fact is

stated by assigning the same labels to the winding terminals as were used in Fig. 2.50 c. Therefore the total mmf of the primary and secondary windings is equal to $i_1 w_1 - i_2 w_2$. This mmf induces a common magnetic flux Φ in the core. Also, in analysing the operation of the transformer, we should include the leakage flux linkage of the primary, $\Psi_{leak,1}$, and of the secondary, $\Psi_{leak,2}$, each proportional to the respective current, i_1 or i_2.

Figure 8.5 b shows an equivalent circuit of the transformer; it is made up of the resistances r_1 and r_2 of the primary and secondary windings, and of the respective leakage inductances, $L_{leak,1} = \Psi_{leak,1}/i_1$ and $L_{leak,2} = \Psi_{leak,2}/i_2$ (similarly to Fig. 7.4).

It is useful to introduce the concept of an *ideal transformer*, that is, one in which the primary and secondary windings have neither resistance nor leakage flux linkage. In Fig. 8.5 b it is enclosed in the dashed box. The positive directions for e_1 and i_1 in its primary are the same as they are in an iron-cored coil (see Fig. 7.4) to which the transformer reduces when its secondary circuit is opened. Since the primary emf

$$e_1 = -w_1 \, d\Phi/dt$$

and the secondary emf

$$e_2 = -w_2 \, d\Phi/dt$$

are induced by the same magnetic flux Φ in the core, their positive directions relative to the like terminals of the two windings are the same. While in the primary winding e_1 and i_1 are in the same direction, in the secondary e_2 and i_2 are assigned opposite directions. This corresponds to the physical interpretation of the role played by the emf: in the former case it opposes any change in the current; in the latter case, it gives rise to a current.

8.3 The Equation of an Ideal Single-Phase Transformer

To begin with, we will consider an ideal single-phase transformer in which the core is fabricated from a ferromagnetic material having a linear B/H relation, $B = \mu_r \mu_0 H$ (see Fig. 6.6 c).

As has been noted in Sec. 6.3, the magnetic field in a core is not the same over its cross-sectional area S. For simplicity, we will ignore this fact and assume that both the magnetic induction and the magnetic intensity in the transformer core are everywhere the same as they are at the central magnetic line. If so, the electric circuit of the transformer is linear, and it can be analysed using the complex-number method.

In Fig. 8.6 an ideal single-phase transformer is shown connected between a source of emf $\dot{E}$ and a load of complex impedance $\underline{Z}_2 = z_2 \angle \varphi_2$.

Let us write the values of $\dot{E}_1$ and $\dot{E}_2$ induced in the primary and secondary windings of the ideal transformer by the magnetic flux $\dot{\Phi}$

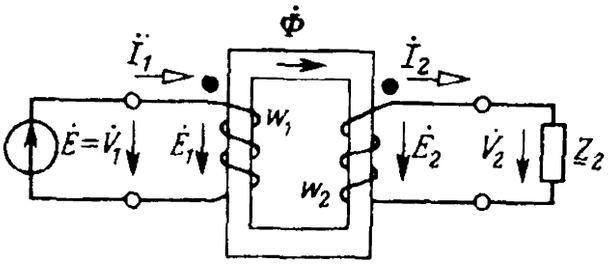


FIGURE 8.6

in the core. By the law of electromagnetic induction in complex form, Eq. (2.33)

$$\dot{E}_1 = -j\omega w_1 \dot{\Phi} = -j\omega w_1 \dot{B} S = -j\omega w_1 \mu_r \mu_0 \dot{H} S \tag{8.1a}$$

$$\dot{E}_2 = -j\omega w_2 \dot{\Phi} = -j\omega w_2 \dot{B} S = -j\omega w_2 \mu_r \mu_0 \dot{H} S \tag{8.1b}$$

where $\dot{B}$ is the complex magnetic induction, and $\dot{H}$ is the complex magnetic intensity.

If the complex currents in the primary and secondary windings of an ideal single-phase transformer are $\dot{I}_1$ and $\dot{I}_2$, then, by virtue of Eq. (6.2), the average magnetic intensity, that is, one at the centre line of the core, is

$$\dot{H} = \dot{I}_1 w_1 / l_{av} - \dot{I}_2 w_2 / l_{av} \tag{8.2}$$

By definition, the source emf is $\dot{E} = \dot{V}_1$, the primary emf is $\dot{E}_1 = -\dot{V}_1$, and the secondary emf is $\dot{E}_2 = -\dot{V}_2$ (see Fig. 8.6). Therefore, subject to Eqs. (8.1) and (8.2),

$$\dot{V}_1 = j\omega w_1 \dot{\Phi} = j\omega w_1 \mu_r \mu_0 S \, (\dot{I}_1 w_1 / l_{av} - \dot{I}_2 w_2 / l_{av}) \tag{8.3a}$$

$$\dot{V}_2 = \underline{Z}_2 \dot{I}_2 = j\omega w_2 \dot{\Phi} \tag{8.3b}$$

Notably, at no-load, or in the open-circuit condition, (which implies that the secondary circuit is open and $\dot{I}_2 = 0$)

$$\dot{V}_1 = j\omega w_1 \mu_r \mu_0 S \, (\dot{I}_{1.oc} w_1 / l_{av}) \tag{8.3c}$$

where $\dot{I}_{1.oc}$ is the *open-circuit* (or *no-load*) *current* also called the *magnetizing current.*

Since the source emf, $\dot{E} = \dot{V}_1$, is a constant quantity, then, by virtue of Eqs. (8.3a) and (8.3c)

$$\dot{I}_1 w_1/l_{av} - \dot{I}_2 w_2/l_{av} = \dot{I}_{1.oc} w_1/l_{av} = \text{const} \tag{8.4}$$

On dividing Eq. (8.3b) by (8.3a) termwise, we get

$$V_2/V_1 = w_2/w_1 = n_{21} \tag{8.5}$$

which is the *turns* (or *transformation*) *ratio* of an ideal single-phase transformer. On substituting for $\dot{\Phi}$ from (8.3b) into (8.3a), we get

$$\dot{V}_1 = \underline{Z}_2 (w_1/w_2) \dot{I}_2 \tag{8.6}$$

Now we multiply and divide the right-hand side of Eq. (8.6) by w_1/w_2

$$\dot{V}_1 = \underline{Z}_2 (w_1/w_2)^2 (w_2/w_1) \dot{I}_2 = \underline{Z}'_2 \dot{I}'_2 \tag{8.7}$$

where

$$\underline{Z}'_2 = \underline{Z}_2 (w_1/w_2)^2 = \underline{Z}_2/n_{21}^2 \tag{8.8}$$

is the complex impedance of the secondary circuit referred (or transferred) to the primary side or, simply, the *referred* (*transferred*) *impedance*, and

$$\dot{I}'_2 = (w_2/w_1) \dot{I}_2 = n_{21} \dot{I}_2 \tag{8.9}$$

is the complex current in the secondary circuit, referred (or transferred) to the primary side or, simply, the *referred* (*transferred*) *current.*

By invoking the concepts of transferred current and impedance we may re-cast Eqs. (8.4) and (8.3) as

$$I_1 - \dot{I}'_2 = I_{1.oc} \tag{8.10a}$$

$$\dot{V}_1 = j \frac{\omega w_1^2 \mu_r \mu_0 S}{l_{av}} (\dot{I}_1 - \dot{I}'_2) = j\omega L_1 \dot{I}_{1.oc} = j x_{L1} \dot{I}_{1.oc} \tag{8.10b}$$

$$\dot{V}'_2 = (w_1/w_2) \dot{V}_2 = \dot{V}_2/n_{21} = \underline{Z}'_2 \dot{I}'_2 \tag{8.10c}$$

where $w_1^2 \mu_r \mu_0 S/l_{av} = L_1$ is the inductance of the primary winding in an ideal single-phase transformer, and $\dot{V}'_2$ is the complex secondary voltage of an ideal single-phase transformer, referred (or transferred) to the primary side or, simply, the *referred* (*transferred*) *voltage.*

Equations (8.10) correspond to the equivalent circuit shown in Fig. 8.7 in which the ideal transformer model is enclosed in a dashed box.

A further assumption made for an ideal transformer is that at $\mu_r = \infty$, the inductive reactance x_{L1} is infinitely large and the magnetizing current is $I_{1.oc} = 0$.

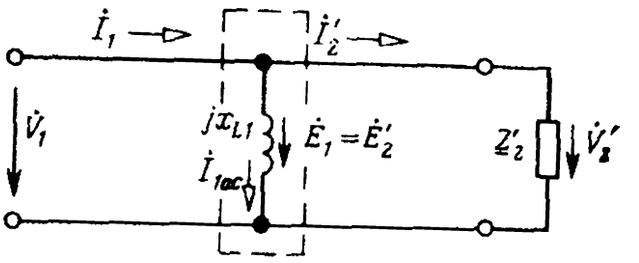


FIGURE 8.7

With the secondary circuit opened, an ideal single-phase transformer reduces to an ideal iron-cored coil. As a consequence, an ideal single-phase transformer at no-load has the same equivalent circuit as an ideal iron-cored coil (see Fig. 7.4) on the condition that the coil and the primary winding of the transformer have the same number of turns and the same cores.

8.4 The Equivalent Circuit and Phasor Diagram for an Ideal Single-Phase Transformer

At this stage we will consider an ideal single-phase transformer in which the core is fabricated from a ferromagnetic material whose hysteresis has to be included in consideration (see Fig. 6.6 b).

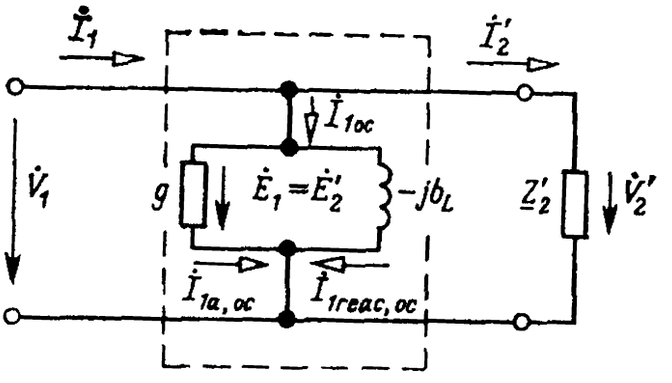


FIGURE 8.8

With the secondary circuit open-circuited, this ideal single-phase transformer has the same equivalent circuit as the ideal coil enclosed in the dashed box in Fig. 7.7 b. On replacing the hysteresis loop of the core with an equivalent ellipse (see Fig. 7.6), we are in a position to determine the conductance g and the inductive susceptance b_L of the ideal coil (see Sec. 7.4). When the secondary circuit is completed through a load (the transformer is in the on-load condition), its equivalent circuit becomes like that shown in Fig. 8.8. Here the ideal single-phase transformer proper is enclosed in a dashed

box, and the referred (transferred) secondary circuit is the same as has been considered earlier (see Fig. 8.7).

The equivalent-circuit parameters g and b_L of an ideal single-phase transformer with a core whose hysteresis has to be included in consideration are frequency-dependent. This is because the area of the dynamic hysteresis loop of the core depends on the frequency of the magnetizing current (see Sec. 7.5).

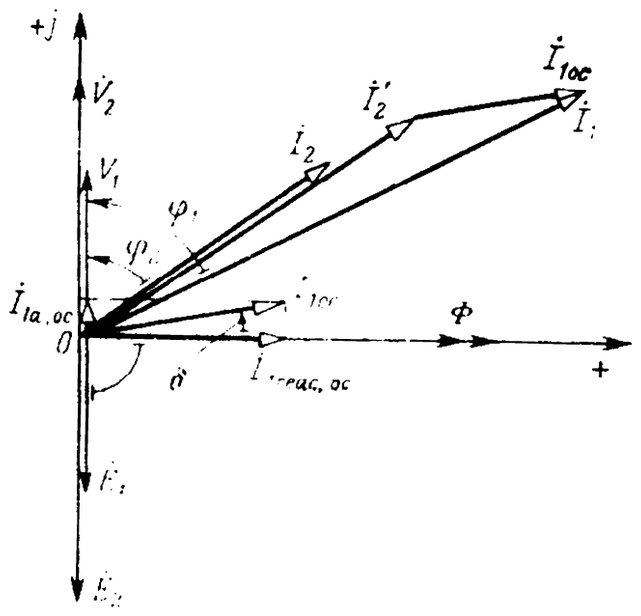


FIGURE 8.9

Figure 8.9 shows the phasor diagram of a loaded ideal single-phase transformer. The magnetic flux phasor $\dot{\Phi}$ is assigned an initial phase of zero. The magnetizing current phasor $\dot{I}_{1,oc}$ leads the magnetic flux phasor $\dot{\Phi}$ by the loss angle δ, much as the current phasor $\dot{I}$ does on the phasor diagram for an iron-cored coil (see Fig. 7.8). The phasors representing the emfs $\dot{E}_1$ and $\dot{E}_2$ induced in the primary and secondary windings of an ideal transformer lag behind the magnetic flux phasor by an angle $\pi/2$, as they should in accord with Eq. (8.1). The phasors $\dot{V}_1$ and $\dot{V}_2$ respectively representing the voltages between the terminals of the primary and secondary windings are the same in length as the emf phasors $\dot{E}_1$ and $\dot{E}_2$, but, as follows from Eq. (8.3), the voltage phasors lead the flux phasor $\dot{\Phi}$ by an angle $\pi/2$.

Given the complex load impedance of an ideal transformer, $\underline{Z}_2 = z_2 \angle \varphi_2$, we can apply Ohm's law to find the secondary current $\dot{I}_2 = \dot{V}_2/\underline{Z}_2$ (in the diagram the current phasor $\dot{I}_2$ is plotted for $\varphi_2 > 0$, that is for an inductive load impedance) and the primary current

$$\dot{I}_1 = \dot{I}_2' + \dot{I}_{1,oc} = (w_2/w_1)\,\dot{I}_2 + \dot{I}_{1,oc}$$

8.5 Equations, Equivalent Circuit and Phasor Diagram for a Real Single-Phase Transformer

Now we will set up an equivalent circuit for a real single-phase transformer (see Fig. 8.5 b) which has an ideal single-phase transformer as an integral part.

The equivalent circuit of a real single-phase transformer is shown in Fig. 8.10 where r_1 is the resistance of the primary winding, $x_{\text{leak},1} = \omega L_{\text{leak},1}$ is its inductive leakage reactance, $r_2' = r_2 \times (w_1/w_2)^2$ is the referred resistance of the secondary winding, and $x'_{\text{leak},2} = \omega L'_{\text{leak},2} = \omega L_{\text{leak},2} \times (w_1/w_2)^2$ is the referred inductive leakage reactance of the secondary winding. The part corresponding to an ideal single-phase transformer is enclosed in the dashed box.

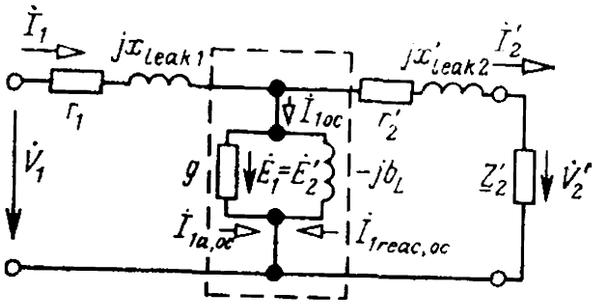


FIGURE 8.10

Referring to the above equivalent circuit, we can write the following Kirchhoff voltage law equations:

$$\dot{V}_1 = -\dot{E}_1 + r_1\dot{I}_1 + jx_{\text{leak},1}\dot{I}_1 = -\dot{E}_1 + \underline{Z}_1\dot{I}_1 \tag{8.11a}$$

$$\dot{V}_2 = -\dot{E}_2 - r_2\dot{I}_2 - jx_{\text{leak},2}\dot{I}_2 = -\dot{E}_2 - \underline{Z}_2\dot{I}_2 \tag{8.11b}$$

where $\underline{Z}_1 = r_1 + jx_{\text{leak},1}$ and $\underline{Z}_2 = r_2 + jx_{\text{leak},2}$ are the complex impedances of the primary and secondary windings, including their resistances and leakage reactances.

Figure 8.11 shows a phasor diagram for a real single-phase transformer. It is similar to that for an ideal single-phase transformer (see Fig. 8.9).

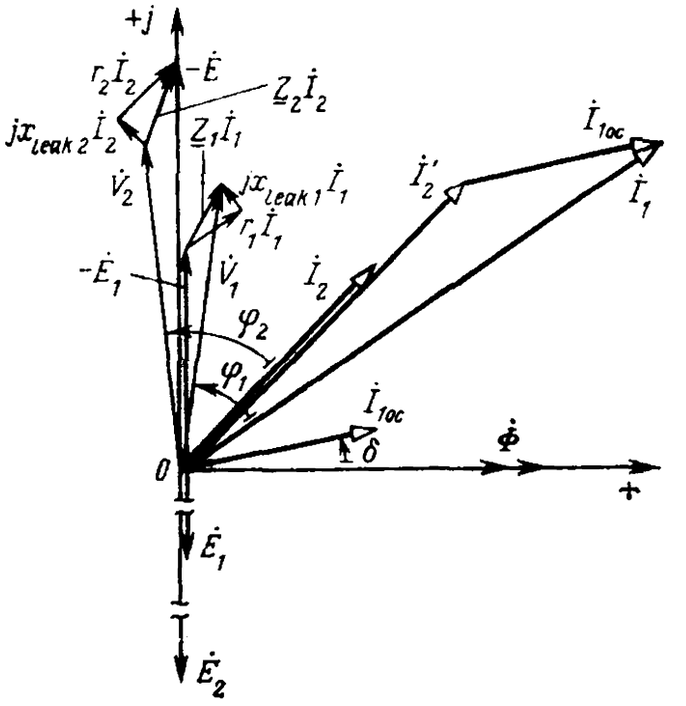


FIGURE 8.11

It follows from the equations of and the phasor diagram for a real single-phase transformer that the relation between the rms voltages across the primary and secondary windings is not the same as that between the rms emfs induced in the same windings by the core magnetic flux Φ. The rms voltages z_1I_1 and z_2I_2 are called the *total internal voltage drops* across the respective windings. It should be remembered that a referred (or transferred) phasor diagram is correct only as far as the qualitative relations between the quantities are concerned. In most practical cases the internal voltage drop tri-

angles are small, that is, $V_1 \approx E_1$ and $V_2 \approx E_2$, so it may be deemed that

$$V_2/V_1 \approx E_2/E_1 = w_2/w_1 = n_{21} \tag{8.12}$$

Note that the magnetizing current in a real transformer depends on its load, i.e. on the current $\dot{I}_2$. This may be explained by the fact that the current in and the total internal voltage drop, $z_1 I_1$, across the primary winding vary with the load. In most cases, however, the voltage drop, $z_1 I_1$, is much lower than the supply voltage, V_1, therefore the magnetizing current may be considered equal to the open-circuit current of the transformer at $\dot{I}_2 = 0$.

A transformer for which the apparent power rating is $S_{rtd} = S_{1.rtd} = V_{1.rtd} I_{1.rtd}$ may be in any one of several conditions, namely:

(1) The *rated condition* when $V_1 = V_{1.rtd}$ and $I_1 = I_{1.rtd}$.

(2) The *operating condition* when the primary voltage is very close or equal to the rated voltage, $V_1 \approx V_{1.rtd}$, and the primary current I_1 depends on the load sustained by the transformer.

(3) The *open-circuit condition* when the secondary of the transformer is open ($I_2 = 0$) or connected across a load of a very high impedance (such as a voltmeter).

(4) The *short-circuit condition* in which its secondary is short-circuited ($V_2 = 0$) or connected across a load of a very low impedance (such as an ammeter).

The open-circuit and short-circuit conditions may be produced deliberately for testing purposes, or occur spontaneously as a result of a fault.

8.6 The Open-Circuit Condition of a Transformer

In the open-circuit condition (Fig. 8.12) a transformer is effectively turned into an iron-cored coil whose winding w_1 is connected across a source of a sinusoidal voltage. For this reason, the phasor diagram for a transformer in the open-circuit condition (Fig. 8.13) is not unlike that of an iron-cored coil (see Fig. 7.8), differing, as it does, only in some symbols and in the added secondary emf phasor $\dot{E}_{2.oc}$.

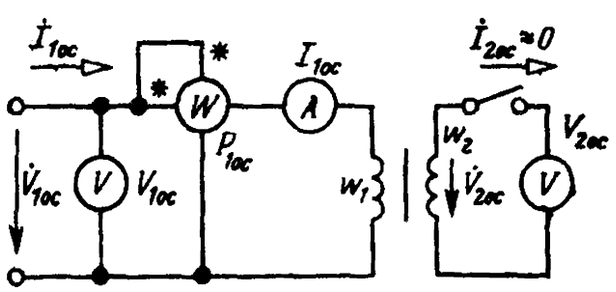


FIGURE 8.12

The open-circuit condition is used for testing purposes in which case it is known as the *open-circuit test* (or the *core loss and magnetizing-current test*). In this form of test, the normal rated voltage, $V_{1.oc} = V_{1.rtd}$, is applied to the primary winding, while the secon-

dary winding is open-circuited. On the basis of the test data, one finds the turns (or transformation) ratio and the core losses. The open-circuit test is one of the mandatory tests carried out at the maker's works on a finished transformer.

When the voltage applied to the primary is $V_{1,oc} = V_{1,rtd}$, the open-circuit current $I_{1,oc}$ is anywhere between 3% and 10% of the rated primary current $I_{1,rtd}$ (this value being the lower, the higher the apparent power rating of the transformer). So, by Eq. (8.11a), $V_{1,oc} \approx E_{1,oc}$. Also, when the secondary winding is open-circuited, $V_{2,oc} = E_{2,oc}$. Therefore, once $V_{1,oc}$ and $V_{2,oc}$ are measured with a voltmeter, it is a simple matter to find the transformation (turns) ratio

$$n_{21} = w_2/w_1 = E_{2,oc}/E_{1,oc} = V_{2,oc}/V_{1,oc} \qquad (8.13)$$

The transformation (or turns) ratio is indicated on the nameplate of each transformer as the ratio of the rated voltages at no-load, such as "6000/230 V".

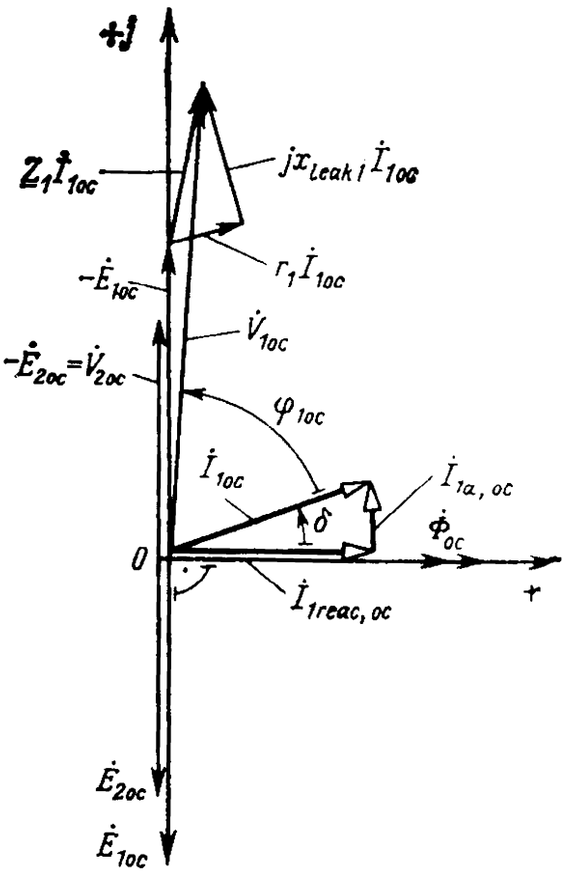


FIGURE 8.13

The losses in the open-circuit condition are the sum of the core losses and the losses in the primary winding, $r_1 I_{1,oc}^2$. However, $I_{1,oc}$ is only a small fraction of $I_{1,rtd}$, so the second term in the sum of losses is negligible in comparison with the core losses (except, perhaps, for transformers rated at less than 1000 V A). This is the reason why the test is referred to, among other things, as the *core loss test.*

The core losses in the open-circuit conditions are small. For transformers rated at 5-50 kV A, they are a mere 0.9-1.4% of the rated power. For transformers rated at 1-10 MV A the figure is as low as 0.3-0.5%. Nevertheless, the core losses are an important factor because power transformers are disconnected from power lines only seldom.

The open-circuit, or magnetizing, current $\dot{I}_{1,oc}$, is, as is the case with an iron-cored coil, a sum of two components, one active, $\dot{I}_{1a',oc}$, and the other reactive, $I_{1reac,oc}$. The active component is associated with hysteresis and eddy currents and stray losses*. It is useful

* These include losses due to the vibration of core laminations at the joints and around the bolts due to variations in induction, losses in structural parts due to the leakage flux, etc. The total is 15-20% of the overall core losses.

to remember that in most cases $I_{1a,oc}$ is less than 0.1 $I_{1reac,oc}$, so it is legitimate to take it that the magnetizing current is practically in quadrature lagging with the primary voltage, that is, $\varphi_{1,oc} \approx \pi/2$ (see Fig. 8.13).

The core loss and magnetizing current test as described above is the principal form of test in the open-circuit condition. In some cases, however, such as when the cooling capability is limited it is important to know how the open-circuit condition will change due to changes in the primary voltage. The relations thus obtained, $P_{1,oc}=f(V_{1,oc})$ and $I_{1,oc}=F(V_{1,oc})$ are called the *open-circuit characteristics* of a transformer (Fig. 8.14). At the beginning of this test the primary voltage is zero. As it is gradually raised, $I_{1,oc}$ is rising at first (so long as the core is not saturated) in proportion to the applied voltage. Some time later, the core reaches saturation (this happens at approximately $V_{1,oc} > 0.8\ V_{1,rtd}$), and $I_{1,oc}$ builds up rapidly.

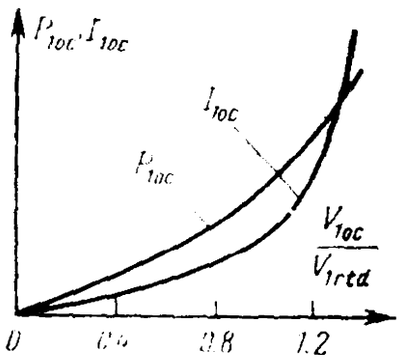


FIGURE 8.14

The core losses are proportional to B^2 (see Eq. (7.11) and Eq. (7.12)), and, as a consequence, to Φ^2. Since the rms emf $E_{1,oc}$ is proportional to $\Phi = \Phi_m/\sqrt{2}$ (see Eq. (8.1a)) and $E_{1,oc} \approx V_{1,oc}$, it follows that $P_{1,oc}$ is proportional to $V_{1,oc}^2$.

For most transformers, the upper limit of voltage in the open-circuit condition is 1.1 of the rated voltage, which fact must be remembered when experimenting with transformers.

8.7 The Short-Circuit Condition of a Transformer

This condition may arise as a result of a fault fraught with grave consequences (overheating and break-down), or it may be produced for testing purposes in which case it is called the *short-circuit* (or *load loss* and *impedance*) *test*.

The short-circuit test is carried out by short-circuiting the secondary winding and applying a low voltage to the primary winding sufficient to cause the rated full-load current, $I_{1,sc} = I_{1,rtd}$, to flow. The objective of the test (Fig. 8.15) is to measure the power dissipated in the conductors, the impedance voltage, etc. It is one of the mandatory tests applied at the maker's works to a finished transformer.

In the short-circuit test when $V_2 = 0$, the emf $E_{2,sc}$ induced in the secondary winding is, as follows from Kirchhoff's voltage law, Eq. (8.11b), equal to the sum of the voltages across the resistance

and inductive leakage reactance of the secondary winding

$$\dot{E}_{2,sc} = -(r_2 + jx_{leak,2})\,\dot{I}_{2,sc}$$

while under the operating conditions

$$\dot{E}_2 = -(\dot{V}_2 + r_2\dot{I}_2 + jx_{leak,2}\dot{I}_2)$$

The primary voltage applied in the short-circuit test, $V_{1,sc}$, in order to cause a full-load rated current, $I_{1,sc} = I_{1,rtd}$, to flow amounts approximately to 5-10% of the rated primary voltage, $V_{1,rtd}$. Therefore the rms secondary emf, $E_{2,sc}$, is a mere 2% to 5% of the rms emf, E_2, under normal operating conditions. The magnetic flux in the core, Eq. (8.1b), decreases in proportion to the emf, and so do the magnetizing current and the core losses which are proportional to Φ^2. In consequence, it is legitimate to think that during a short-circuit test all of the primary power in the transformer, $P_{1,sc}$, is equal to the copper (conductor) losses in the primary and secondary windings

$$P_{1,sc} = r_1 I_{1,sc}^2 + r_2 I_{2,sc}^2 = r_1 I_{1,sc}^2 + r_2' I_{2,sc}'^2$$

The magnetizing current $I_{1,oc}$ is negligibly small, therefore, by Eq. (8.10a), $I_{1,sc} = I'_{2,sc}$, and

$$P_{1,sc} = (r_1 + r_2')\, I_{1,sc}^2 = r_{sc} I_{1,sc}^2 \tag{8.14}$$

As the apparent power rating, S_{rtd}, of a transformer is increased, the short-circuit active power, $P_{1,sc}$, goes down in relative terms. For S_{rtd} = 20-5 kV A, the ratio $P_{1,sc}/S_{rtd}$ is 3-3.7%, while for S_{rtd} = 320-5600 kV A, the figure is 2-1%.

Using the value of load loss found in a short-circuit test, $P_{1,sc}$, and the full-load rated current $I_{1,sc} = I_{1,rtd}$, we can, on the basis of Eq. (8.14), determine the *short-circuit resistance* of the transformer

$$r_{sc} = P_{1,sc}/I_{1,sc}^2 \tag{8.15}$$

To keep the size of a transformer to a reasonable minimum, the designer has to choose the current density in the conductors and the magnetic flux density (magnetic induction) in the core such that the temperature rise of the transformer will remain within the safe limits. For this reason, in finding the load loss, the value of r_{sc} found by a short-circuit test must be re-scaled to a temperature of 75°C (see Table 1.2).

The *short-circuit inductive reactance* of a transformer may be deemed independent of temperature. Therefore, it is found directly from the results of a short-circuit test

$$x_{sc} = \sqrt{z_{sc}^2 - r_{sc}^2} = \sqrt{(V_{1,sc}/I_{1,sc})^2 - r_{sc}^2} \tag{8.16}$$

Hence the *short-circuit impedance* of a transformer, as re-scaled to an operating temperature of 75°C, is

$$z_{sc} = \sqrt{r^2_{sc,\,75°} + x^2_{sc}} \tag{8.17}$$

Figures 8.16 a and b show an impedance triangle and a geometrically similar *short-circuit triangle* in which the sides represent the active and reactive components of the short-circuit primary voltage, $V_{1,sc}$, as a percentage of the primary rated voltage, $V_{1,rtd}$. The two components are found at $I_{1,sc} = I_{1,rtd}$ so

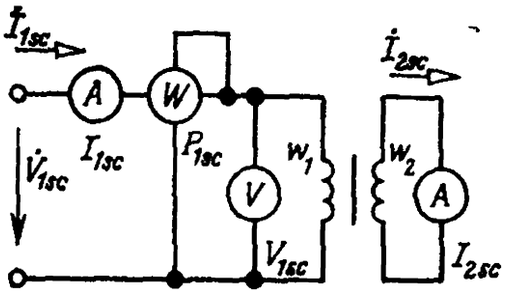


FIGURE 8.15

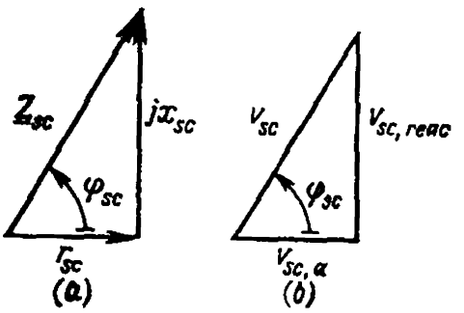


FIGURE 8.16

$$v_{sc,\,a} = (r_{sc} I_{1,\,rtd}/V_{1,\,rtd}) \times 100\%$$

$$v_{sc,\,r} = (x_{sc} I_{1,\,rtd}/V_{1,\,rtd}) \times 100\%$$

and the hypotenuse of the short-circuit triangle is

$$v_{sc} = (z_{sc} I_{1,\,rtd}/V_{1,\,rtd}) \times 100\%$$

where v_{sc} is called the *impedance voltage* of a transformer. It is a very important parameter on the basis of which we can determine variations in the secondary voltage of a loaded transformer (see Sec. 8.8). It is usual to state the impedance voltage of a transformer on its nameplate.

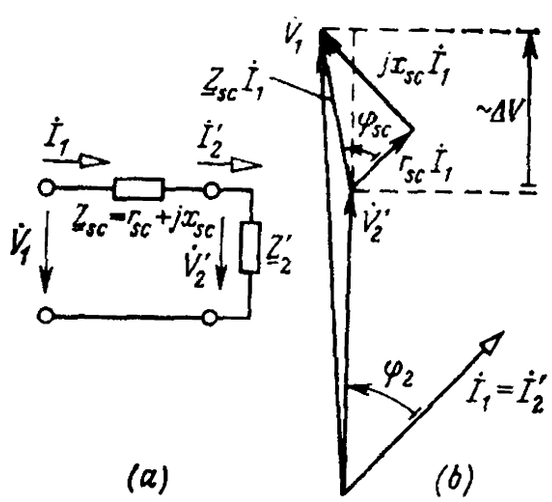


FIGURE 8.17

The value of v_{sc} increases with increasing rated voltages across the transformer windings. The reason is that at an increase in the thickness of conductor insulation, the leakage flux linkages go up, and so do the two leakage reactances, $x_{leak,1}$ and $x_{leak,2}$. In a transformer rated at anywhere between 5000 and 5600 kVA and with a rated voltage of 6.3 and 10 kV on the H.V. side, the impedance voltage is 5.5%. In a transformer rated at anywhere between 3200 and 4200 kVA and

with a rated voltage of 35 kV on the H.V. side, the impedance voltage is 7%.

The short-circuit test can also provide a check on the turns (or transformation) ratio. In the short-circuit condition the term $I_{1,oc}w_1$ in Eq. (8.4) is negligible in comparison with the other two terms, so

$$w_1 I_{1,sc} \approx w_2 I_{2,sc} \tag{8.18}$$

Hence the transformation (turns) ratio is

$$n_{21} = w_2/w_1 \approx I_{1,sc}/I_{2,sc} \tag{8.19}$$

If we neglect the open-circuit (magnetizing) current $I_{1,oc}$ of a transformer and find its r_{sc} and x_{sc} by a short-circuit test, we can draw up a simplified equivalent diagram (Fig. 8.17 a) for which a phasor diagram is drawn up in Fig. 8.17 b.

8.8 External Characteristics of a Transformer

Consider the performance of a transformer for several values of the complex load impedance $\underline{Z}_2 = z_2 \angle \varphi_2$. If the primary voltage is constant and equal to its rated value, $V_1 = V_{1,rtd}$, variations in the complex load impedance will bring about changes in the primary and secondary currents $\dot{I}_1$ and $\dot{I}_2$ and in the secondary voltage $\dot{V}_2$ of the transformer.

The difference between the no-load and full-load secondary voltages expressed in terms of the former, with the primary voltage held constant, is called the *voltage regulation* of a transformer. To find it, it is usual to refer (transfer) the secondary voltage to the primary side. Then, $V_2' = (w_1/w_2)\, V_2$. The no-load secondary voltage is practically equal to $V_{1,rtd}$. Therefore, we may write $V_{1,rtd} - V_2'$. Ordinarily, the voltage regulation of a transformer is expressed as a *percentage of the rated primary voltage*

$$\Delta V\,\% = \frac{V_{1,rtd} - V_2'}{V_{1,rtd}} \times 100\% \tag{8.20}$$

From the simplified equivalent circuit diagram of a transformer (see Fig. 8.17 a) and from its phasor diagram (see Fig. 8.17 b) it follows that

$$V_{1,rtd} - V_2' \approx (r_{sc} \cos \varphi_2 + x_{sc} \sin \varphi_2)\, I_1$$

or, by Eq. (8.20),

$$\Delta V\,\% = (I_1/V_{1,rtd})\,(r_{sc} \cos \varphi_2 + x_{sc} \sin \varphi_2) \times 100\% \tag{8.21}$$

A plot of voltage regulation as a function of the load power factor, $\cos \varphi_2$, for $I_1 = \text{const}$, is drawn up in Fig. 8.18 a. The voltage regulation is a maximum when $\cos \varphi_2 = \cos \varphi_{sc}$, in which case the internal voltage drop phasor $\underline{Z}_{sc}\dot{I}_1$ is in the same direction as the primary voltage phasor $\dot{V}_1$ (see Fig. 8.17 b). In consequence,

$$V_2' = V_1 - z_{sc}I_1$$

By plotting variations in the secondary voltage V_2 as a function of the load current I_2 for a constant load power factor and the rated primary voltage, we obtain what is known as the *external characteristic*

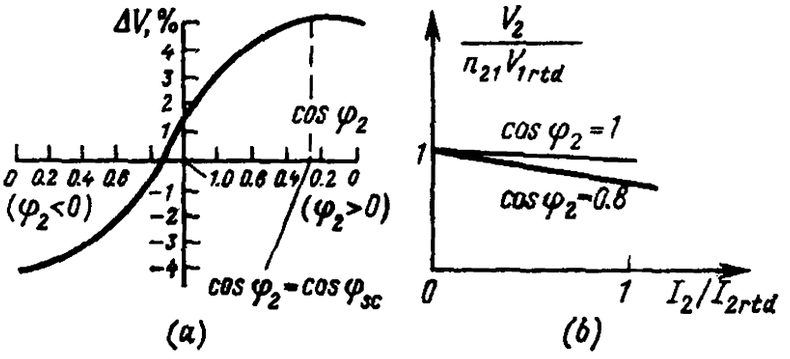


FIGURE 8.18

of a transformer. As often as not, the external characteristic of a transformer is constructed on a per-unit basis, that is, in terms of the ratio of the secondary voltage to its rated value, $V_{2,rtd} = n_{21}V_{1,rtd}$ in the open-circuit condition, and its load current ratio

$$k_{load} = I_2/I_{2,rtd} \tag{8.22}$$

where $I_{2,rtd} = I_{1,rtd}/n_{21}$ is the load current at the rated primary current $I_1 = I_{1,rtd}$. In view of Eqs. (8.20) and (8.21), the external characteristic of a transformer on a per-unit basis is defined by the following equation

$$V_2/n_{21}V_{1,\,rtd} = (1 - \Delta V\,\%/100) = 1 - k_{load}\,(I_{1,\,rtd}/V_{1,\,rtd}) \times (r_{sc}\cos\varphi_2 + x_{sc}\sin\varphi_2)$$

For $\varphi_2 > 0$, the external characteristic is shown in Fig. 8.18 b.

8.9 Transformer Losses

The ratio of the active power P_2 at the output of a transformer to the active power P_1 at its input (see Fig. 8.4) is called its efficiency

$$\eta = P_2/P_1$$

or

$$\eta\% = (P_2/P_1) \times 100 \tag{8.23}$$

When a transformer is operating at its rated primary voltage, $V_1 = V_{1,\text{rtd}}$, its rated primary current, $I_1 = I_{1,\text{rtd}}$, and a load power factor ($\cos \varphi_2$) of over 0.8, the efficiency is very high, being in excess of 99% for power transformers. For this reason it is never found by directly measuring P_1 and P_2. The point is that one would have then to measure P_1 and P_2 to an extremely high level of accuracy not attainable at the existing state of the art. Fortunately, the efficiency of a transformer can be found indirectly from the transformer losses which can be measured directly. Since the transformer losses are

$$\Delta P = P_1 - P_2$$

it follows then that the efficiency is

$$\eta = \frac{P_2}{P_2 + \Delta P} = \frac{P_1 - \Delta P}{P_1} = 1 - \Delta P/P_1 = 1 - \frac{\Delta P}{P_2 + \Delta P} \tag{8.24}$$

The transformer losses are the sum of the core losses, P_c, and the winding losses, P_w. Given $V_1 = V_{1,\text{rtd}}$ and $I_1 = I_{1,\text{rtd}}$, the core losses are equal to the active power of the transformer in the open-circuit test (see Sec. 8.6), and the winding losses (traditionally called the copper losses) are equal to the active power of the transformer in the short-circuit test (see Sec. 8.7).

Consider how the efficiency of a transformer varies in operation at the rated primary voltage $V_1 = V_{1,\text{rtd}}$, into a load whose impedance z_2 is varying and whose power factor $\cos \varphi_2$ remains constant. A change in z_2 brings about a change in the load power, in the winding currents and, as a consequence, in the winding losses and the transformer efficiency. The winding losses are also referred to as the *transient losses* of a transformer, and the core losses are called its *steady-state losses*.

The winding losses (see Fig. 8.17 a) are given by

$$r_{sc}(I_2')^2 = k_{\text{load}}^2 r_{sc}(I_{2,\text{rtd}}')^2 = k_{\text{load}}^2 P_{sc,\text{rtd}}$$

where k_{load} is the load current ratio, Eq. (8.22), and $P_{sc,\text{rtd}}$ is the winding losses at rated currents.

As the secondary current is varied from zero to its rated value, it may be assumed that

$$V_2 \approx \text{const} \approx V_{2,\text{rtd}} = V_{1,\text{rtd}}\, n_{21}$$

The output active power of a transformer is given by

$$P_2 = V_2 I_2 \cos \varphi_2 = k_{\text{load}} V_2 I_{2,\text{rtd}} \cos \varphi_2 \approx k_{\text{load}} S_{\text{rtd}} \cos \varphi_2$$

and the efficiency, in accord with Eq. (8.24), is

$$\eta\% = \left(1 - \frac{\Delta P}{P_2 + \Delta P}\right) \times 100$$

$$\approx \left(1 - \frac{k_{\text{load}}^2 P_{\text{sc. rtd}} + P_c}{k_{\text{load}} S_{\text{rtd}} \cos \varphi_2 + k_{\text{load}}^2 P_{\text{sc. rtd}} + P_c}\right) \times 100$$

As is seen, the efficiency of a transformer depends on the load power factor $\cos \varphi_2$ and the load current ratio k_{load}. If we hold the load power factor constant and equate to zero the derivative of η with respect to k_{load}, we will find that the efficiency of a transformer is a maximum at

$$k_{\text{load}} = \sqrt{P_c / P_{\text{sc. rtd}}}$$

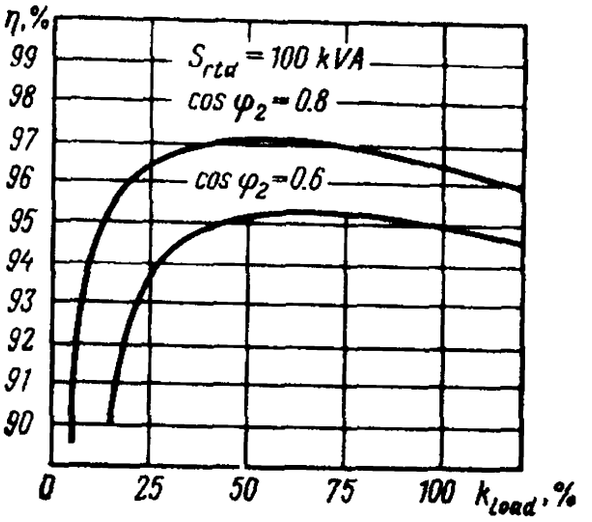


FIGURE 8.19

It follows then that in the case of a maximum load current ratio ($k_{\text{load}} = 1$) a maximum efficiency can be obtained when the core losses are equal to the winding (or copper) losses.

Actually, the load current ratio of a real transformer is never unity. For this reason, transformers are designed so that their efficiency (Fig. 8.19) would be a maximum at some average load. For example, for $P_c/P_{\text{sc,rtd}} = 0.25\text{-}0.5$ the efficiency will be a maximum at $k_{\text{load}} = \sqrt{P_c/P_{\text{sc,rtd}}} = 0.5\text{-}0.7$

8.10 Three-Phase Transformers

All that has been said about a single-phase transformer may be carried over to each phase of a three-phase transformer operating into a balanced load. Let us take a closer look at how a three-phase transformer is arranged and operates.

A three-phase system of currents or voltages might be handled by three single-phase transformers operating as a coordinated unit (Fig. 8.20 a). The actual practice, however, is to combine the three single-phase transformers into a single three-phase unit. In doing so, a good deal of materials is saved, and we will show how this is achieved.

To begin with, we build the core of a three-phase transformer by joining together the cores of the single-phase units and leaving unchanged their parts carrying the windings (see Fig. 8.20 b). This arrangement may be likened to a star connection of three electric circuits (see Fig. 3.4). As will be recalled, no neutral wire is needed for

a three-phase system operating into a balanced load, because no current would flow in it. By doing well without a neutral wire, we save a fairly good quantity of copper. In our combined magnetic circuit, a neutral wire roughly corresponds to the common middle limb (or leg) (see Fig. 8.20 b). In a balanced three-phase system, no common limb is needed and may be dispensed with (Fig. 8.20 c), because the

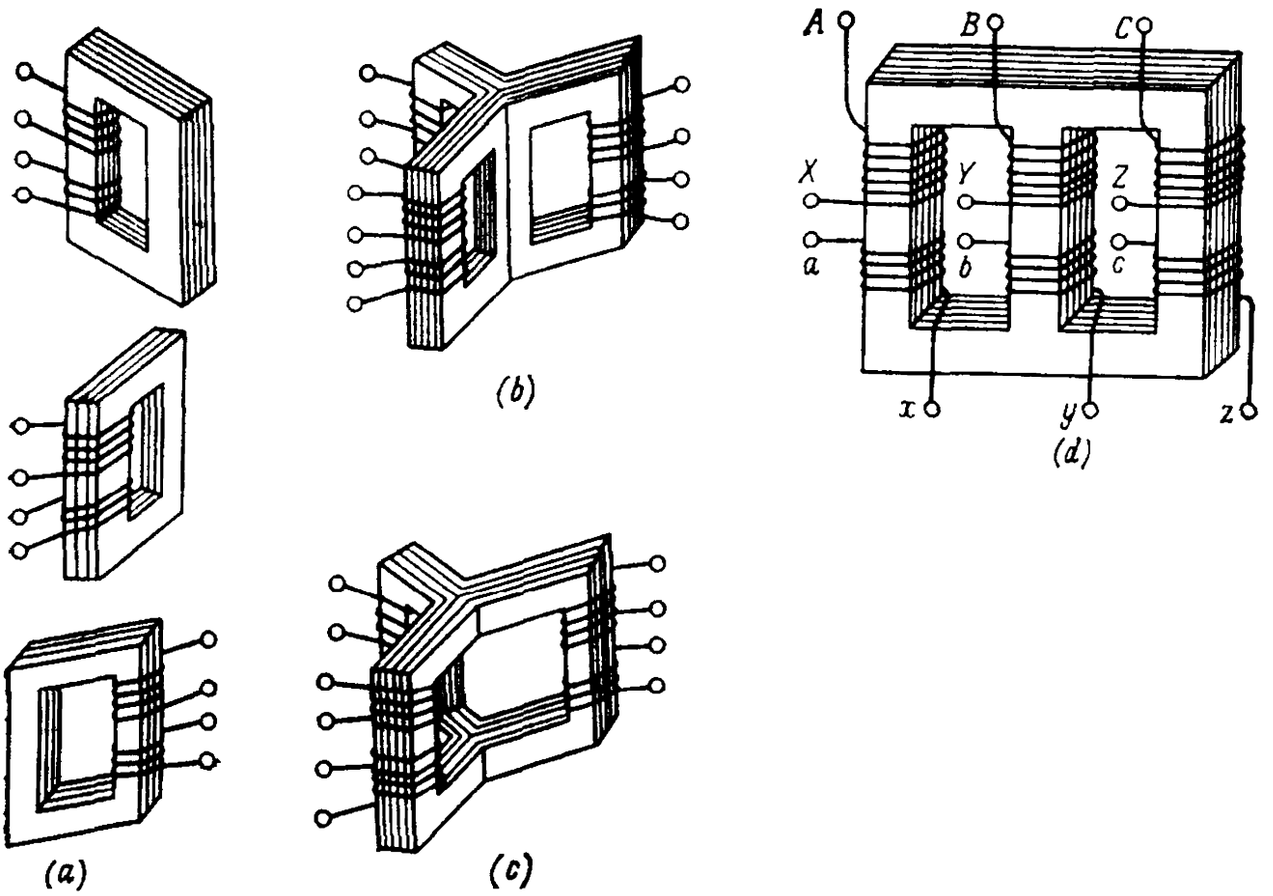


FIGURE 8.20

sum of the instantaneous values of the three magnetic fluxes is zero at any instant.

The symmetrical core of Fig. 8.20 c is poorly adapted to fabrication and is usually replaced with an unsymmetrical one (see Fig. 8.20 d) which may be looked upon as a straightened version of a symmetrical core. In this unsymmetrical core the balanced (symmetrical) system of primary currents induces a symmetrical (balanced) system of magnetic fluxes (see Sec. 7.1). Since, however, the reluctances of the core limbs are all different, the magnetizing currents in the phases are different too. Fortunately, this difference in magnetizing currents between the phases is immaterial.

An assemblage of three single-phase transformers is more expensive than one three-phase transformer of the same power rating; it takes up a larger floor area, and its efficiency is somewhat lower. On the other hand, only one single-phase transformer needs to be kept in reserve for use in an emergency or during repairs, because it is

highly improbable that a fault will occur in the three units at the same time, whereas scheduled maintenance can be carried out on each in turn. With a three-phase transformer, a similar three-phase unit has to be kept in reserve. Thus, with all other conditions being equal, an assemblage of three single-phase transformers is more reliable in service. Last but not least, it is simpler and less expensive to move and set up three single-phase transformers than one three-phase unit of the same aggregate power rating.

Practically all small- and medium-sized transformers are built as three-phase units. In cases involving high-power ratings, the choice is decided in the light of the prevailing conditions. In Soviet practice, three-phase transformers are built in sizes up to 1000 MVA. Beginning at 1.8 MVA, it is allowed to use banks of single-phase transformers for three-phase service.

According to the convention adopted in the USSR, the starts of the phases are labelled A, B, and C on the H.V. side, and a, b, and c on the L.V. side. Their finishes are marked X, Y, and Z on the H.V. side, and x, y, and z on the L.V. side (see Fig. 8.20 d).

The windings of a three-phase transformer may be connected in a star (Y) or in a delta (Δ), the former being simpler and less expensive. In a star-connected transformer with a solidly grounded neutral point, each winding and its insulation must be designed to carry the line current. Since the number of turns in a transformer winding is directly proportional to voltage, Eq. (7.2b), each winding in a star connected transformer needs fewer turns, but heavier conductors. Star connection is widely practised in small and medium transformers (not over 1.8 MVA in size). This arrangement is preferable for applications involving high voltages as the winding insulation has only to carry the phase voltage.

The delta connection of the transformer windings is warranted in cases involving heavy currents. A star-delta (Y-Δ) connection is preferable in high-power transformers if no neutral wire is needed on the L.V. side.

As follows from the relations applicable to a three-phase system (see Secs. 3.2 and 3.3), in three-phase transformers only the phase voltage ratio V_{1ph}/V_{2ph} is approximately equal to the turns ratio, w_1/w_2, and the line voltage ratio depends on how the windings are interconnected. In a star-to-star or a delta-to-delta connection, the line voltage ratio is equal to the phase transformation ratio. In a star-to-delta or a delta-to-star connection the line voltage ratio is smaller or greater than the transformation ratio by a factor of $\sqrt{3}$. Hence, by changing the interconnection of its windings, we can always vary the secondary line voltage of a transformer.

8.11 Phase Displacement (Reference Phasor) Groups

Under normal operating conditions, the primary and secondary voltages v_1 and v_2 are practically in phase with the respective currents, i_1 and i_2, for the positive directions chosen in Fig. 8.21. For the chosen positive directions of the load current i_{load} and load voltage v_{load} it follows that $i_{load} = i_2$ and $v_{load} = v_2$ if the switch *Sw* is in position 1, or $i_{load} = -i_2$ and $v_{load} = -v_2$ if the switch is in position 2. In the former case the load current and voltage are in phase with the secondary current and voltage of the transformer; in the latter case they are in anti-phase. These phase relations are very important when transformers are to be run in parallel, or when using instrument transformers, etc.

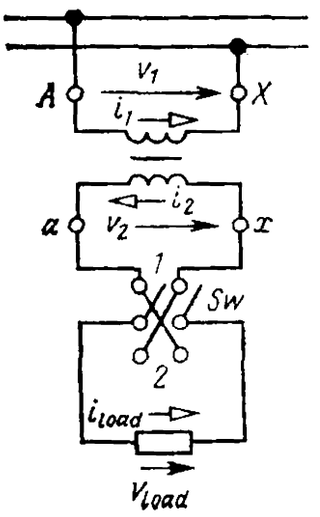


FIGURE 8.21

In terms of the phase difference between the secondary and primary voltages, transformers are classed into phase displacement (or reference phasor) groups. In assigning a phase displacement group to a transformer, its voltage on the H.V. side is assumed to be primary, and that on the L.V. side, secondary.

The phase difference used as a basis for this classification is that between the line high and low voltages. For two transformers falling in the same reference group this phase displacement is the same, and the two are assigned the same reference number.

A convention adopted internationally is to indicate phase displacement as a clock figure representing the hour read by a clock when the minute hand takes the place of the primary line voltage phasor in a complex plane and is set at 12 o'clock, and the hour hand represents the secondary line voltage phasor. Since an angle of 30° is exactly the central angle between adjacent hour markings on a clock dial, it is taken as the unit of phase displacement. The phase displacement is counted from the minute to the hour hand in the same direction as they rotate. The "time" thus read is the reference number assigned.

In a single-phase transformer, the secondary voltage phasor can make with the primary voltage phasor an angle of about 0° or of about 180°. In the former case, the respective clock-hour figure is 12 (or 0), and this phase displacement group is assigned reference number 0. In the latter case, the respective clock-hour figure is 6, and this phase displacement group is assigned reference number 6. In this case the primary and secondary windings are wound in opposite directions with respect to that of the magnetic flux. In the USSR, single-phase transformers come only in reference group 0.

With three-phase transformers, all the 12 clock-hour groups are possible in principle. For standardization purposes, however, only two have been chosen. They are (at least in the Soviet Union) reference group 11 and reference group 0. Transformers in reference group 11 are used with their windings connected star to delta (Y/Δ-11) and star (with the neutral point brought out) to delta (Y_n/Δ-11).

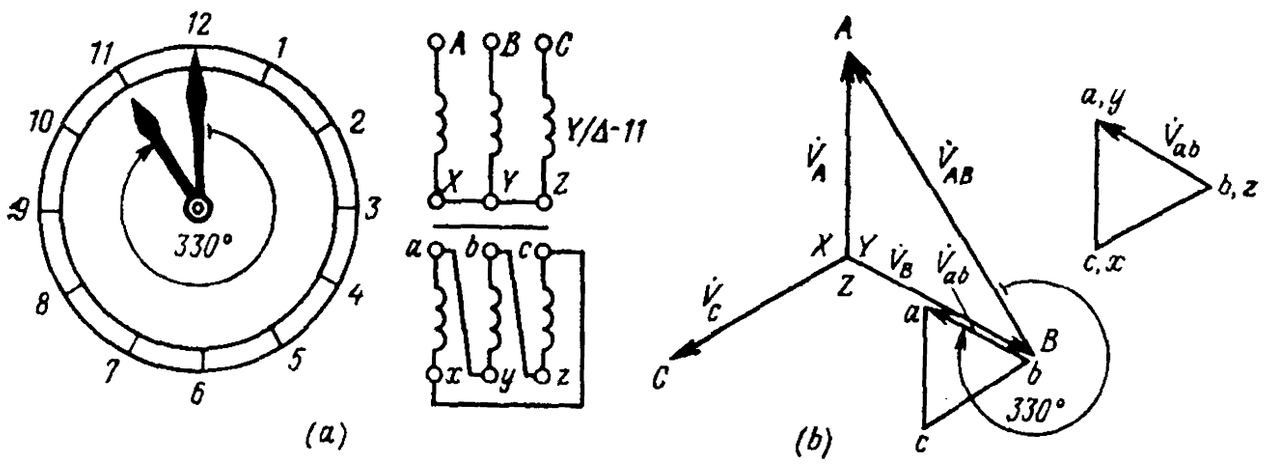


FIGURE 8.22

Transformers in reference group 0 have their windings connected only star to star with the neutral point brought out (Y/Y_n-0). The numerator always refers to the H.V. winding.

The Y/Y_n-0 connection is used for transformers in which the high voltage is up to 35 kV and the low voltage is 230 V, the power rating being up to 560 kVA, or with the same high voltage, a low voltage of 400 V, and a power rating up to 1.8 MVA. Both forms of connection in reference group 11 are used for bigger transformers and at higher voltages.

As an example, Fig. 8.22 shows a star-to-delta connection in which the low (that is, secondary) voltage phasor $\dot{V}_{ab}$ makes an angle of 330° with the high (that is, primary) voltage phasor $\dot{V}_{AB}$. This phase displacement corresponds to a clock-hour figure of 11, so this connection falls in reference phasor group 11.

8.12 Transformers in Parallel

When two (or more) transformers are used in parallel, their primary windings are energized from a common source (a common power line in Fig. 8.23), and their secondary windings are connected across a common load (a common line in Fig. 8.23).

Transformers to be brought in for parallel operation should meet three conditions of which one is mandatory and the remaining two permit some variations. These conditions are as follows.

(1) The transformers should have the same phase displacement, that is belong to the same reference group (for three-phase transformers, this should be either reference group 0 or 11).

(2) The transformers should have the same rated primary and secondary voltages, and the difference in transformation ratio ought not to exceed 0.5%.

(3) The transformers should have the same impedance voltage (to within $\pm$ 10%).

Meeting the first two conditions avoids heavy circulating currents in the transformer windings which would otherwise flow at no-load if the paralleled transformers belonged to different phase displacement groups or differed in secondary emfs.

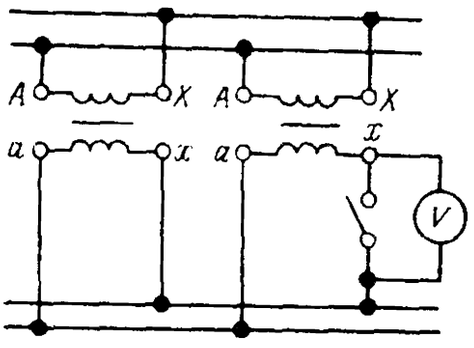


FIGURE 8.23

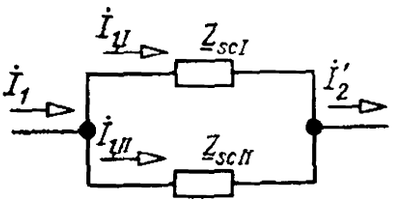


FIGURE 8.24

For single-phase units, a further requirement is the correct polarity on the secondary side. This requirement is checked as follows. One single-phase transformer is connected in parallel with another as shown in Fig. 8.23, and a voltmeter is placed in parallel with the knife-blade switch which serves to turn on the second transformer. If the secondary connection is correct, the voltmeter will read zero; otherwise the voltmeter will read twice the secondary voltage of the transformer.

In the case of failure to meet the first requirement when connecting in parallel two three-phase units, the phase displacement between the secondary line voltages will be 30°. Therefore a resultant emf will be induced in each loop formed by two phase windings (of the two transformers). Since the resistance of the windings is small, this emf will produce a very heavy current detrimental to the transformer windings.

It is important to satisfy the third condition in order that the load will be shared between the parallel transformers in proportion to their apparent power ratings. In the simplified equivalent circuit of Fig. 8.17 a a transformer is represented by a circuit in which the complex short-circuit impedance is $\underline{Z}_{sc}$ (see Sec. 8.7). Two transformers in parallel may be shown in a common equivalent circuit as two parallel branches containing short-circuit impedance $\underline{Z}_{sc,I}$ and

$Z_{sc,II}$ (Fig. 8.24). With this arrangement, the rms currents $I_{1,I}$ and $I_{1,II}$ are inversely proportional to the impedances of the parallel branches

$$I_{1,I}/I_{1,II} = z_{sc,II}/z_{sc,I} \tag{8.25}$$

The impedance voltage of a transformer is proportional to the product of the rated primary current $I_{1,rtd}$ by its short-circuit impedance z_{sc}. If the two paralleled transformers have the same impedance voltage, then

$$z_{sc,I} I_{1,I,rtd} = z_{sc,II} I_{1,II,rtd}$$

Since the transformers in parallel should have the same primary rated voltage, $V_{1,rtd}$, it follows that, on fulfilling the above condition, we have

$$z_{sc,II}/z_{sc,I} = I_{1,I,rtd}/I_{1,II,rtd} = V_{1,rtd} I_{1,I,rtd}/V_{1,rtd} I_{1,II,rtd} = S_{I,rtd}/S_{II,rtd}$$

and, by Eq. (8.25),

$$I_{1,I}/I_{1,II} = z_{sc,II}/z_{sc,I} = S_{I,rtd}/S_{II,rtd}$$

That is, the currents are shared between the paralleled transformers in proportion to their apparent power ratings. In other words, the equality of impedance voltages provides for the distribution of load between the paralleled transformers in proportion to their apparent power ratings.

8.13 Autotransformers

In power transmission it is sometimes necessary to interconnect electric networks whose rated voltages differ in a 2-to-1 ratio at most. An example is an interconnection of a 110-kV network and a 220-kV network. This form of interconnection is effected with the aid of a transformer or, to be more specific, an autotransformer which has a higher efficiency and a smaller size than the conventional transformer of the same power rating.

An *autotransformer* differs from a transformer in that it has only one electrically continuous winding that is intended for use in such a manner that part of the winding is common to both the primary (usually H.V.) and the secondary (usually L.V.) circuits, such as shown in Fig. 8.25 a, although the reverse arrangement, such as shown in Fig. 8.25 b, is also possible.

Given the primary voltage of an autotransformer and its number of turns w_1, the amplitude of the magnetic flux, Φ_m, in the core may

be deemed fully specified, because in accord with Eq. (8.3a) and Eq. (7.2c)

$$V_1 \approx 4.44 f w_1 \Phi_m$$

The above approximate equality turns into a full equality in the case of an ideal transformer. In each turn of the winding, this flux induces an emf practically independent of the current in the winding.

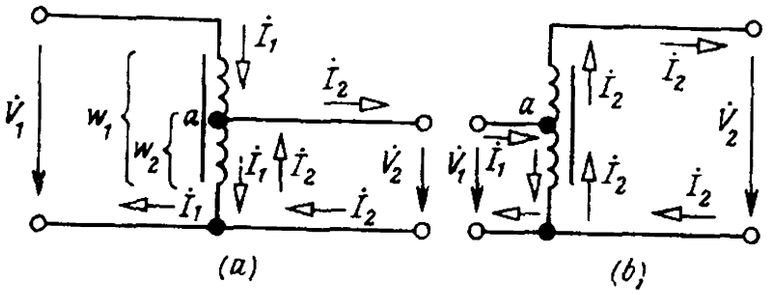


FIGURE 8.25

In consequence, the voltages between the various parts of the winding are maintained constant as well.

The voltages and currents existing in an autotransformer are connected by the same approximate relations as applied to a conventional transformer [see Eqs. (8.12) and (8.19)]

$$V_1/V_2 \approx w_1/w_2 \approx I_2/I_1$$

The current in the common part of the winding is equal to the difference between the primary current $\dot{I}_1$ and the secondary current $\dot{I}_2$ (Fig. 8.25). As in a conventional transformer (see Fig. 8.11), these two currents are nearly in phase. Therefore we may neglect the effect of the magnetizing (or open-circuit) current $\dot{I}_{1,oc}$, and deem that the rms current in the common part of the winding is $|I_1 - I_2|$.

If the transformation (turns) ratio, $n_{12} = w_1/w_2$, only slightly differs from unity, the rms currents I_2 and I_1 are nearly the same and their difference, $|I_2 - I_1|$, is small in comparison with each of them. Therefore the part of the winding common to the primary and secondary circuits may be wound with a finer wire, so that the cost of the windings of an autotransformer is lower than in the case of a conventional transformer and it takes up less space.

The size of a conventional transformer depends on its apparent power

$$S_T = V_1 I_1 \approx V_2 I_2$$

or, which is the same, on the rated apparent load power. In the case of an autotransformer, its apparent power S_{AT} is always smaller than the load power.

The apparent power in the common part of the winding in an autotransformer (see Fig. 8.25 a) is

$$S' \approx V_2 (I_2 - I_1) \approx V_2 I_2 (1 - w_2/w_1)$$

and that of the remaining part of the winding is

$$S' \approx (V_1 - V_2) I_1 = V_1 I_1 (1 - w_2/w_1)$$

Since approximately $V_2 I_2 \approx V_1 I_1$, it follows that $S' \approx S'' \approx S_{AT}$.

The apparent power in each winding of a conventional transformer is

$$S_T = V_1 I_1 \approx V_2 I_2$$

Thus, given the same apparent load power, the ratio of the design apparent powers for an autotransformer and a conventional transformer is

$$S_{AT}/S_T = 1 - w_2/w_1$$

We may interpret it like this: The advantage arising from the use of an autotransformer increases as the difference between w_2 and w_1 is decreased.

To sum up, the advantages of an autotransformer decrease as the transformation (turns) ratio increases. Also, the use of an autotransformer is warranted when the interconnection is between two voltage levels only slightly different. For example, an autotransformer may not be used for interconnecting a 6-kV bulk supply line to a 220-V distribution network. If an attempt were ever made to do so, the insulation of the 220-V distribution network would have to be designed for 6 kV and, what is more important, such a distribution network would be an increased hazard source.

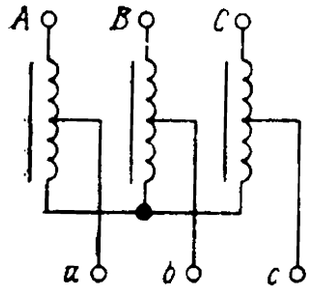


FIGURE 8.26

In a three-phase autotransformer it is usual to interconnect the windings into a star with the neutral brought out or to dispense with the neutral altogether (Fig. 8.26).

The position of the common electrical connection between the primary and secondary sides (point *a* in Fig. 8.25) can be changed at will, thereby providing a ready means for adjusting the secondary voltage, as is done in laboratory autotransformers where one of the terminals in the secondary circuit is a movable contact arm.

8.14 Multiwinding Transformers

In many electrical installations it is desired to have an interconnection between several circuits (or networks) operating at different voltage levels. Most conveniently this can be done with a multiwinding transformer which has one or several primary windings and several secondary windings. The simplest unit in this class is a three-winding transformer widely used in high-voltage networks.

A three-winding transformer has three electrically isolated windings. These are the H.V. winding with w_1 turns, the M.V. (medium-voltage) winding with w_2 turns, and the L.V. winding with w_3 turns. The respective voltages may, for example, be 220 kV, 38.5 kV, and 11 kV. The three windings (one primary and two secondaries) are put on a common core which does not differ from that of a double-winding transformer. The magnetizing current in the primary winding of a three-winding transformer excites in the core a magnetic flux which induces in all the three windings emfs proportional to their turns. If the load currents in the M.V. and L.V. windings are $\dot{I}_2$ and $\dot{I}_3$, the mmf in the primary must be such as to balance the demagnetizing effect of mmfs of the two secondary currents and also to supply a magnetizing mmf. Therefore

$$\dot{I}_1 w_1 = \dot{I}_2 w_2 + \dot{I}_3 w_3 + \dot{I}_{1,\,oc} w_1 \tag{8.26}$$

and the primary current may be treated as a sum of the referred (or transferred) currents in the second and third windings and of the magnetizing current [see Eq. (8.10a)]

$$\dot{I}_1 = \dot{I}'_2 + \dot{I}'_3 + \dot{I}_{1,\,oc} \tag{8.27}$$

where

$$\dot{I}'_2 = (w_2/w_1)\,\dot{I}_2$$

and

$$\dot{I}'_3 = (w_3/w_1)\,\dot{I}_3$$

As is seen, power in a three-phase transformer is simultaneously transferred to two secondary circuits, one associated with the second winding and the other with the third winding.

It is highly improbable that the load currents in the two secondary circuits, $\dot{I}_2$ and $\dot{I}_3$, could be at their maximum values and in phase at the same time. Therefore it is usual to design the primary winding for a rated power which is smaller than the sum of the rated powers of the secondary windings. The rated power of a three-winding transformer is the apparent power of the highest-rated winding.

A three-winding transformer has **three transformation** (or turns) ratios

$$n_{12} = w_1/w_2, \quad n_{13} = w_1/w_3, \quad n_{23} = w_2/w_3 = n_{13}/n_{12}$$

The above ratios are found at the open-circuit voltages, as for a double-winding transformer, Eq. (8.13).

A more recent trend has been to use three-winding transformers in which the H.V. and M.V. windings are interconnected as an auto-transformer, with the usual transformer connection applied only to the L.V. winding electrically insulated from the H.V. and M.V. windings.

8.15 Core and Winding Designs

The service conditions in which transformers are used have a direct bearing on the design of their cores, coils, and (in the case of oil-immersed units), oil tanks.

The core of a transformer is built up from electrical-sheet steel laminations 0.35 mm or 0.5 mm thick. The steel usually contains

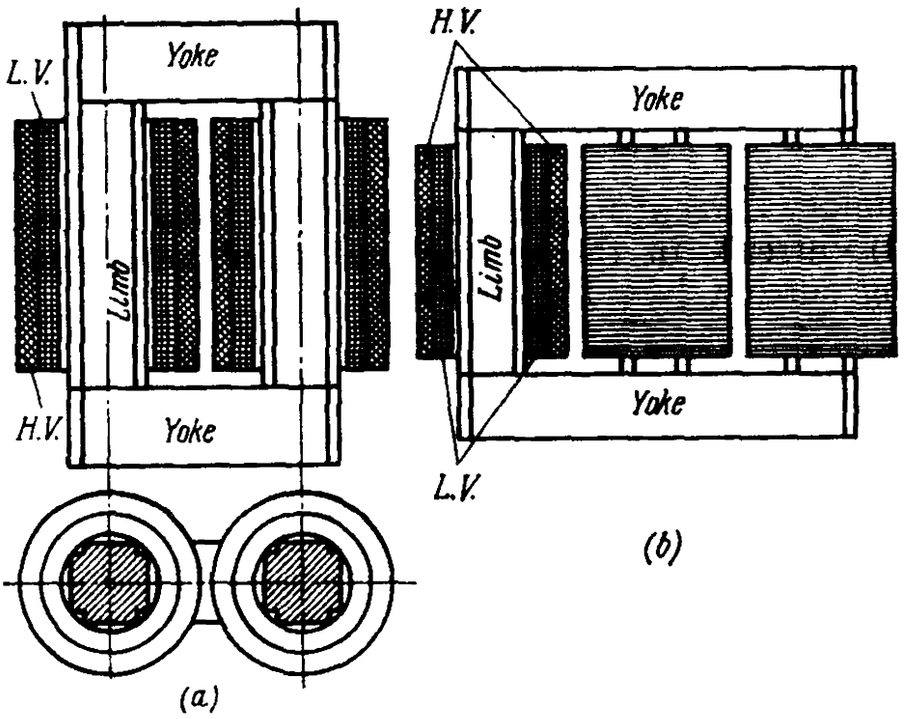


FIGURE 8.27

4-5% silicon so as to minimize eddy-current losses. Prior to assembly, the punchings are given a coat of varnish on each side for better insulation in the core. The laminations are assembled into blocks in which they are held together by steel bolts insulated from the core material so as not to form closed loops.

According to the core design, two types of transformers are in common use. When the magnetic circuit takes the form of a single ring

encircled by two or more groups of primary and secondary windings distributed around the periphery of the ring, the transformer is termed a *core-type transformer* (Fig. 8.27). When the primary and secondary windings take the form of a common ring which is encircled by two or more rings of magnetic material distributed around its periphery, the transformer is termed a *shell-type transformer* (Fig. 8.28).

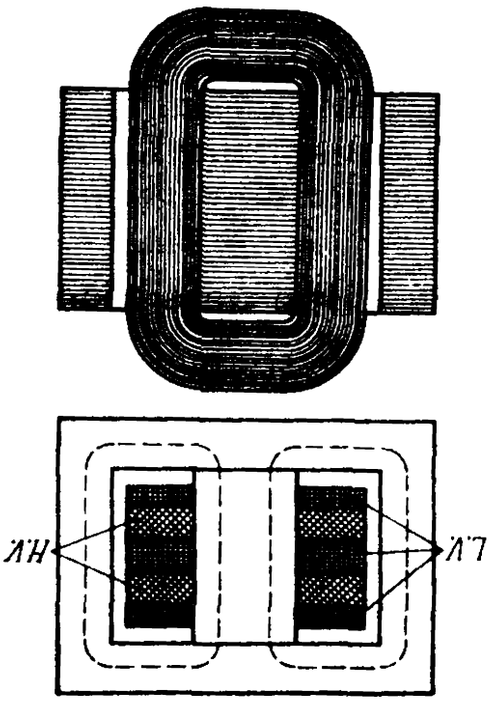


FIGURE 8.28

The magnetic circuit in a core-type transformer is made up of limbs (or legs) which carry the windings, and two yokes which complete the magnetic circuit. This form of magnetic circuit is simpler in design than the shell type, and it is easier to provide the right level of insulation for the windings. For these reasons, core-type transformers are the more commonly used of the two designs.

The vertical limbs in a core-type transformer are of multistep cruciform section approaching a circle. This shape has been chosen for better utilization of the window area in the core, and the number of steps increases with increasing power rating of the transformer. It is only in small-size transformers that the limbs are rectangular or square in cross section.

To improve cooling and ventilation in medium- and large-size transformers it is usual to provide oil ducts between the blocks of punchings. The yokes of a core-type transformer may be of a multistep cruciform or rectangular cross section.

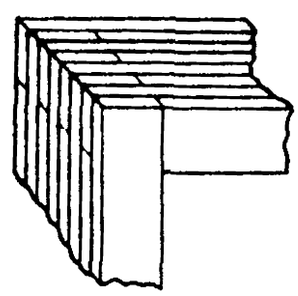


FIGURE 8.29

An advantage of a shell-type transformer is that its magnetic circuit protects the windings against mechanical injury to some extent. The shell-type core is used in small-size dry transformers.

The joints between the limbs and the yokes in a transformer core may be either of the butt or of the interleaved (or imbricated) type (Fig. 8.29). The latter type is preferable as it reduces the reluctance at the joints.

The transformer windings, H.V. or L.V., may be concentric or interleaved (or sandwich) types. *Concentric windings* are used in most transformers. The simplest variety is in the form of cylindrical coils taking up the whole length of, and arranged coaxially on, the same

limb (labelled H.V. and L.V. in Fig. 8.27). The L.V. cylinder is placed next to the limb as it is easier to insulate from the core, and the H.V. cylinder encloses the L.V. winding. The two windings are separated by a synthetic-resin-bonded paper (S.R.B.P.) sleeve or cylinder.

Interleaved (or *sandwich*) *windings* are used relatively seldom, and then mostly in shell-type transformers. This type of winding is a series or pancake or disc coils with the primary and secondary sections alternating in an interleaved or sandwich pattern (Fig. 8.28). The outer disc coils next to the yoke are those on the L.V. side.

8.16 Heat Generation in and Cooling of Transformers

The losses in a transformer are small in relative terms, but they are appreciable in absolute terms, especially in large-size units. Therefore, a major task in transformer design is to provide for proper withdrawal of the heat generated in the core and coils to the surroundings. The task grows progressively more complicated as the transformer size increases. Given the magnetic induction in the core and the current density in the coils, the losses in a transformer rise in proportion to its volume (that is, the third power of its linear dimensions), whereas the cooling area increases only in direct proportion to the square of the increase in linear dimensions.

In consequence, the increase in transformer rating entails a rapid increase in the required rate of heat transfer from its surface. This can be done by increasing the cooling surface area of a transformer, or by raising the rate of heat transfer by some artificial means.

On the whole, the cooling systems applied to transformers may be classed into air natural cooling, oil natural cooling, oil-natural air-blast cooling, forced oil natural cooling.

Air natural cooling is applied to dry transformers. In this case, the heat generated in a transformer is directly transferred to the surroundings. Because heat transfer is poor, the temperature distribution in a dry transformer may be all but uniform. Also the low electric strength of air (2.1 MV m^{-1}) impairs the insulation level in dry transformers. Furthermore, account must be taken of the dust which lodges on the windings and impairs their insulation still more. For these reasons, air natural cooling is mostly used on small-sized, low-voltage transformers.

At present the most commonly used type of transformer is an oil-immersed transformer. In it the core and coils assembly is placed in a steel tank filled with a carefully purified mineral oil. On absorbing heat, the oil rises in the tank, flows over the core and coils, cools them, transfers the heat thus absorbed to the tank walls, and sinks (this is known as convection). Transformer oil has a high heat capacity, so a higher level of core and copper losses may be tolera-

ted. Also, since transformer oil has a high electric strength, the spacing between the conductors and the core may be substantially reduced.

In transformers up to 20-30 kVA in size, a sufficient cooling surface area is obtained with a plain tank. Larger units (up to 15-20 MVA) call for the use of tubular tanks (see Fig. 8.3). For still larger transformers it is usual to employ radiators with air natural or air blast cooling. Units from 90 MVA and bigger in size mostly use radiators in combination with forced oil, air blast cooling.

When a transformer is in operation, the volume of oil in its tank increases or decreases, according as the transformer is heated or cooled. In transformers rated up to 100 kVA at 6.6 kV, there is enough space left under the tank cover to accommodate the expanding oil as it displaces air from the tank through what is called the breather. When the oil contracts, moist air inevitably finds its way inside the tank, and the oil picks up moisture. As a result, a layer of water accumulates at the tank bottom with time, and the oil loses a good deal of its electric strength. Also, atmospheric oxygen oxidizes the oil and brings down its electric strength still more. All of this is avoided in large transformers whose tanks are filled with oil to the brim. To accommodate excess oil on thermal expansion, each transformer in this class is fitted with an oil conservator (see Fig. 8.3) which is a sheet-steel cylinder set up on the tank cover and is connected with the tank insides by a tube which stops some distance short of the conservator bottom. The volume of the oil conservator is about 10% of that of the tank. Atmospheric moisture and sludge are mostly collected at the conservator bottom from which they are removed by means of a drain cock. With a conservator, the oil surface exposed to air is substantially smaller than it is in a transformer without a conservator. Also, since the temperature in the conservator is lower than it is in the tank, the oil is less oxidized.

A major drawback of oil cooling is that transformer oil is combustible (it ignites at around 160 °C), so it presents both a fire and an explosion hazard. The gases forming as transformer oil burns can tear the cover off the tank, and expel a large amount of oil, and also cause damage to the tank itself. In large units this is prevented from happening by providing a relief or explosion stack (see Fig. 8.3) fitted with a disc of thin glass. Should a fault inside the transformer entail the evolution of a large volume of gas, it will burst the disc and escape into the atmosphere.

In especially critical applications transformer tanks may be filled with quartz sand or an incombustible synthetic liquid; this liquid and its vapour are toxic.

The use of liquid coolants markedly complicates the use of transformers since this necessitates a constant watch on their condition and a periodic purification or renewal.

8.17 Instrument Transformers

These may be current transformers and voltage transformers. They are used, firstly, for the purpose of changing currents or voltages in a power circuit to values which render them convenient for measurement, and secondly in order to extend the range of the associated instruments.

Voltage instrument transformers are used in conjunction with voltmeters, frequency meters, the potential (shunt) circuits of wattmeters, electricity meters and phase meters, and relays. Current instrument transformers are used in conjunction with ammeters and the current (series) circuits of some instruments, and relays.

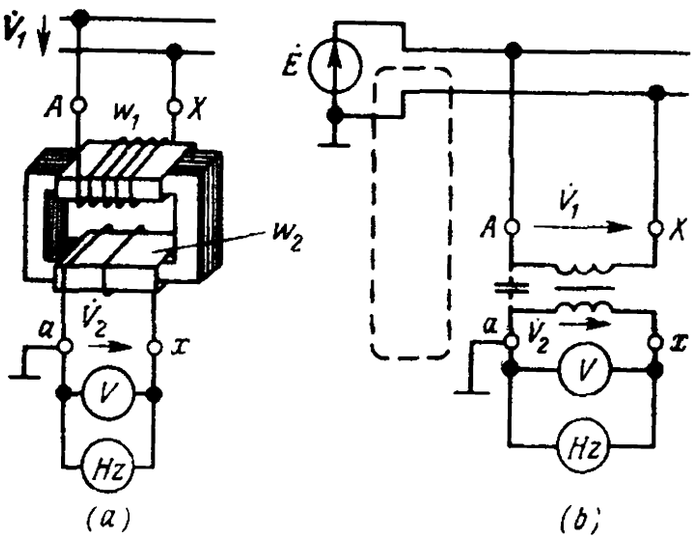


FIGURE 8.30

A schematic diagram of a *voltage* (or *potential*) *transformer* is shown in Fig. 8.30 a, and its symbolic representation in Fig. 8.30 b. This type of transformer is similar to a small-power transformer. Its primary, or H.V., winding with w_1 turns is connected in the circuit whose voltage V_1 is to be measured, and its secondary, or L.V., winding at a voltage V_2 and with substantially fewer turns w_2 is connected in parallel to a voltmeter and the potential circuits of other instruments. As a rule, both windings are of the concentric type, with the H.V. winding enclosing the L.V. winding as in a power transformer (in Fig. 8.30 a they are shown separately). One terminal of the secondary winding and the transformer frame are grounded. This is done as a precaution in case of a damage to insulation and also to return to earth the current path shown by the dashed line in Fig. 8.30, via the capacitance between the primary and secondary windings. Unless returned to ground, this current might impair the accuracy of measurement.

The voltmeter and the potential circuits of the other instruments connected to a voltage transformer have a relatively high resistance (of the order of several thousand ohms), which implies that a voltage transformer operates under conditions very close to the open-circuit condition of a power transformer. Therefore the voltage drop across the primary winding, $z_1 I_1$, and across the secondary winding, $z_2 I_2$, of a voltage transformer is very small, so we may deem that $V_1 \approx$

$\approx E_{1,oc}$ and $V_2 \approx E_{2,oc}$. Also, since by analogy with Eq. (8.13) $E_{1,oc}/E_{2,oc} = w_1/w_2 = n_{12}$

where n_{12} is the transformation (turns) ratio, then

$$V_1 = n_{12}V_2 \tag{8.28}$$

As is seen, the secondary voltage is related to the primary voltage by a constant ratio. In consequence, by measuring the low voltage V_2 we immediately obtain the high voltage V_1.

When the chosen positive directions of voltages (see Fig. 8.30) are the same relative to the like transformer terminals, the secondary and primary voltages must be in phase. Therefore, the windings of a voltage transformer are always connected to give phase displacement group 0 (see Sec. 8.11), and the winding terminals are labelled *A-X* and *a-x*. The same phase voltages in a voltage transformer and in the associated measuring instrument(s) are made equal by suitably connecting the secondary terminals of the transformer to the instrument terminals. Of course, a correct phase relation is important for a wattmeter and an electricity meter and not for a voltmeter or a frequency meter. Most voltage transformers are wound for a standard secondary voltage of 100 V.

The ratio of the primary to the secondary voltage would always be the same if the voltage drops across the transformer windings were zero. Actually these drops are anything but zero, and are a source of two errors, the voltage ratio error and the phase angle error.

The *voltage ratio error* is usually expressed as a percentage and is defined as the ratio of the error in the measured primary voltage to the true value of that voltage

$$f_v\,\% = \frac{V_{1,\,meas} - V_1}{V_1} \times 100$$

where $V_{1,meas}$ is the measured primary voltage, and V_1 is the true value of primary voltage.

The *phase angle error* is defined as the angle, δ_v, between the primary voltage phasor and the secondary voltage phasor (Fig. 8.31) on a phasor diagram drawn up similarly to that in Fig. 8.13. The phase angle error is expressed in minutes of arc, and is taken as positive if the secondary voltage phasor leads the primary voltage phasor.

For voltages up to 6 kV, potential transformers come as air-natural-cooled dry units. For voltages in excess of 6 kV they are oil-immersed units. Often, potential transformers are made three-phase. A general view of a three-phase potential transformer is shown in Fig. 8.32.

A *current transformer* is connected on its primary side as an ammeter, that is, in series with the line whose current is to be measured

or controlled (Fig. 8.33), while its secondary winding is connected directly to an ammeter or the current (series) coil of a wattmeter (or similar instruments). The total resistance of an ammeter or of the current coil of a meter is very small (being less than 2 Ω), so a current transformer operates under conditions approaching the short-circuit condition of a conventional transformer. The secondary voltage of a current transformer depends on the voltage drop across

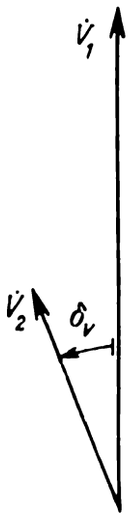


FIGURE 8.31

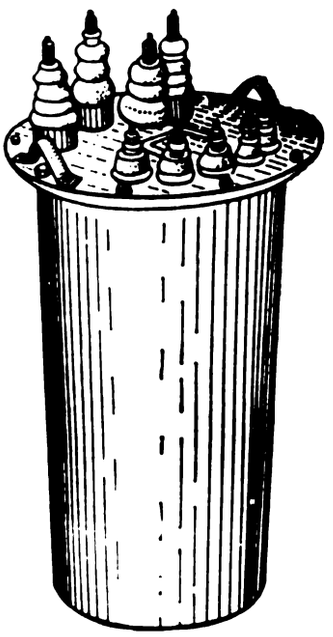


FIGURE 8.32

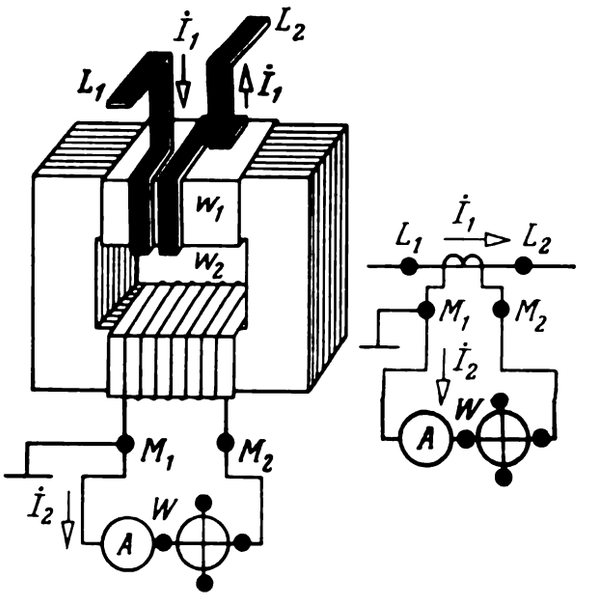


FIGURE 8.33

the relatively small resistance of the instrument load and connecting wires (being usually 1-12 V). The low secondary voltage corresponds to a small secondary emf, E_2, and, in consequence, to a small value of magnetic flux in the core of the current transformer

$$E_2 = 4.44 f w_2 \Phi_m \tag{8.29}$$

It takes an insignificant mmf, $I_{1,oc} w_1$, to excite such a flux, so in the equation

$$\dot{I}_1 w_1 = \dot{I}_2 w_2 + \dot{I}_{1,oc} w_1 \tag{8.30}$$

this quantity may be neglected, and we may write

$$\dot{I}_1 w_1 = \dot{I}_2 w_2$$

or

$$I_1 = (w_2/w_1)\, I_2 = n_{21} I_2 \tag{8.31}$$

Thus, the primary current can be determined by multiplying the secondary current by a constant transformation (or turns) ratio n_{21}. In this way we can measure the current in an H.V. power line by determining a small current without running any safety hazard.

Current transformers are often used to measure heavy currents in plants or lines operating at below 1 kV. With a proper connection of the current-transformer leads to the meter terminals, the current in the instrument load will be in phase with that in the current-transformer primary. When an ammeter is intended for permanent use with one and the same current transformer, its scale can be directly laid out in values of the primary current. The rated secondary current is the same for all current transformers, the standard value being 5 A (or 1 A in some special cases).

The current ratio in current transformers is not always the same owing to the effect of the mmf $\dot{I}_{1,oc}w_1$ which has been so far neglected. This effect is the source of two errors, the *current ratio error*, f_i, and the *phase angle error*, δ_i. They are defined in about the same manner as for potential transformers, except that current is inserted in the applicable equations instead of voltage.

In most cases the primary current of a current transformer is many times the secondary current. For this reason the number of primary turns, w_1, is small, being a fraction of the number of secondary turns w_2. When large currents are to be measured, the primary is most often a single turn which is formed by running the line through the window of the core. Since $w_1 < w_2$, the primary voltage is a few hundredths of a volt for a secondary voltage of several volts.

The value of $\dot{I}_1$ depends on the circuit in which it is measured. Any increase in the resistance of the current-transformer secondary cannot practically affect the value of $\dot{I}_1$, but, as follows from Eq. (8.30), this can raise the mmf $\dot{I}_{1,oc}w_1$ due to a reduction in $\dot{I}_2$. Because of this, as the resistance of the secondary circuit is increased, the actual condition deviates progressively more from that essential to the accuracy of a current transformer: $I_{1,oc}w_1 \ll I_2w_2$. This is why it is usual to state the maximum resistance of the instrument load that can be put on the secondary without exceeding the limits of ratio and phase angle errors.

When current is passing through the primary, the secondary of a current transformer must always be shorted either through the current coils of meters or through the shorting switch provided as part of the transformer. The reason for this precaution is as follows. When the secondary circuit of a current transformer is open, the secondary current I_2 is zero, while the primary current I_1 remains practically unaffected. Now all of the primary current is acting as a magnetizing (or exciting) current, that is, by virtue of Eq. (8.30), $I_{1,oc}w_1 = I_1w_1$, whereas under normal conditions $I_{1,oc}w_1$ accounts for as little as 0.5% of I_1w_1. This manifold increase in mmf leads to a very large build-up of the magnetic flux (solely limited by the saturation of the core). Since, by Eq. (8.29), the secondary emf E_2 is pro-

portional to the magnetic flux, it may run as high as several hundred volts, while in current transformers for large currents it may be as high as 1.5 kV. This is obviously a source of hazard to human life. Also, the core losses go up [see Eqs. (7.11) and (7.12)], more heat is generated, and the core expands. Either is highly detrimental to the insulation and might finally lead to its break-down and a fault to ground on the H.V. side.

The mmf required to excite a magnetic flux in a core decreases as the core reluctance is brought down. For this reason the cores for precision current transformers are made continuously wound (to avoid joints) from materials such as Permalloy. An example is the multi-range portable current transformer shown in Fig. 8.34 a.

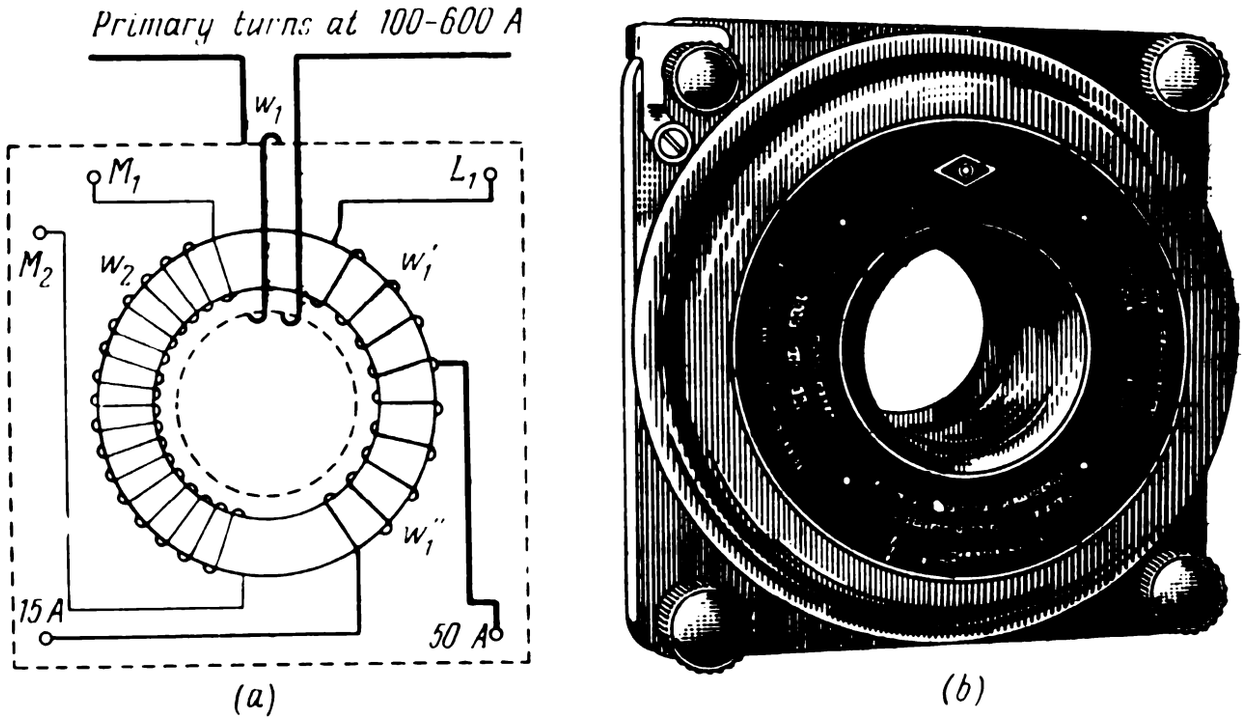


FIGURE 8.34

It has one secondary winding w_2 and three primaries: w_1 for measuring currents up to 600 A, w_1' for currents up to 50 A, and $w_1' + w_1''$ for up to 15 A. How this current transformer looks like is shown in Fig. 8.34 b.

In addition to accuracy, current transformers are often required to have a high stability towards short-circuits because their primaries are connected in power lines where short-circuits are likely to develop, and also because current transformers themselves are connected to relays used to disconnect a power line or an electric installation in the case of a short-circuit. In other words, a current transformer must be capable of withstanding (momentarily) the short-circuit current and operate the relay(s) that will de-energize the faulty equipment.

One way to enhance the accuracy of a current transformer is to keep the effect of its exciting mmf, $I_{1.oc}w_1$, to a minimum. This can be done, among other things, by making its rated primary mmf, $I_{1.rtd}w_1$, as large as practicable. For precision current transformers it should be no less than 500 A. If the rated primary current is less than 500 A, the primary must have several turns. As an example, for a rated primary current of 100 A the primary should have at least five turns. If high accuracy is not a factor (such as when a current transformer is connected to an ammeter or an overcurrent relay), the rated primary mmf may be substantially smaller.

Currents of 500 A and more are measured with single-turn through-type current transformers which include the split-core type shown

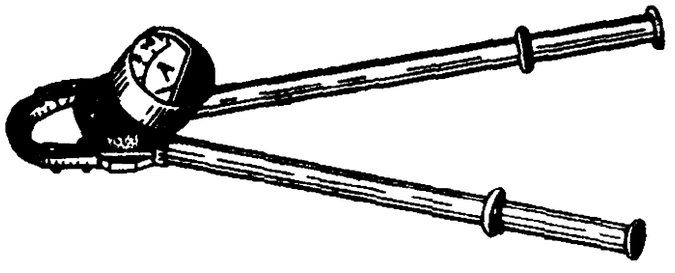


FIGURE 8.35

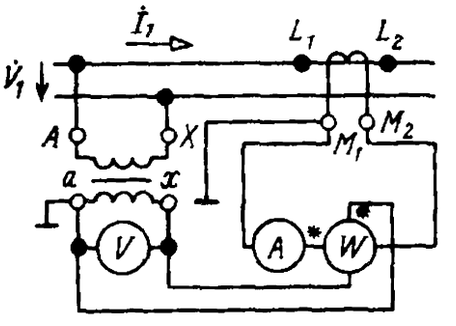


FIGURE 8.36

in Fig. 8.35. A split-core current transformer* is used to measure approximately currents from 20 to 1000 A at low voltage. Its core is made up of two U-shaped parts pulled together by a strong spring and fabricated from electrical-sheet steel punchings stacked up and polished at the mating faces. When using a split-core current transformer, the operator moves the handles together, opens the tongs, puts the instrument around the line conductor, and releases the handles—the spring will then force the tongs to close. Now the line conductor passing through the tongs acts as the primary. The secondary is wound on the core and is connected to an ammeter.

Instrument (both potential and current) transformers are grouped into accuracy classes on the basis of their limits of ratio error. For a potential transformer in accuracy class 0.5, the limit of ratio error is $\pm$ 0.5% and the limit of phase angle error is $\pm 20'$, with the primary voltage being from 80% to 120% of the rated value. For a current transformer in accuracy class 1 the limit of ratio error is $\pm 1.0\%$ and the limit of phase angle error is $\pm 90'$, with the load impedance being from 25% to 100% of the rated value and with the primary current being from 120% to 10% of the rated value.

Connections between an instrument transformer and its load should be made according to the markings on their terminals which on

* Many authors class this type of current transformer as a split-electromagnetic ammeter.—*Translator's note.*

potential transformers are labelled in the same way as on power transformers (A-X, a-x, etc.). In the case of current transformers the start and finish of the primary are labelled (at least in the Soviet Union) L_1 and L_2 (where the letter L stands for "line"), while the start and finish of the secondary are marked M_1 and M_2 (where the letter M stands for "meter").

The manner in which instruments are connected via potential and current transformers into a single-phase circuit is illustrated in Fig. 8.36. For measurements in three-phase, three-wire lines several instrument transformers are required in the general case. For example, in order to measure active power (see Fig. 3.13 a) one will need two current transformers and two potential transformers (or one three-phase potential transformer).

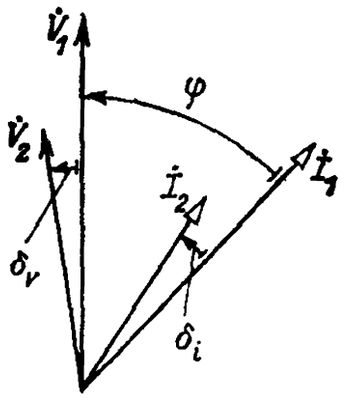


FIGURE 8.37

Readings of a wattmeter (or a watthourmeter) connected via potential and current transformers must be multiplied by the product of the transformation ratios of the two transformers. The errors associated with instrument transformers affect the accuracy of wattmeters. The phase angle error has an especially strong effect mainly at large phase angles between the primary voltage and current. Because of this, wattmeter readings are proportional not to the power factor $\cos \varphi$, but to $\cos (\varphi + \delta_v - \delta_i)$ (Fig. 8.37). The phase angle errors may combine because it often happens that $\delta_v < 0$ and $\delta_i > 0$.

9 Saturable Reactors and Magnetic Amplifiers

9.1 The Saturable Reactor

In the simplest case a *saturable reactor* is a coil wound on a core which is fabricated from a rectangular B/H loop material (see Fig. 6.6 a).

For purposes of analysis, we assume that the core of a saturable reactor is magnetized along the static hysteresis loop. Given the B/H relation for the core material, we can plot (similarly to Fig. 7.1) a weber-ampere curve (Fig. 9.1 b), $\Psi(F)$, where $\Psi = w\Phi$ is the flux linkage with the coil turns, w is the number of turns, $\Phi = BS$ is the magnetic flux in the core (the leakage flux is ignored), $F = iw = Hl_{av}$ is the mmf with the current i in the coil, S is the cross-sectional area of the core, and l_{av} is the average (or mean) length of the magnetic line of force. This is in effect the same hysteresis loop but constructed to a different scale.

Let us see what happens in a saturable reactor when its coil is energized from a source of sinusoidal emf

$$e = E_m \sin \omega t$$

(see Fig. 9.1 a). Neglecting the coil resistance, the flux linkage with the coil turns can be found from the law of electromagnetic induction

$$e = E_m \sin \omega t = v_L = d\Psi/dt$$

Hence, on integrating, we get

$$\Psi = -(E_m/\omega) \cos \omega t + \Psi_0 \qquad (9.1)$$

where Ψ_0 is the constant of integration.

Since in our case the direct component of flux linkage, Ψ_0, is non-existent in the steady-state (at $e = 0$, the core is completely demagnetized, $\Psi = 0$, and $iw = 0$), it follows that

$$\Psi = -\Psi_m \cos \omega t$$

where

$$\Psi_m = E_m/\omega$$

is the maximum flux linkage.

Figure 9.1 shows a plot of the current i in the reactor circuit as a function of time or, rather, the product iw as a function of the phase ωt, constructed on the basis of the $\Psi(\omega t)$ relation and the weber-ampere characteristic, $\Psi(F) = \Psi(iw)$, of a saturable reactor. As fol-

lows from the current-vs-time relation, at $\Psi_m > \Psi_r$, where $\Psi_r = wB_rS$ is the remanent flux linkage, a saturable reactor has the properties of a switch. Time intervals during which the current in the coil of a saturable reactor is small alternate with those during which it is high.

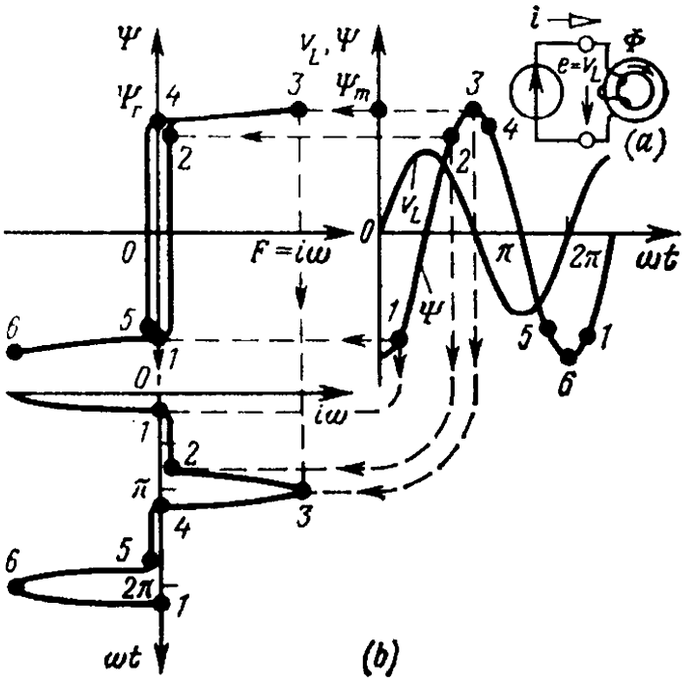


FIGURE 9.1

Now we will make one more simplifying assumption—we will introduce the concept of an *ideal saturable reactor* for which the weber-ampere characteristic $\Psi(F)$ is shown in Fig. 9.2 a where Ψ_s is the *saturation flux linkage*. An ideal saturable reactor has the properties of an ideal switch. A symbol for a saturable reactor as a circuit element is shown in Fig. 9.2 a. Once a saturable reactor reaches saturation, the current in its coil rises without bound. Therefore for a proper analysis of such a reactor, we should consider it connected in series with some other element, say, a resistive element of resistance r (see Fig. 9.2 b).

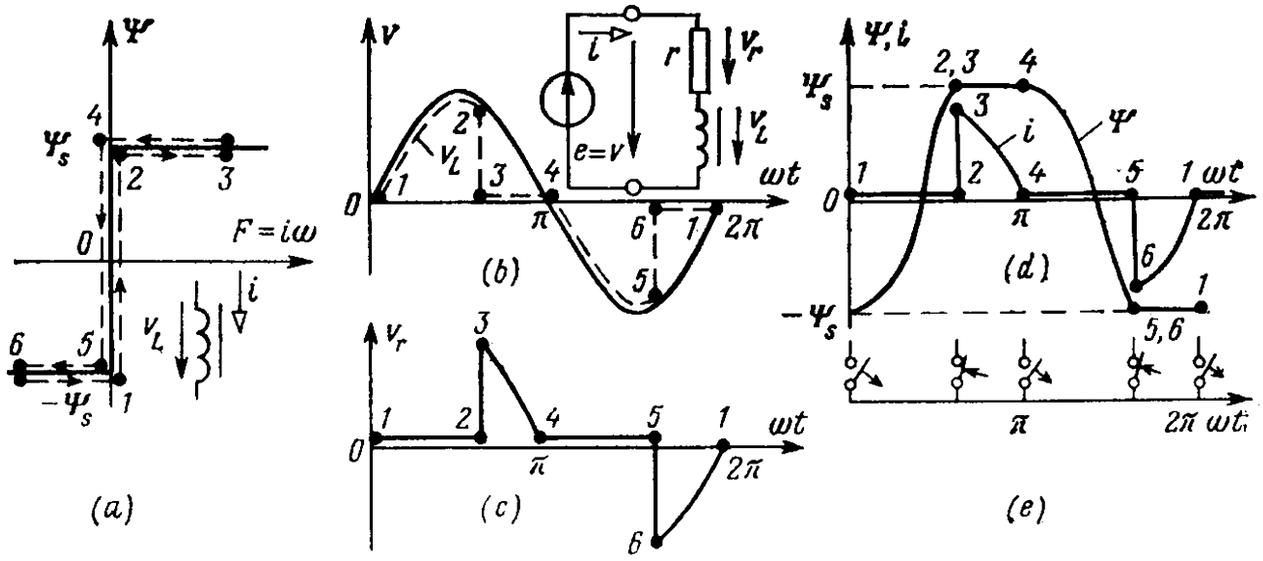


FIGURE 9.2

Let us analyse the steady-state condition in the circuit of Fig. 9.2 b during one period. By Kirchhoff's voltage law at any instant

$$v = v_r + v_L = ri + d\Psi/dt = e = E_m \sin \omega t \qquad (9.2)$$

Suppose that at time $t = 0$ the magnetization of the core is represented by point 1 on the weber-ampere characteristic of an ideal reactor (see Fig. 9.2 a), so that the core is at saturation ($\Psi = -\Psi_s$). From that instant on, the core is magnetized in the reverse sense, and the flux linkage varies along part 1-2 of the weber-ampere characteristic. In this condition the ideal reactor is equivalent to an open switch because the current is zero, $i = 0$ (see Fig. 9.2 a). The voltage across the reactor, v_L, is equal to the source voltage in accord with Eq. (9.2)

$$v_L = d\Psi/dt = v = e$$

This differential equation has already been solved, Eq. (9.1), and now it remains to determine the constant of integration for the process in question. As follows from Eq. (9.1) for $t = 0$

$$\Psi(0_+) = -E_m/\omega + \Psi_0$$

whereas in accord with the conditions specified

$$\Psi(0_-) = -\Psi_s$$

Since the flux linkage cannot change abruptly, the constant is

$$\Psi_0 = -\Psi_s + E_m/\omega$$

and during the magnetization in the reverse sense

$$\Psi = -(E_m/\omega)\cos\omega t - \Psi_s + E_m/\omega \qquad (9.3)$$

as shown in Fig. 9.2 d by the portion of the $\Psi(\omega t)$ graph from point 1 to point 2.

The flux linkage keeps varying in accord with Eq. (9.3) until the core again reaches saturation, $\Psi = \Psi_s$ (point 2 in Fig. 9.2 a and d).

Beginning at that instant and for the duration of the remaining part of the first half-cycle $\Psi = \Psi_s$, the ideal saturable reactor is equivalent to a closed switch, and all of the source voltage is applied across the resistive element so that, by Eq. (9.2)

$$v = v_r = ri = e$$

as shown by the part of the $v_r(\omega t)$ graph from point 3 to point 4 in Fig. 9.2 c.

During the second half-cycle, the events taking place in the circuit are similar to those occurring during the first.

The various steps in the cyclic magnetization of the core of an ideal saturable reactor in Fig. 9.2 a are numbered similarly to the respective portions of the curves relating the flux linkage, current and voltages to time (or to the phase ωt) in Fig. 9.2 b through e.

It follows from Eq. (9.3) that when

$$E_m \leqslant \omega \Psi_s \tag{9.4}$$

no current is flowing in an ideal saturable reactor (the reactor behaves as an open switch for duration of the entire cycle).

The behaviour of a saturable reactor as a switch is utilized in automatic control and instrumentation. One example is the magnetic amplifier.

9.2 The Magnetic Amplifier

As already noted, *magnetic amplifiers* utilize the behaviour of a saturable reactor as a switch by controlling this behaviour in an appropriate manner. In this way, one can readily control the current in the circuit which is external to the reactor. This external circuit may be, say, an electric motor.

The principle of operation. The arrangement of a simple magnetic amplifier with a single controlled saturable reactor is shown in Fig. 9.3 a.

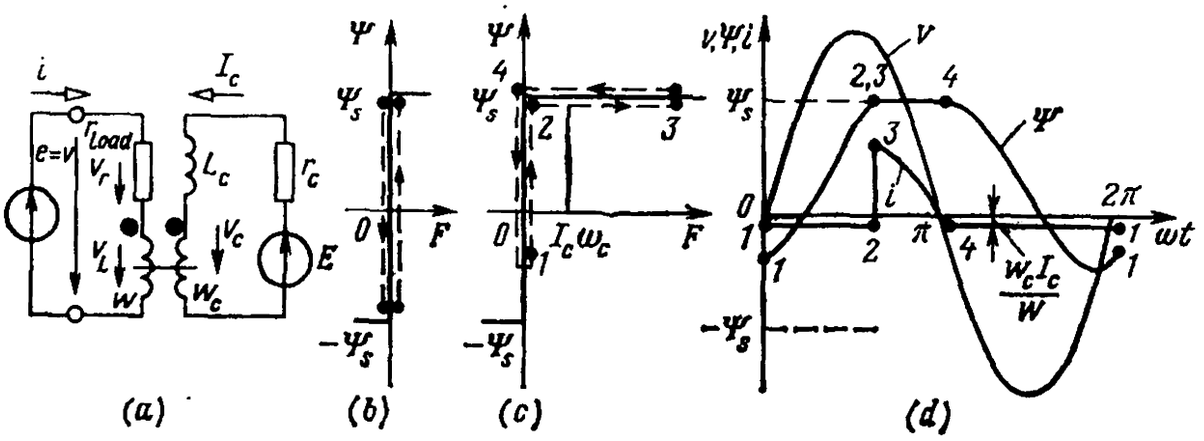


FIGURE 9.3

In contrast to the previously examined arrangement with one uncontrolled saturable reactor (see Fig. 9.2 b), this amplifier has one more circuit—a d.c. control circuit. The current in the control circuit does not vary because it contains an inductor of high inductance. This is *operation with forced magnetization.* In Fig. 9.3 a the inductor is represented by a series combination of an inductive element L_c and a resistive element r_c whose resistance is equal to the d.c. resistance of the entire control circuit, including the resistance associated with the turns of the additional inductor, that of the control winding, etc.

As is the case with a circuit containing only one uncontrolled saturable reactor (see Fig. 9.2 b), the processes in the load circuit of the magnetic circuit in Fig. 9.3 a are described by Eq. (9.2), but the

resultant mmf in the core is equal to the sum of a d.c. mmf and an a.c. mmf

$$F = I_c w_c + iw \tag{9.5}$$

where w is the number of turns in the a.c. or gate winding, w_c is the number of turns in the d.c. control winding, i is the load current, and I_c is the d.c. control current defined as

$$I_c = E/r_c$$

where E is the control signal.

When the condition (9.4) is satisfied and in the absence of a control signal ($E = 0$), the core is cyclically magnetized as shown in Fig. 9.3 b. In this condition, the load current is zero (as it should be in accord with Eq. (9.4)), and all of the supply voltage is applied across the gate winding

$$v_L = v = e$$

At the same time, the voltage across the control winding is

$$v_c = v_L w_c / w$$

In the presence of a control signal ($E \neq 0$), the core of the saturable reactor is cyclically magnetized in a different way (see Fig. 9.3 c), and the current i in the circuit follows a different path. During the time intervals when the magnetization of the core traces out the up-sloping part of the ideal magnetization curve, such that $F = 0$ (see Fig. 9.3 c), the load current has, as follows from Eq. (9.5), a constant value equal to $i = -(w_c/w)I_c$ (see Fig. 9.3 d). On substituting it in Eq. (9.2) and on integrating, we obtain the $\Psi(\omega t)$ relation for magnetization. During the time interval when the core is at saturation, that is, $v_L = 0$, the load current is

$$i = e/r_{load} = v/r_{load}$$

that is, sinusoidal.

The load current averaged over a cycle is zero. This immediately follows from Kirchhoff's voltage law in terms of the average emf and voltages

$$e_{av} = v_{L,av} + v_{r,av}$$

The sinusoidal emf supplied by the source does not contain a d.c. component, that is, $e_{av} = 0$, whereas the d.c. component of the voltage across an inductive element in a steady state is always zero, that is, $v_{L,av} = 0$.

Since the additional inductor has a very large number of turns, the resistance of the control circuit, r_c, is usually very high for which reason the amplifier has a low sensitivity towards the control sig-

nal E. In order to enhance the sensitivity, it is necessary to reduce the resistance of the control circuit, r_c, which could be done by omitting the additional inductor from the circuit. Actually, one uses two saturable reactors and connects their a.c., or gate, windings in series as shown in Fig. 9.4. In this form of magnetic amplifier the control circuit has a very small resistance, $r_c \approx 0$.

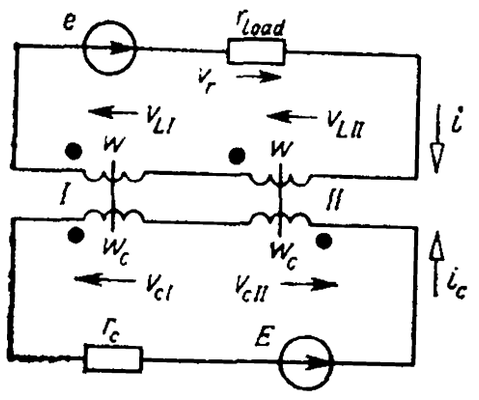


FIGURE 9.4

Let us take a closer look at how this form of amplifier operates. In the absence of a control signal ($E = 0$), the two series-connected saturable reactors are magnetized cyclically in the same manner. Because the emfs induced in the control windings cancel out each other, there is no current flowing in the control circuit ($i_c = 0$). Accordingly, the processes taking place in the circuit of Fig. 9.4 in the absence of a control signal are similar to those taking place in the circuit of Fig. 9.2 b. So long as $E_m < 2\omega\Psi_s$, the cores do not reach saturation, and the current i in the load circuit is zero.

In the presence of a control signal ($E \neq 0$) and for the positive directions of currents shown in Fig. 9.4, the mmf in saturable reactor I is equal to

$$F_I = iw + i_c w_c \tag{9.6a}$$

and that in saturable reactor II is equal to

$$F_{II} = iw - i_c w_c \tag{9.6b}$$

With the mmfs as given above, the magnetization of ideal saturable reactors will follow minor cycles (see Fig. 9.5 a).

Let us examine the steady-state condition during one cycle in the load circuit of our magnetic amplifier. Let at $t' = 0$ the state of magnetization of the first and second saturable reactors be represented by points 1 on their respective weber-ampere characteristics, Ψ_I (F_I) and Ψ_{II} (F_{II}), in Fig. 9.5 a and on their respective time characteristics, Ψ_I (ωt) and Ψ_{II} (ωt), in Fig. 9.5 c.

During the time interval from t' (point 1) to time t'' (point 2) or from $\omega t = 0$ to $\omega t = \omega t''$, the source emf causes the flux linkages in the first and second ideal saturable reactors to change at the same time and by the same amount

$$v_{LI} + v_{LII} = d\Psi_I/dt + d\Psi_{II}/dt = e$$

(see Fig. 9.5 b). Since, at the same time, the two mmfs, F_I and F_{II}, are zero, it follows from Eq. (9.6) that the current i in the gate win-

dings and the current i_c in the control windings are zero during this time interval (see Fig. 9.5 d).

At time t'' the flux linkage in the first ideal saturable reactor reaches saturation, $\Psi_I = \Psi_s$, magnetic coupling between the gate and control windings on the first reactor ceases to exist, and the voltages across the two windings become equal to zero

$$v_{LI} = d\Psi_I/dt = 0$$

$$v_{cI} = (w_c/w)\, d\Psi_I/dt = 0$$

On the other hand, magnetic coupling between the gate and control windings of the second reactor is preserved, and this reactor may be

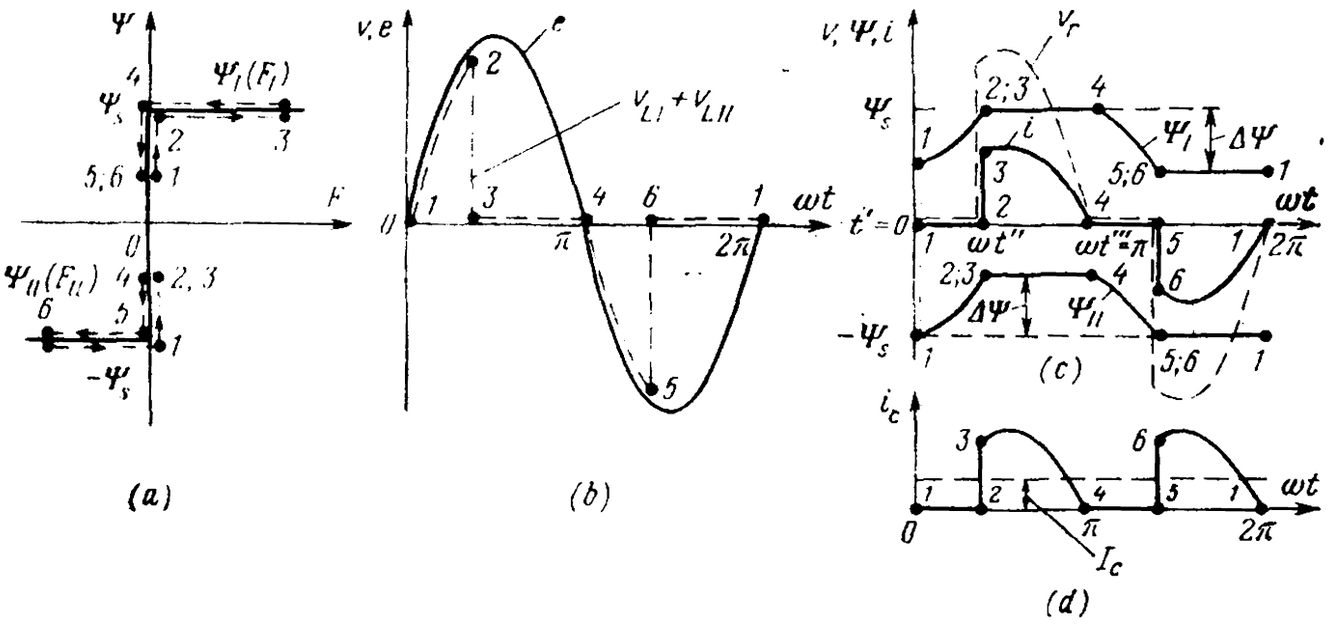


FIGURE 9.5

regarded as an ideal transformer (see Sec. 8.3) with a primary winding w and a secondary winding w_c. Since the control circuit has a small resistance, r_c, the condition in the second reactor is equivalent to that existing in an ideal transformer with its secondary short-circuited

$$iw - i_c w_c = 0 \qquad (9.7)$$

$$v_{LII} = d\Psi_{II}/dt = 0 \qquad (9.8a)$$

$$v_{cII} = (w_c/w)\, d\Psi_{II}/dt = 0 \qquad (9.8b)$$

This is *operation with self-magnetization.* Now an ideal magnetic amplifier with the series-connected gate windings of the two saturable reactors is equivalent to a closed switch. Therefore $e = v_r$ (see Fig. 9.4), and the currents in the load circuit and in the control circuit are respectively given by

$$i = (E_m/r_{load}) \sin \omega t$$

$$i_c = i\,(w/w_c) = (w/w_c)\,(E_m/r_{load}) \sin \omega t$$

At time $t''' = T/2$ (or $\omega t''' = \pi$), the load current decreases to zero (point 4). During the subsequent time interval from t''' to T (or from $\omega t''' = \pi$ to 2π), the electromagnetic processes in an ideal magnetic amplifier are repeated in the same sequence as during the time interval from t' to t''' (or from $\omega t' = 0$ to π), with the only difference that now the first reactor is being magnetized as the second one was and vice versa.

The control characteristic. The behaviour of a magnetic amplifier is defined by its *control characteristic* which relates the average or rms current in the load circuit to the respective current in the control circuit.

Since in an ideal magnetic amplifier with the two gate windings connected in series, the relation between the load and control cur-

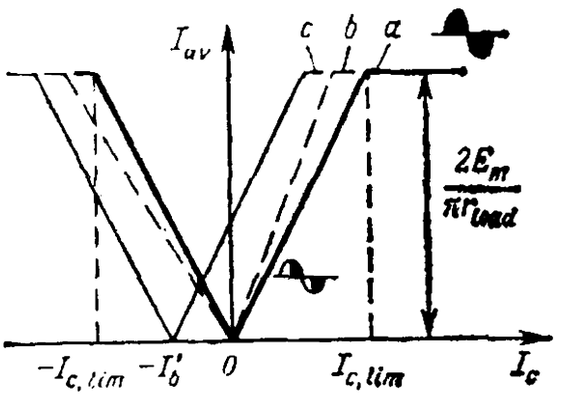


FIGURE 9.6

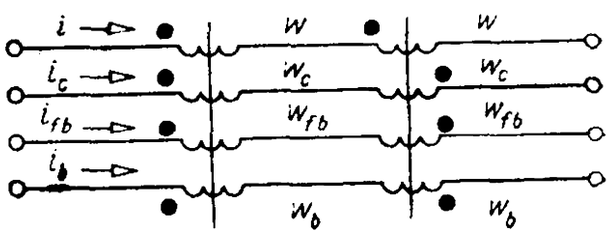


FIGURE 9.7

rents is defined by Eq. (9.7) at any instant, this relation also holds for the average values of these two currents

$$I_{av} w = I_c w_c$$

or

$$\Delta I_{av}/\Delta I_c = w_c/w = K_i \tag{9.9}$$

where K_i is the *current gain* defined as the slope of the control characteristic (curve a in Fig. 9.6). When the control current exceeds some limiting value, $I_c > I_{c,lim}$, an ideal magnetic amplifier loses control, and the average load current remains constant (see Eq. (2.16a))

$$I_{av,max} = 2E_m/\pi r_{load}$$

The control characteristic of a magnetic amplifier is symmetrical about the axis of ordinates.

The *power gain* of an ideal magnetic amplifier is

$$K_P = \frac{\Delta P_{load}}{\Delta P_c} = \frac{r_{load}\Delta I_{av}^2}{r_c \Delta I_c^2} = (w_c/w)^2 (r_{load}/r_c) \tag{9.10}$$

The power gain increases with decreasing resistance of the control circuit. However, an excessive decrease in the resistance of the control circuit extends the duration of transients in the wake of a change in the control signal E, or its response time. This slow-down in the response is due to an increase in the time constant of the control circuit. In practice, K_i may be as high as 100, and K_p as high as 1000.

The current and power gain of a magnetic amplifier can be varied by applying feedback through an additional set of feedback windings, w_{fb} (Fig. 9.7), in which the current i_{fb} is a fraction of the rectified load current. When the mmfs due to the currents in the control and feedback windings are in the same direction, feedback is positive; otherwise it is negative. Considering feedback, the equation for the mmf in the core of a magnetic amplifier in terms of average currents is written as

$$I_{av}w = I_c w_c \pm I_{fb}w_{fb} = I_c w_c \pm \beta I_{av}w \tag{9.11}$$

where the "+" sign applies to positive feedback, and the "—" sign to negative feedback; $\beta = I_{fb}w_{fb}/I_{av}w$ is the feedback ratio.

From Eq. (9.11) it follows that

$$I_{av} = \frac{1}{1 \mp \beta}(w_c/w)\, I_c = K_{i,fb} I_c \tag{9.12}$$

where $K_{i,fb} = K_i/(1 \mp \beta)$ is the closed-loop current gain, or the overall gain with feedback. In the case of positive feedback the current gain of the amplifier is increased $1/(1 - \beta)$ times (curve b in Fig. 9.6). In the case of negative feedback, the current gain is decreased by a factor of $1/(1 + \beta)$.

The control characteristic, $I_{av} = f(I_c)$, of a real magnetic amplifier is nonlinear and differs from the linear control characteristic of an ideal magnetic amplifier owing to the nonlinearity of and difference between the magnetic characteristics of the saturable reactors, hysteresis, and some other factors. The nonlinearity of the control characteristics is especially noticeable at low values of control current. Operation within the linear portion of a nonlinear control characteristic can be assured by applying bias through an additional set of bias windings, w_b, energized with direct current I_b and connected in a manner similar to the control windings (Fig. 9.7). This results in a biased control characteristic (curve c in Fig. 9.6). The amount of bias is given by

$$I'_b = I_b\,(w_b/w_c)$$

9.3 The Self-Saturating Magnetic Amplifier

In magnetic amplifiers using two saturable reactors with their gate windings connected in series, the cores are magnetized through the simultaneous action of the mmfs due to the control and load cur-

rent. In a *self-saturating* (or *self-biasing*) *magnetic amplifier*, the magnetization of the cores is varied owing to the action of the mmf due to the control current during the time intervals when the load current is zero. This improves the performance of the amplifier, notably its power gain and speed of response.

The principle of operation. The circuit diagram of an elementary self-saturating magnetic amplifier is shown in Fig. 9.8 a. It differs from that shown in Fig. 9.3 a in the presence of a rectifying element (a diode) in series with the saturable reactor gate winding (load circuit). We will analyse the operation of this amplifier in the steady state, assuming that the saturable reactor and the diode are ideal.

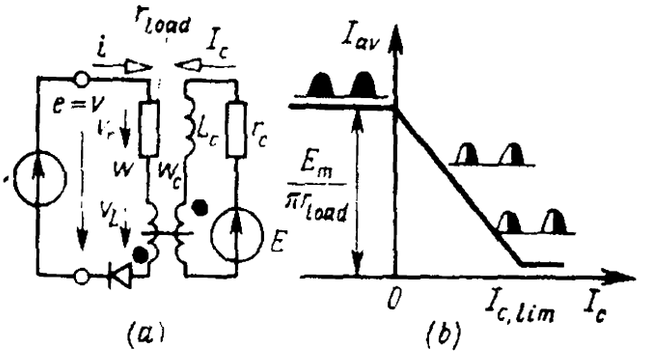


FIGURE 9.8

To begin with, we assume that there is no control signal applied ($E = 0$), so there is no control current flowing ($I_c = 0$). In the circumstances, the core is solely magnetized by the mmf due to the load current, iw, and the diode maintains the load current flow always in the same direction. For a saturable reactor with an ideal weber-ampere characteristic this condition corresponds to core saturation, $\Psi = \Psi_s$, over the entire cycle of the sinusoidal supply emf, so that

$$v_L = d\Psi/dt = 0$$

In consequence, the load circuit of the amplifier operates as a half-wave rectifier (see Sec. 10.6). The load current i is a half-wave rectified current whose average value is $E_m/\pi r_{load}$, Eq. (2.17). This unidirectional current is supplied by the source, and this explains why this form of magnetic amplifier is called self-saturating (or self-saturated).

The half-cycle during which the diode is conducting is called the *gating* or *firing half-cycle*. The half-cycle during which the diode is nonconducting is called the *resetting half-cycle*. During a resetting half-cycle no current is flowing in the load circuit, and the magnetic (that is, switching) properties of the saturable reactor are controlled, or reset, provided a control signal is applied.

Now we will consider the operation of the amplifier in the presence of a control signal ($E \neq 0$). Suppose that at the end of a particular gating half-cycle the core is saturated, $\Psi = \Psi_s$ (point 1 in Fig. 9.9 a and b). During the subsequent resetting half-cycle the diode is turned off, the load current is zero, and the core is magnetized by the control-circuit mmf, $I_c w_c$. Under the assumption of an ideal weber-ampere characteristic, the change in the flux linkage with the

gating winding is constant:

$$\Delta\Psi\ (I_c) = -2\Psi_s = \text{const}$$

for any value of $I_c w_c > 0$ and for any duration of the resetting half-cycle. In such a magnetic amplifier, there is no way of controlling the load current effectively. Actually, the magnetization of the core during a resetting cycle takes a different path owing to hysteresis, eddy currents and some other factors. As a result, the change in the flux linkage with the gating winding is

$$\Delta\Psi\ (I_c) = \Psi\ (I_c) - \Psi_s < 0$$

and, with the resetting half-cycles being all of the same duration, it depends on the value of $I_c w_c$ and the properties of the core material (the parts of the graphs between points 1 and 2).

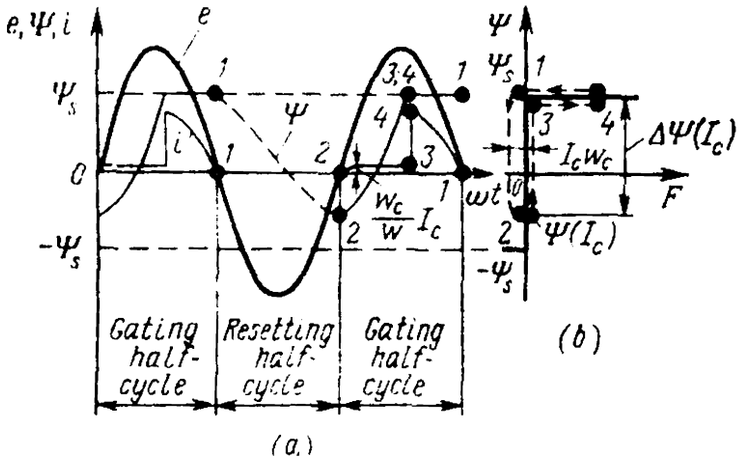


FIGURE 9.9

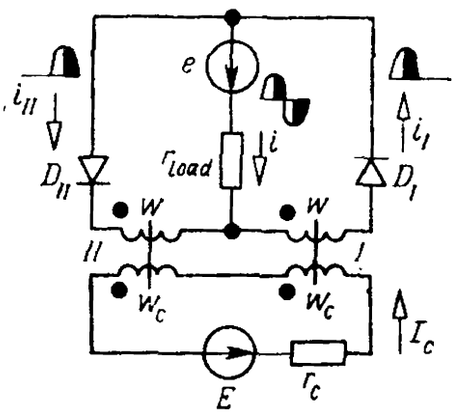


FIGURE 9.10

During the subsequent gating half-cycle the diode is conducting, and the processes occurring in the load circuit are described by the state equation (9.2). At first the core is magnetized along the up-sloping portion of the ideal magnetization curve (see Fig. 9.9 b) with $F = iw - I_c w_c = 0$, that is, at a constant load circuit, $i = I_c\ (w_c/w)$. Now the flux linkage with the gating winding varies from $\Psi\ (I_c)$ to Ψ_s (between points 2 and 3 on the graphs). Once the flux linkage with the gating winding reaches saturation and until the end of the gating half-cycle all of the source voltage appears across the load resistance r_{load}, so that $e = v_r$. The load current abruptly goes up (from point 3 to point 4 on the graphs), following which it begins to vary sinusoidally (from point 4 to point 1 on the graphs).

By combining two half-wave self-saturating magnetic amplifiers, we obtain what is known as a *full-wave self-saturating magnetic amplifier*. One of the likely ways of doing this is shown in Fig. 9.10. All events in amplifier I are shifted from those in amplifier II through a half-cycle of the sinusoidal supply voltage. More specifically,

during a particular half-cycle of the sinusoidal emf when diode D_I is conducting, diode D_{II} is nonconducting, so the gating half-cycle of amplifier I is the resetting half-cycle of amplifier II. During the next half-cycle diode D_I turns off, diode D_{II} turns on, so the resetting half-cycle of amplifier I now is the gating half-cycle of amplifier II.

During the resetting half-cycles the control winding has a very high inductance and, since each half-cycle of the supply emf is the resetting half-cycle of either amplifier I or amplifier II, the inductance of the respective control winding limits the a.c. component in the control circuit. As a result, we may omit the current-limiting inductor which has to be used in the control circuit of a half-wave magnetic amplifier. In this way, the resistance of the control circuit, r_c, is brought down and the amplifier sensitivity is enhanced.

The control characteristic. We will analyse the control characteristic of a half-wave self-saturating magnetic amplifier (see Fig. 9.8 a) by reference to the weber-ampere characteristic in Fig. 9.9 b and the time dependences of voltages, currents and flux linkages in Fig. 9.9 a. Since the processes taking place during the gating half-cycles are described by the state equation (9.2), we can integrate it over the entire gating half-cycle and obtain

$$r_{\text{load}} \int_0^{T/2} i\,dt + \int_{\Psi(I_c)}^{\Psi_s} d\Psi = r_{\text{load}} T I_{\text{av}} - \Delta\Psi(I_c)$$

$$= \int_0^{T/2} E_m \sin \omega t\,dt = (T/\pi) E_m$$

Hence, in analytic form, the control characteristic may be defined as

$$I_{\text{av}} = \frac{(T/\pi) E_m + \Delta\Psi(I_c)}{r_{\text{load}} T} \tag{9.13}$$

where I_{av} is the average load current.

Let us assume that variations in flux linkage, $\Delta\Psi$, are a linear function of the control current I_c, which is nearly so in reality. Then the control characteristic for an ideal half-wave self-saturating magnetic amplifier will also be linear. Since $\Delta\Psi(I_c) < 0$, it will be analogous to that shown in Fig. 9.8 b. If we allow the control current to exceed a certain limiting value, $I_c > I_{c.lim}$, the magnetic amplifier will lose its amplifying properties.

As we have seen, the gating and resetting half-cycles in a self-saturating magnetic amplifier are separated. Therefore, given the same power delivered to the load, the control current may be substantially smaller than it is in a magnetic amplifier in which the gating

windings of two saturable reactors are connected in series and in which the operating cycle is not separated into a gating and a resetting half-cycle. For this reason we may markedly reduce the power required for control purposes and raise the power gain to as high as 10^4. In self-saturating magnetic amplifiers, the response time (that is, the time it takes the core magnetization to change in response to the control signal E) is determined by the duration of the resetting half-cycle. The speed of response can be increased by reducing the duration of the resetting half-cycle, that is, by raising the frequency of the supply emf.

9.4 The Reversible Magnetic Amplifier

In a *reversible magnetic amplifier* each time the control signal reverses in polarity, there is a reversal of phase in the case of a.c. load current or a reversal of direction in the case of d.c. load current. This type of magnetic amplifier can be based on both magnetic am-

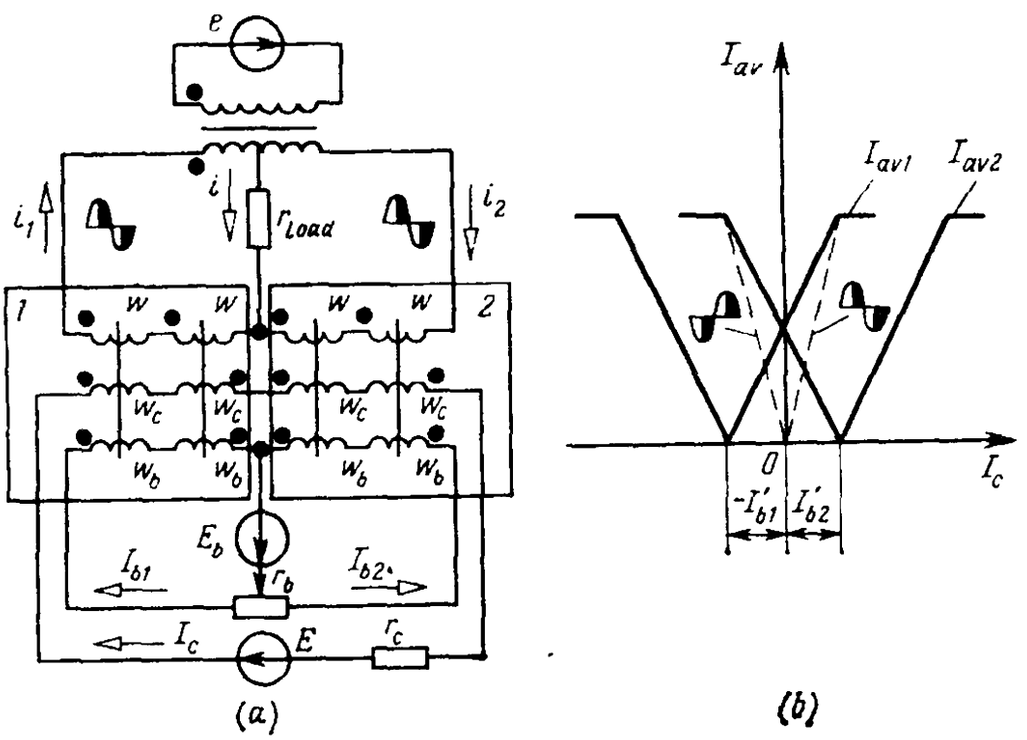


FIGURE 9.11

plifiers with the gating windings of the two saturable reactors connected in series, and self-saturating magnetic amplifiers.

An elementary reversible magnetic amplifier with an alternating current in its load circuit is shown in Fig. 9.11 a. It consists of two nonreversible magnetic amplifiers, 1 and 2, with the gating windings of their saturable reactors connected in series in each amplifier. The two magnetic amplifiers are energized from two identical sec-

ondary windings of a transformer in such a way that the mmfs due to the control and bias currents are combined in the first amplifier

$$I_c w_c + I_{b1} w_b$$

and are subtracted in the second amplifier

$$I_c w_c - I_{b2} w_b$$

For this reason, the control characteristic of the first amplifier is shifted along the axis of abscissae through

$$-I'_{b1} = -(w_b/w_c) I_{b1}$$

and that of the second by

$$I'_{b2} = (w_b/w_c) I_{b2}$$

(see Fig. 9.11 b).

The function of the variable resistor r_b is to set the load current to zero in the absence of the control current ($I_c = 0$). As the control current rises ($I_c > 0$), the alternating current rises in the gating windings of the first amplifier (i_1 increases) and falls in the gating windings of the second amplifier (i_2 decreases). When the direction of the control current is reversed ($I_c < 0$), the a.c. load current, $i = i_1 - i_2$, undergoes a phase reversal. The control characteristic, $I_{av}(I_c)$, where $I_{av} = I_{av1} - I_{av2}$, for such an amplifier is shown dashed in Fig. 9.11 b.

Reversible magnetic amplifiers with a.c. and d.c. load are used to reverse electric motors, in control systems, etc.

10 Semiconductor Devices and Circuits

10.1 General

For insight into the operation of semiconductor devices, we will first look into the basic electrophysical processes that take place in a semiconducting material.

The term 'semiconductor' applies to a wide class of materials whose resistivity is in the range 10^{-6}-10^{8} Ω m. These are the chemical elements silicon (Si) and germanium (Ge), the oxides Cu_2O and ZnO, and a number of other compounds. Most often semiconductor devices are fabricated from silicon and germanium.

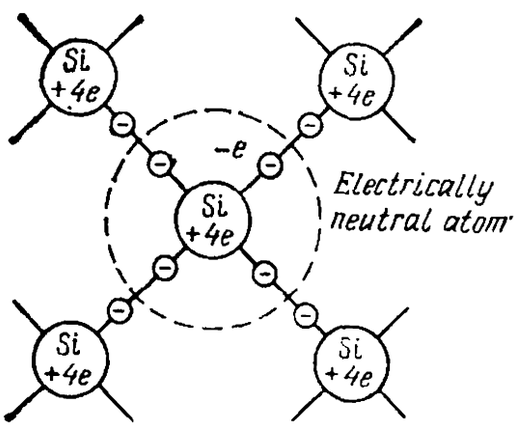


FIGURE 10.1

Figure 10.1 shows the electron structure of an ideal tetrahedral crystal of silicon in which each of the four valence electrons in any atom forms an electron-pair or covalent bond with the valence electrons of the four neighbouring atoms. So long as the atoms of this ideal silicon crystal are not exposed to external sources of energy (light or heat) capable of destroying its electron structure, all the atoms remain electrically neutral. Such an ideal silicon crystal does not conduct electric current.

However the electric properties of an ideal silicon crystal can be substantially altered by adding some impurity (or impurities) to it. The impurities usually employed for this purpose are the elements antimony (Sb) and phosphorus (P) belonging to Group V of the Periodic System, or the elements gallium (Ga) and indium (In) belonging to Group III of the Periodic System.

Figure 10.2 a shows the electronic structure of a tetrahedral silicon crystal doped with a controlled amount of phosphorus which has five valence electrons. Four of the valence electrons of the impurity are seen to form four covalent bonds with the valence electrons of the four neighbouring silicon atoms. The fifth valence electron of the impurity is left in excess. With an input of a small amount of external energy (heat at room temperature), this excess electron breaks loose from its impurity atom to become a free electron. On losing it,

the parent impurity atom becomes a positive ion, while the free electron forms conduction current. Thus in such a material conduction is by electrons and, since an electron carries a negative (n) charge it is classed as an *n-type semiconductor*. The symbol for an ideal *n*-

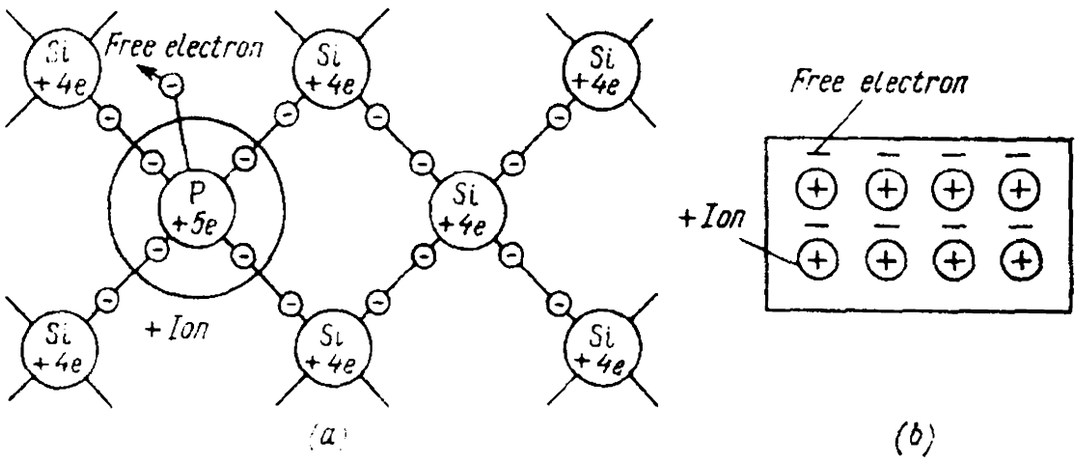


FIGURE 10.2

type semiconductor is shown in Fig. 10 2 b where a "+" sign stands for a fixed positive ion, and a "—" sign for a free electron.

When the impurity is indium which has three valence electrons, the situation is different (Fig. 10.3 a). Now one of the four possible covalent bonds remains incomplete, and the missing electron con-

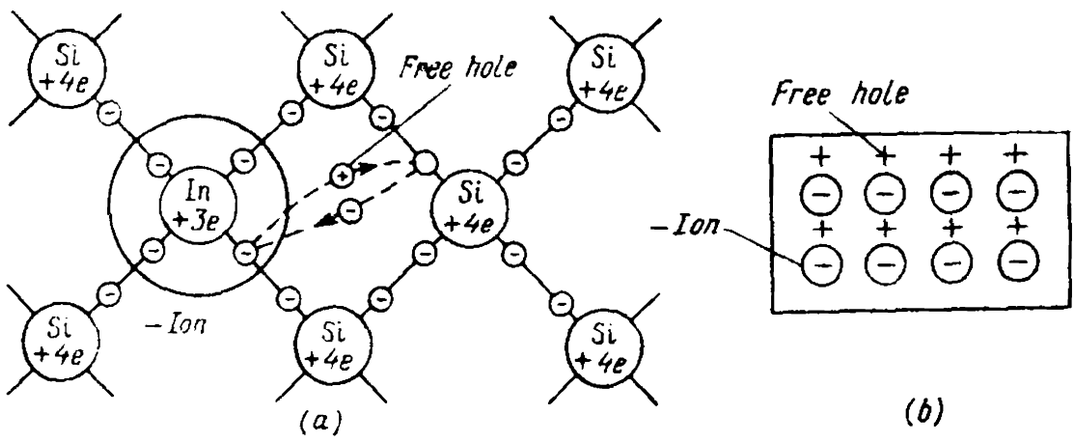


FIGURE 10.3

stitutes what is known as a hole. The hole may be neutralized electrically by an electron jumping into it from a nearby bond (under the action of, say, room temperature) and thus effectively moving the hole to a new position in the silicon. The indium atom is thus turned into a stable fixed negative ion, and the hole may again be neutralized by another electron jumping from another nearby bond, and so on. From an electrophysical point of view, this process may be visualized as a chaotic motion of holes through the crystal. Since in such a material conduction current is constituted by holes and each

hole acts the same as a positive (p) electron charge, we have what is known as a *p-type semiconductor*. The symbol for a p-type semiconductor is shown in Fig. 10.3 b.

For their operation, most semiconductor devices depend on the processes that occur at the boundary, or junction, between an n-type and a p-type material. We will look into them by taking as an example an idealized plane-parallel structure (Fig. 10.4 a).

To begin with, let the circuit containing a d.c. voltage source be open (the switch *Sw* in Fig. 10.4 is in position 3). Because there is a difference in hole and free electron concentration (or, which is the same, because there is a hole and free electron density gradient) across the junction, some holes diffuse from the p-type into the n-type material; similarly some electrons diffuse from the n-type into the p-type material. This goes on until the region next to the junction becomes depleted of holes on the p-side and of electrons on the n-side. This results in the formation of a layer of fixed negative ions on the p-side and a layer of fixed positive ions on the n-side. Together they form what is known as the *depletion region* because it is depleted of mobile charge carriers (holes and electrons). The electric field set up between the two ion layers, of intensity $\mathscr{E}$, tends to oppose the motion of holes and free electrons across the junction. At a certain value of electric intensity their diffusion ceases completely. This is the reason why the depletion region is also called the barrier region. If we take the electric potential at the junction ($x = 0$) equal to zero, $\varphi(0) = 0$, then the potential distribution on either side of the junction will be given by (see Fig. 10.4 b)

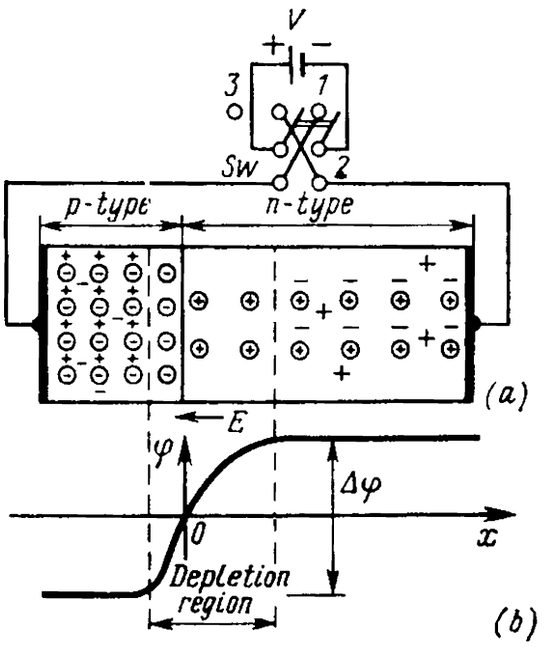


FIGURE 10.4

$$\varphi = \int_0^x \mathscr{E}\, dx$$

This variation in electrostatic potential across the depletion (or barrier) region, $\Delta\varphi$, constitutes a *potential-energy barrier*, and its value is known as the *potential-barrier height*. Because the electrostatic-potential variation in the deplection region is gradual, the region is alternatively called the *transition region*.

Now we close the switch *Sw* (by moving it to position 1 in Fig. 10.4 a). This connects the negative side of the d.c. voltage source to

the *p*-type side of the junction and its positive side to the *n*-type side of the junction. This condition is known as *reverse bias* (or biasing). As a result, both the electric field intensity $\mathscr{E}$ and the potential-barrier height are increased, and no current can flow around the circuit. Finally, we place the switch *Sw* in position 2. This connects the positive side of the source to the *p*-side of the junction and the negative side of the source to the *n*-side of the junction. This condition is known as *forward bias* (or biasing). The forward-bias voltage brings down the potential-barrier height, and an electric current begins to flow around the circuit. Thus, in the case of an ideal transition region an electric current can flow across the junction in one direction only.

In real semiconductors and real transition regions the basic processes described above are accompanied by other, secondary events which may, however, have a marked effect on the performance of a semiconductor device.

It may be noted that a transition region can be produced in some other cases, such as between two *n*-type regions differing in resistivity, between a semiconductor and a metal, etc. The processes occurring in and/or across such a *transition region* are utilized in various semiconductor devices.

10.2 Semiconductor (Crystal) Diodes

A *semiconductor diode* is a device with two terminals. For its operation it depends on the property of a *p-n* junction to conduct current in one direction and to block its flow in the reverse direction. Accordingly, there is a *forward* and a *reverse current*, a *forward* and a *reverse voltage* associated with a semiconductor diode.

The performance of a semiconductor diode is usually specified by giving its current-voltage characteristic the shape of which depends on the manner of making its *p-n* transition region, the concentration of holes and free electrons, mechanical design, and so on.

Figure 10.5 shows a graphic symbol for a semiconductor diode and its current-voltage characteristic. The forward current is that which flows from the *positive terminal*, or *anode*, towards the *negative terminal*, or *cathode*. The ratings of a semiconductor diode include its maximum forward current I_f, the respective forward voltage V_f, the maximum reverse voltage V_{rev} and the respective reverse current I_{rev}, and the maximum temperature (which is up to 50 °C for germanium diodes and up to 140°C for silicon diodes).

Semiconductor diodes may be the point-contact type and the junction type. In a *point-contact diode* the transition region is very small in comparison with the semiconductor region surrounding it and controlling the physical processes at the junction. A point-contact transition region can be produced, for example, by welding the tip of a

metal needle, called the whisker, into the crystal surface by means of a "shot" of current. Atoms of a suitable impurity in the wire tip diffuse into the semiconductor crystal and form a *p-n* transition region close to the point of contact. Because of the small surface area of the junction, point-contact diodes are low-power devices mainly used at microwave frequencies. The maximum power dissipation for

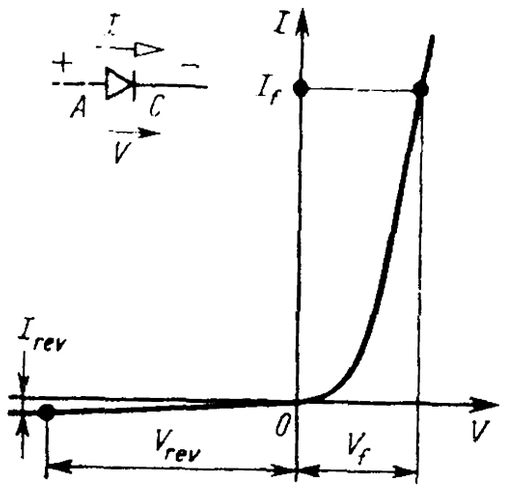


FIGURE 10.5

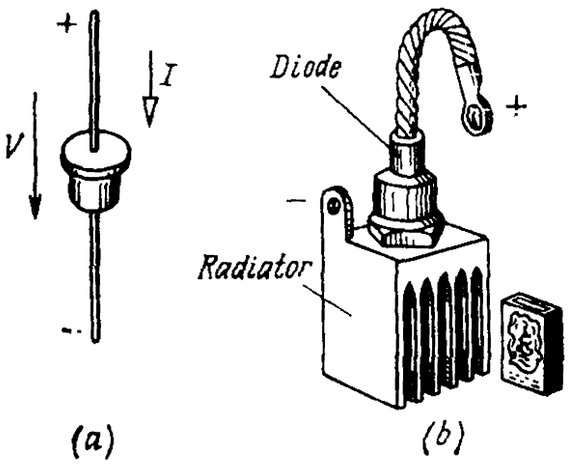


FIGURE 10.6

point-contact diodes is about 10 mW for a forward current of 10-20 mA.

In a *junction diode*, the transition region is spread over a surface area of up to 1000 mm^2 (such as in power diodes). The junction is produced either by joining together two wafers of a *p*-type and an *n*-type material, or by diffusing suitable impurity atoms into a wafer. Since the transition region has a large surface area, the maximum power dissipation for naturally cooled low-power junction diodes (Fig. 10.6 a) may be as high as 1 W for a forward current of up to 1 A. Such diodes are often used in automatic control circuits and instrumentation.

In the case of high-power junction diodes (Fig. 10.6 b) either fitted with heat sinks or radiators, or artificially cooled (with air or water), the maximum power dissipation may be as high as 10 kW for a forward current of up to 1 kA and a reverse voltage of up to 1.5 kV. High-power junction diodes are mainly used in rectifiers and a.c.-to-d.c. converters.

There is a special variety of semiconductor diodes known as *Zener diodes* widely used for voltage regulation in electric circuits. A Zener diode owes its regulating characteristics to the nondestructive electric breakdown of its *p-n* junction, at certain values, V_{bd}, of the reverse voltage (Fig. 10.7 a). The value of breakdown voltage depends on the design of the *p-n* junction and the electrophysical properties of the semiconductor material.

Figure 10.7 b shows the circuit of an elementary **voltage regulator** built around a Zener diode. As the input voltage V_{in} varies so

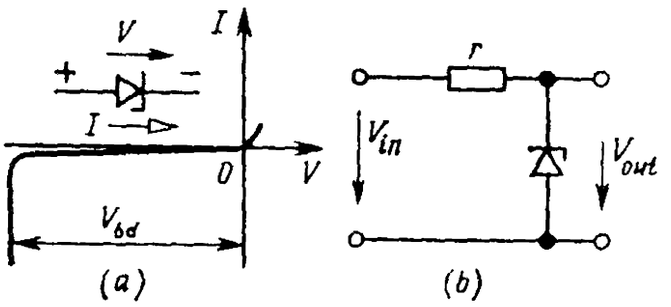


FIGURE 10.7

that it exceeds the breakdown value, $V_{in} > V_{bd}$, the output voltage, $V_{out} = V_{bd}$, varies very little.

10.3 Transistors

Transistors are semiconductor devices mainly used to amplify electric signals in power. They may be classed into bipolar transistors and field-effect transistors (or FETs).

A *bipolar transistor* is composed of two *p-n* junctions on a common piece of semiconductor material in very close proximity to each other

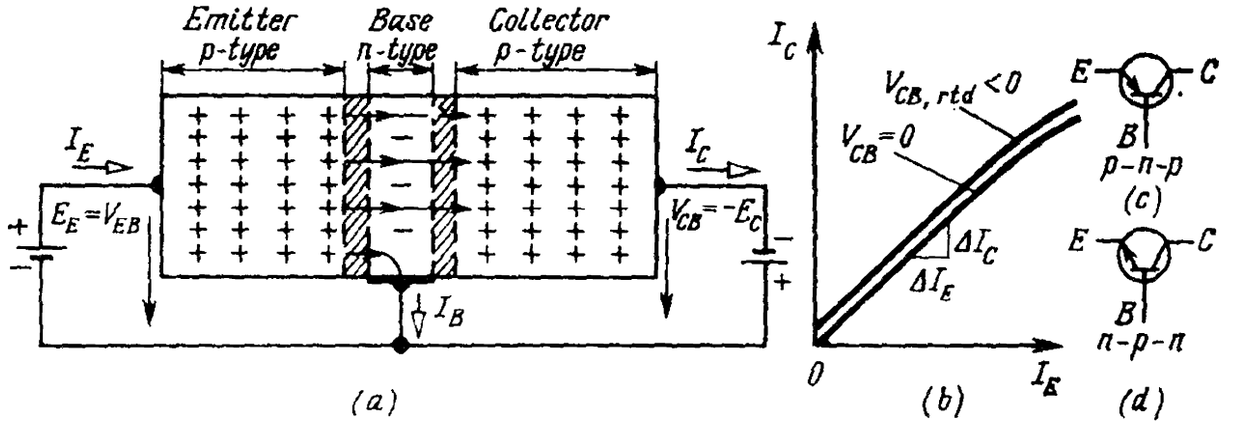


FIGURE 10.8

so that the carriers which flow through one *p-n* junction may reach the other junction before they recombine and disappear.

Bipolar transistors may be the point-contact type and the junction type. In a point-contact transistor, the transition regions are very small in area and are similar in arrangement to that in a point-contact diode (see Sec. 10.2). They have not found any appreciable use.

A *junction transistor* is a three-layer structure (Fig. 10.8 a) in which an *n*-type layer is sandwiched between two *p*-type layers.

Accordingly, it is called a *p-n-p* structure. When a *p*-type layer is sandwiched between two *n*-type layers, we have an *n-p-n* structure. Symbols for the two types of junction structures are shown in Fig. 10.8 c and d.

This type of transistor is called bipolar because it uses both holes (positive charge carriers) and electrons (negative charge carriers) for its operation.

The middle layer is called the *base*, that on one side of the base is called the *collector*, and that on the other side is called the *emitter*. Each layer, or region, has a terminal of its own. With its three terminals, a bipolar transistor can be connected in an electric circuit.

A bipolar transistor may operate in any one of the following regions.

(1) The *active region* when the emitter-base junction is forward-biased and the collector-base junction is reverse-biased.

(2) The *inversion region* when the emitter-base junction is reverse-biased and the collector-base junction is forward-biased.

(3) The *cut-off region* when both junctions are reverse-biased.

(4) The *saturation region* when both junctions are forward-biased.

The active region is the basic form of operation for a bipolar transistor. We will consider it in more detail, taking an ideal bipolar transistor as an example.

Referring to Fig. 10.8 a, it is seen that the circuit contains two voltage sources, one being in the emitter lead, $E_E = V_{EB}$, and the other in the collector lead, $E_C = -V_{CB}$. If we connect the negative side of the E_E source to the base and its positive side to the emitter, the potential-barrier height at the *p-n* junction between the base and emitter will be brought down, and holes will be injected from the emitter region into the base region, thereby constituting the emitter current I_E. If we connect the positive side of the E_C source to the base, the potential-barrier height at the *p-n* junction between the base and collector will be increased. As a result, most of the holes injected from the emitter region into the base region will be pulled in by the strong reverse electric field of that *p-n* junction, thereby giving rise to the collector current I_C. It is to be noted that an electric field exists in the collector-base junction even when the E_C source is disconnected (see Fig. 10.4 b). Therefore, the collector current depends but little on $V_{CB} < 0$, down to $V_{CB} = 0$ (Fig. 10.8 b). A small fraction of the holes injected from the emitter into the base constitutes the base current I_B. When practically none of the holes injected from emitter to base remain available ($E_E = 0$ and $I_E \approx 0$), there still remain small currents in the collector and base leads, $I_C \approx -I_B$, each equal to the reverse diode current, that is, the current through the base-collector junction, when the emitter-base circuit is shorted and $V_{CB} < 0$. The value of this current specified at $V_{CB,rtd}$ is a service rating of transistors.

Several conclusions may be drawn from the analysis of an idealized transistor model carried out above.

The number of holes injected from emitter to base strongly depends on V_{EB}. Nearly all of them reach the collector irrespective of $V_{CB} \leqslant 0$. Therefore, V_{EB} may be regarded as a control signal for the collector current I_C at V_{CB} held constant.

The direct currents in the collector, emitter and base leads of a transistor determine its *static* mode of operation. When transistors are used in circuits such as amplifiers, it is important to know how a small change in the emitter-to-base (that is, control) voltage, ΔV_{EB}, is related to a small change in these currents, ΔI, at V_{CB} held constant. The change in the emitter-to-base voltage, ΔV_{EB}, is the *a.c.* (*dynamic* or *incremental*) *component* of the control voltage. We will consider these relations, while neglecting the transients that are likely to occur in a transistor owing to the finite velocity of holes and electrons and some other factors. Under this assumption, all transistor currents may change suddenly as well.

By Kirchhoff's current law

$$\Delta I_E = \Delta I_C + \Delta I_B$$

For small values of the alternating component of the control voltage a circuit containing a transistor may be treated as linear with one voltage source, ΔV_{EB}. By virtue of Eq. (1.35) the currents in the various branches of the circuit are connected by a linear relation (see Fig. 10.8 b)

$$\Delta I_C = \alpha \Delta I_E \tag{10.1}$$

$$\Delta I_C = \alpha \Delta I_B/(1 - \alpha) = \beta \Delta I_B \tag{10.2}$$

where

$$\alpha = dI_C/dI_E \,|_{V_{CB}=\text{const}}$$

is called the a.c. (dynamic or incremental) emitter current transfer ratio or gain, and β is the a.c. (dynamic or incremental) base current transfer ratio or gain. Since they are measured for small values of the control voltage, they are called the small-signal emitter current gain and the small-signal base current gain, respectively, and the transistor itself is said to operate in the *small-signal condition.*

The two gains, alpha and beta ($\alpha = 0.98$-0.99 and $\beta = 50$-100), are service ratings of bipolar transistors.

The assumption that a transistor in a circuit is free from transients limits the applicability of the parameters alpha and beta to real

circuit calculations. If the a.c. component of the control signal varies sinusoidally

$$\Delta v_{EB} = V_m \sin \omega t$$

then $\Delta i_C/\Delta i_E = \alpha$ and $\Delta i_C/\Delta i_B = \beta$ are only valid at low frequencies. At high frequencies transients may not be neglected any longer, and the relations (10.1) and (10.2) are violated. This limitation must be borne in mind when carrying out design calculations with the use of the parameters alpha and beta.

As is with junction diodes, bipolar junction transistors may be fabricated by the alloy process and the diffusion process. Several other processes are in use, and this fact is reflected in the name of the transistors thus fabricated. Thus we have planar transistors, micro-alloy-type transistors, mesa-type transistors, etc.

When fitted with heat sinks and radiators, bipolar junction power transistors may operate at a power dissipation of as high as 50 W at currents up to 10 A.

The *field-effect transistor* is a semiconductor device which depends for its operation on the control of current by an electric field. There are two types of field-effect transistors, the junction field-effect transistor (abbreviated JFET, or simply FET) and the metal-oxide-semiconductor field-effect transistor (abbreviated MOSFET). We will limit ourselves to the FET.

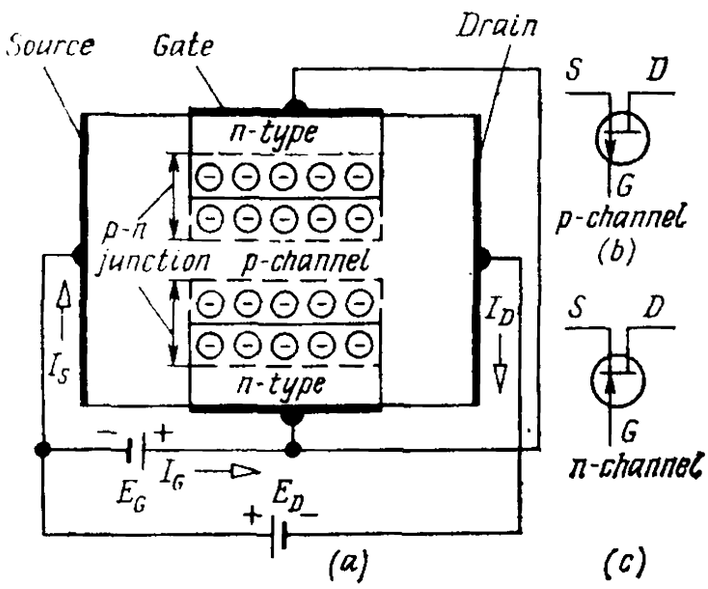


FIGURE 10.9

The basic structure of what is called a *p-channel FET* is shown in Fig. 10.9 a. It is essentially a semiconductor bar of *p*-type material, called the *channel*, to the two ends of which two ohmic contacts are made, one called the *source* and the other the *drain*. Current is caused to flow along the channel by the voltage supply E_D connected between the source and the drain. The value of current through the channel depends on the latter's conductivity, and this is in turn dependent on the channel width which can be varied at will. To this end, there is a third electrode, called the *gate*, and one more voltage supply, E_G, connected to the gate on the positive side and to the source on the negative side. Therefore, the *p-n* junction between the *p*-channel and the *n*-type material at the gate is reverse-biased. The width of each *p-n* junction affects the width of the *p*-channel and,

in the final analysis, its conductivity. When a bar of *n*-type material is used for the channel and the gate is a *p*-type material, the device is referred to as an *n-channel FET*. For the *p-n* junction between the channel and the gate to be reverse-biased, the positive side of the E_G supply must be connected to the source.

Since FETs depend for their operation upon the flow of majority carriers only (holes in a *p*-channel or electrons in an *n*-channel), they are alternatively called *unipolar devices*.

An important feature of FETs is that they retain their ability to function properly even upon reversal of polarity between the source and the gate.

Symbols for *p*-channel and *n*-channel FETs are shown in Fig. 10.9 b and c.

FETs are essentially low-power devices and are mainly used in instrumentation. The current in a conducting channel is 1-10 mA for a source-to-drain voltage of 5-10 V.

10.4 Thyristors

A *thyristor* is a bistable device comprising three or more junctions in which at least one of the junctions can switch between reverse- and forward-voltage polarity. The most commonly used type of thyristor is one with four or more alternating *p* and *n* regions.

Thyristors may be two-terminal and three-terminal. The former are called diode thyristors and, similarly to a semiconductor diode (see Fig. 10.5), have each an anode terminal and a cathode terminal. Three-terminal thyristors, or triode thyristors, additionally have a third terminal called the *gate* (G).

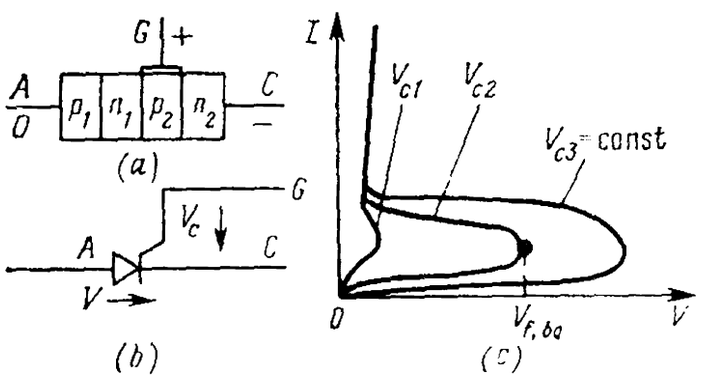


FIGURE 10.10

Let us consider the static characteristics of a three-terminal thyristor in which the three *p-n* junctions are arranged as shown in Fig. 10.10 a. (It is assumed that the anode is at zero potential.) The graphic symbol for a three-terminal thyristor is shown in Fig. 10.10 b.

It is to be noted that the current in the gate lead is usually small and that the anode and cathode currents may be assumed to be the same equal to *I*.

For the four-layer structure of Fig. 10.10 a, the current-voltage characteristics (the dependence of the current *I* on the voltage between anode and cathode *V*, with the voltage between the gate and cathode, V_G, held constant) has three characteristic regions (Fig.

10.10 c). Within the first region from $V = 0$ to the forward breakover voltage $V = V_{f.bo}$ the thyristor is turned off, and the current I is small. Within this region the dynamic resistance r_d of the thyristor is positive

$$r_d = dV/dI$$

Within the second region r_d is negative. An increase in I brings about a decrease in V, thereby leading to a further increase in I, and so on. This is an unstable region, and the thyristor can spontaneously pass into the third region within which the thyristor is turned on (fully conducting), and the dynamic resistance becomes positive again.

Since the forward breakover voltage, $V_{f.bo}$, depends on the control voltage, V_c, a thyristor can be used as a controlled switch. Thyristors are mainly used in converters, inverters, motor control, etc. (see Sec. 16.6). The rated "ON" current I for some thyristors is as large as 5 kA, and the breakover voltage as high as 5 kV.

10.5 Semiconductor Circuits

Semiconductor devices are widely used in a variety of automatic-control, telemetry, data-processing and similar applications. Our discussion will be limited to rectifiers, amplifiers, pulse circuits, and logic gates, as they are most important industrially.

Rectifiers serve to convert alternating (mostly sinusoidal) currents and voltages to direct currents and voltages. They may, for example, be used in d.c. drives when these draw their power from an a.c. power line.

Amplifiers boost a particular parameter of signals to a value sufficient for operation of the end element. For example, the signals picked up by a radio antenna are amplified to a level sufficient to drive a loudspeaker.

Pulse circuits and *logic gates* are essential elements of various automatic control systems. Pulse circuits coordinate the operation of control system's elements on a time basis, and logic gates on a logic basis.

It is to be noted that the above classification of semiconductor circuits is arbitrary. As often as not, a semiconductor circuit may combine the functions of several types and include oscillators, voltage regulators, etc.

10.6 Rectifiers

Rectifiers can be uncontrolled and controlled. In the former, the rectifying elements used are semiconductor diodes; in the latter, they are thyristors.

An elementary half-wave rectifier (Fig. 10.11 a) contains a source of a sinusoidal voltage $v = V_m \sin \omega t$, a semiconductor diode, and a load resistor r_{load}.

For simplicity, let us assume that the circuit uses an ideal diode, that is, one with a forward resistance of zero and a reverse resistance of infinity. Such a diode acts as a short-circuit for the forward current and as an open-circuit for the reverse current. Therefore, the current flowing through the load (Fig. 10.11 b) is a periodic sequence of half-sinewaves

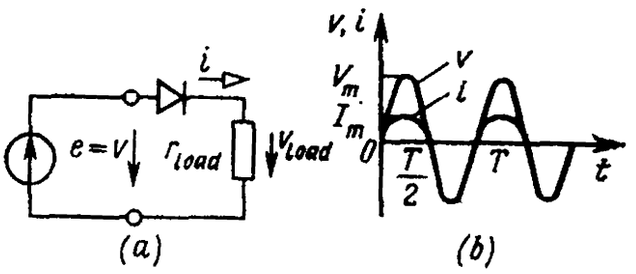


FIGURE 10.11

$$i = I_m \sin \omega t$$

where

$$I_m = V_m / r_{load}$$

during the half-cycles of supply voltage when $v > 0$, and is zero when $v < 0$.

In the case of half-wave rectification, the direct component of the load current, that is, the average current is

$$I_0 = \frac{1}{T} \int_0^{T/2} i \, dt = \frac{1}{T} \int_0^{T/2} I_m \sin \omega t \, dt = (1/\pi) I_m \approx 0.32 I_m \tag{10.3}$$

The root-mean-square (rms) current is

$$I = \sqrt{\frac{1}{T} \int_0^{T/2} i^2 \, dt} = \sqrt{\frac{1}{T} \int_0^{T/2} I_m^2 \sin^2 \omega t \, dt} = I_m/2 \tag{10.4}$$

The voltage across the load resistor

$$v_{load} = r_{load} i = r_{load} I_m \sin \omega t$$

is equal to the supply voltage during the half-cycles when $v > 0$ and the current is nonzero; and it is zero at $v < 0$ and the current is zero. This is the *rectified voltage*. Its d.c. component is

$$V_0 = r_{load} I_0 = (r_{load}/\pi) I_m = V_m/\pi \tag{10.5}$$

That is, this voltage is half as great as the average of the magnitude of a sinusoidal voltage having the same amplitude, Eq. (2.15).

The active power delivered to the load is equal to that supplied by the source

$$P = r_{load} I^2 = r_{load} I_m^2/4 = V_m I_m/4 \tag{10.6}$$

In *a full-wave rectifier* (Fig. 10.12), the load resistor r_{load} is connected between the common node of two diodes and the centre tap on the secondary of a transformer. If, in each half of the secondary

winding, we assume that the positive direction of current is that in which the respective diode is conducting, a current will be flowing in both half-windings. The current through the respective diode will then be sinusoidal during the positive half-cycles (of that half-winding) and zero during the negative half-cycles. In the load, the positive directions of the two currents are the same. Now the direct

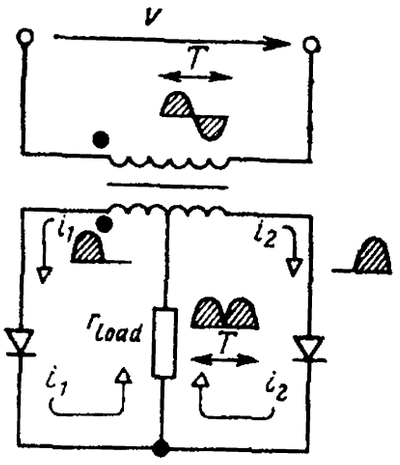


FIGURE 10.12

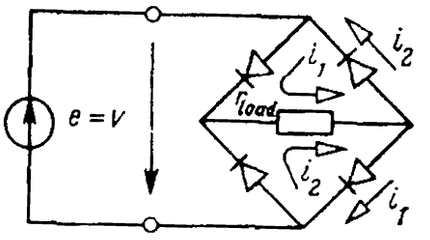


FIGURE 10.13

component of load current is twice as large as it is in the case of half-wave rectification. Given ideal diodes, it would be

$$I_0 = (2/\pi)\, I_m \approx 0.64 I_m \tag{10.7}$$

The value of the integral $\int_0^{T/2} i^2\, dt$ will be the same during either half-cycle, and so the rms load current will be

$$I = \sqrt{\frac{2}{T}\int_0^{T/2} i^2\, dt} = \sqrt{\frac{2}{T}\int_0^{T/2} I_m^2 \sin^2 \omega t\, dt} = I_m/\sqrt{2} \tag{10.8}$$

That is, the rms value of rectified current is $\sqrt{2}$ times the value obtained for half-wave rectification, and is equal to the rms value of a sinusoidal current of the same amplitude, Eq. (2.18a).

The transformer in a full-wave rectifier acts as a voltage divider (see Fig. 10.12). The same function can be served by an autotransformer. In some cases it may be preferable to use a *full-wave bridge rectifier* (Fig. 10.13). This circuit uses four diodes which form the four arms of a bridge. The sinewave voltage source is placed across one pair of opposite junctions, and the load resistor across the other pair. During one half of the alternating current wave two diodes in the opposing arms are conducting a current i_1, while the other two diodes are turned off. During the second half of the cycle, the diodes that

were off are now conducting a current i_2, and the previously conducting diodes are off. The current through the load resistor is the same as in a full-wave rectifier, so we may apply all the relations derived earlier to the bridge rectifier as well.

Semiconductor diodes show poor rectifying properties at low values of voltage (see Fig. 10.5). In a bridge rectifier, two diodes in series are operating at any instant. So at low input voltages this form of rectifier is less efficient in its performance than a full-wave rectifier (see Fig. 10.12).

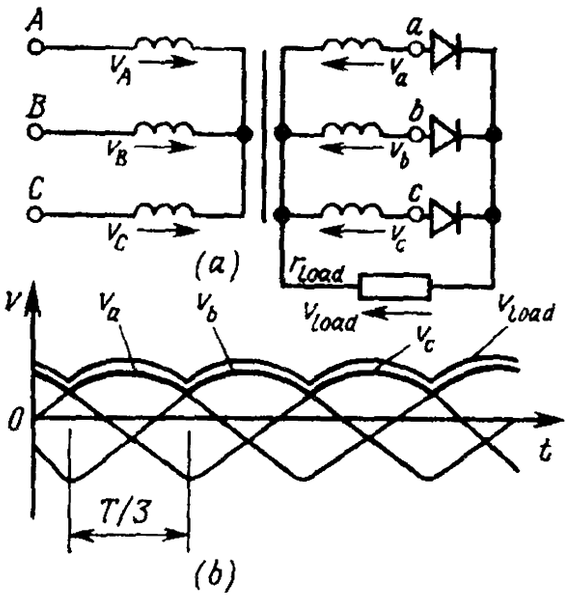


FIGURE 10.14

A full-wave rectifier produces a load current whose value varies from a maximum to zero, and this introduces pulsations or the ripple. The ripple can effectively be reduced through the use of *polyphase rectification*. Figure 10.14 a shows the circuit of a three-phase rectifier. In this arrangement only that diode can conduct at any instant which is connected to the transformer secondary (a, b, or c) across which the voltage (v_a, v_b, or v_c) is positive and the highest. The waveform of the rectified voltage v_{load} is the same as the envelope of the positive half-waves of the secondary voltages (Fig. 10.14 b).

The three-phase bridge rectifier shown in Fig. 10.15 a combines the features of a single-phase bridge rectifier and of a polyphase rectifier. In this arrangement the secondaries of the transformer need not be joined at a common junction, so they may equally be connected in a delta as well as in a star. One half of the diodes make up a group in which all the cathodes are tied together; the remaining half of the diodes form a group in which all the anodes are tied together. The ripple in the rectified voltage is reduced still more as compared with a three-phase bridge rectifier, and the ripple frequency is twice the number of rectifier phases. For a three-phase rectifier system, the ripple occurs six times every cycle (see Fig. 10.15 b).

Consider what happens in this case. Suppose that the three-phase system of voltages is supplied by an imaginary star-connected balanced source as shown dashed in Fig. 10.15 a. At any instant only that diode in the first group is conducting whose anode is at the highest positive potential relative to the neutral point N of the imaginary generator, and so is the diode in the second group whose cathode is at the highest negative potential (in absolute value) relative to the same neutral point N.

We set out to determine the dependence of the average rectified voltage (the d.c. component), V_0, on the ripple frequency, m. To this end we integrate the instantaneous voltage across the load resistance,

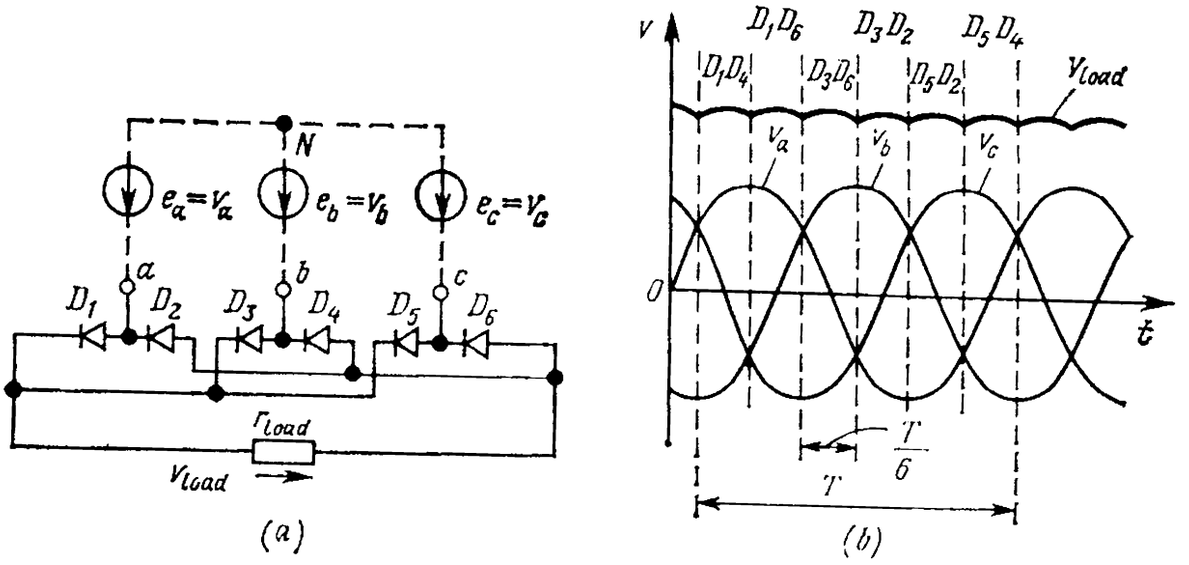


FIGURE 10.15

$v_{load} = V_m \cos \omega t$ (Fig. 10.16) over the interval from $-T/2m$ to $+T/2m$ and find the average value

$$V_0 = (m/T) \int_{-T/2m}^{T/2m} V_m \cos \omega t \, dt = (m/2\pi) \int_{-\pi/m}^{\pi/m} V_m \cos \omega t \, d\omega t$$

$$= V_m \frac{\sin(\pi/m)}{\pi/m} \tag{10.9}$$

As is seen, an increase in m causes the d.c. component V_0 to approach the maximum value V_m.

Now we will turn to the operation of a controlled half-wave thyristor rectifier (Fig. 10.17 a) whose diode counterpart is shown in Fig. 10.11 a, assuming that the voltage to be rectified is $v = V_m \sin \omega t$.

In order to maintain the same operating conditions for the thyristor during the positive half-cycles of the a.c. voltage, it is required that the repetition period of control voltage pulses, v_c (see Fig. 10.17 c), which turn on the thyristor, should be equal to the period $T = 2\pi/\omega$ of the voltage being rectified. If the time interval between the beginning of a positive half-cycle of the input voltage and the arrival of a control pulse is $\Delta t = \alpha/\omega$, where α is the delay angle, then the average voltage across the load, v_{load} (see Fig. 10.17 b) will be

$$V_0 = (V_m/2\pi) \int_{\alpha}^{\pi} \sin \omega t \, d\omega t = V_m \frac{1+\cos \alpha}{2\pi} \tag{10.10}$$

By varying the delay angle α, we can achieve a high efficiency of rectification (95-99%) and control voltages at will.

Controlled polyphase rectifiers are built similarly to the single-phase unit we have discussed above.

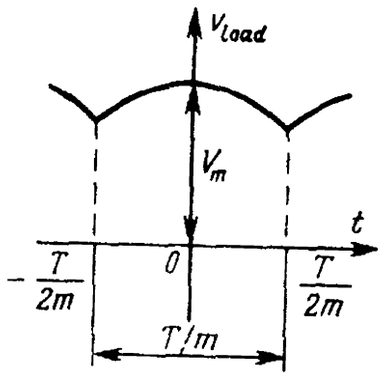


FIGURE 10.16

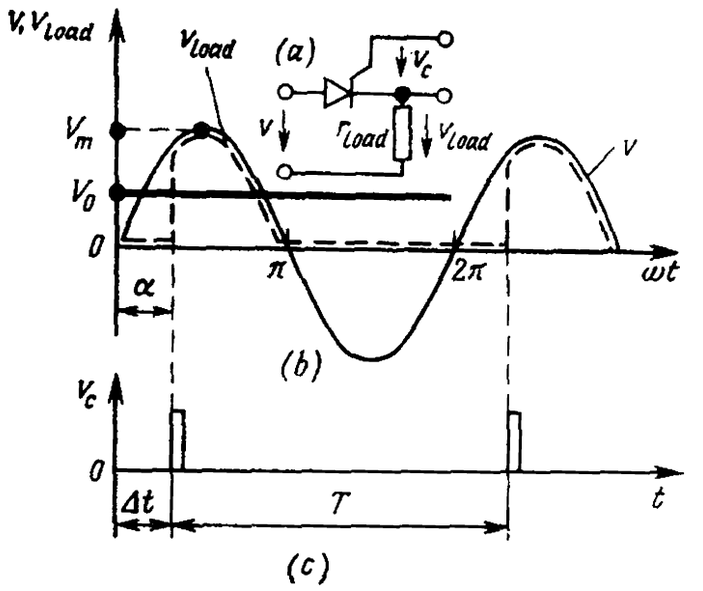


FIGURE 10.17

10.7 Amplifiers

Amplifiers may be classed in many ways. Thus there are d.c. amplifiers to handle constant or slowly varying signals; in them only resistive coupling is used between the stages (if there are several of them). There are amplifiers adapted to handle rapidly varying signals; in them the stages are coupled capacitively.

Amplifiers can boost a signal in current, voltage, or power. Accordingly they are called current amplifiers, voltage amplifiers, or power amplifiers.

For their operation present-day amplifiers mostly depend on the specific properties of bipolar and field-effect transistors which can be connected into single- and multistage amplifiers. We will limit ourselves to amplifiers based on bipolar transistors.

In amplifiers, transistors are mostly operated in the active region, and they may be connected in any one of three circuit configurations, namely *common-base* (Fig. 10.18 a), *common-emitter* (Fig. 10.18 b), and *common-collector* (Fig. 10.18 c). The word "common" in the definition of a circuit configuration refers to the fact that the terminal so defined is common both to the input and output of the amplifier. The most commonly used circuit configuration for transistor amplifiers is the common-emitter circuit (see Fig. 10.18 b).

Let us see what happens in a transistor connected in a common-emitter circuit. We will do this by comparing it with the common-

base configuration examined earlier (see Sec. 10.3). In each case we use the same voltage, $V_{BE} = -V_{EB} < 0$ to control the collector current which only slightly depends on $V_{CB} \leqslant 0$ (see Fig. 10.8 b). By Kirchhoff's voltage law, $V_{CB} = V_{CE} - V_{BE}$. Therefore, the collector current is only slightly dependent on V_{CE}, too, if $V_{CE} \leqslant$

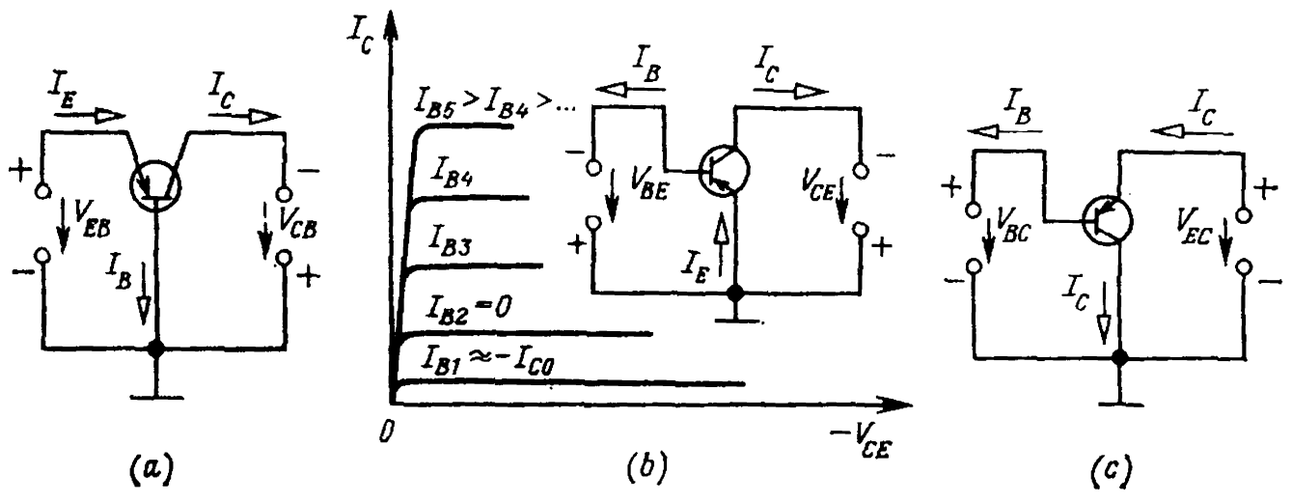


FIGURE 10.18

$\leqslant V_{BE}$. This condition is always satisfied in amplifiers. In consequence, the relation

$$I_C = f(I_E)_{V_{CB}=\text{const}}$$

(see Fig. 10.8 b) which applies to a transistor in a common-base circuit is practically the same as the relation

$$I_C = f(I_E)_{V_{CE}=\text{const}}$$

which applies to a transistor in a common-emitter circuit. To state this differently, the static (d.c.) mode of operation is the same in each case.

In the small-signal mode of operation, both circuit configurations, common-base and common-emitter, have practically the same emitter current gain, Eq. (10.1)

$$\left.\frac{dI_C}{dI_E}\right|_{V_{CE}=\text{const}} \approx \left.\frac{dI_C}{dI_E}\right|_{V_{CB}=\text{const}} = \alpha$$

The same goes for the base current gain, β, Eq. (10.2).

A major advantage of a common-emitter amplifier is that its small-signal input resistance defined as

$$|\Delta V_{BE}/\Delta I_B|$$

is by a factor $\beta/\alpha = 50$ to 100 greater than in a common-base amplifier. The signal source power is reduced, too.

A common-emitter amplifier can conveniently be designed, using the static (d.c.) collector characteristics of the transistor, $I_C\,(-V_{CE})_{I_B=\text{const}}$, shown in Fig. 10.18 b.

Two regions can be noted on the collector characteristics. As $-V_{CE}$ rises from zero, the collector current first rapidly rises, then levels off and remains practically constant. In the no-signal condition, $V_{BE} = 0$, when the base and the emitter are shorted together, the collector current is practically the same as the base current, $I_C \approx -I_B$, and both are equal to the reverse collector (or saturation) current. When measured at $V_{CE,rtd}$, it is designated as I_{C_0} and is one of the parameters of a transistor. Using a set of collector characteristics or, rather, their portions parallel to the axis of abscissae, it is easy to determine the small-signal base current gain

$$\beta = \Delta I_C/\Delta I_B \quad (V_{CB} = \text{const})$$

which is used in amplifier design.

10.8 The Single-Stage Transistor Amplifier

An elementary single-stage common-emitter transistor amplifier is shown in the circuit diagram of Fig. 10.19 a. In the diagram the source of the signal to be amplified is represented by a voltage gene-

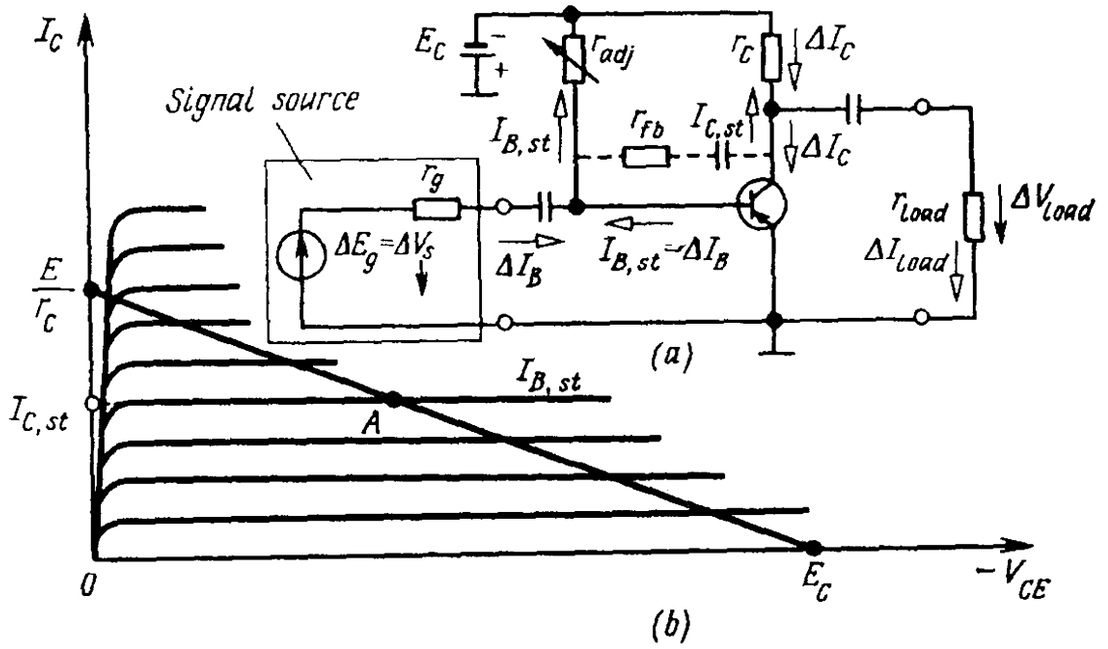


FIGURE 10.19

rator for which $\Delta V_s = \Delta E_g$ and which has an internal resistance r_g.

In the no-signal state ($\Delta V_s = 0$), the amplifier is in a static condition, and both the base current, $I_{B,st}$, and the collector current, $I_{C,st}$, are constant. By varying the base current with a variable resistor, r_{adj}, we can set any value of collector current. The circuit in-

cludes d.c. blocking capacitors which block the passage of direct current to the signal source and the load of resistance r_{load}.

Assuming that no transients take place in the transistor, a sudden change in the input signal by $\Delta V_s \neq 0$ will cause the base current to change as suddenly by ΔI_B. For the chosen positive directions of $I_{B,st}$ and ΔI_B, the base current will become equal to $I_{B,st} - \Delta I_B$. The direction of ΔI_B is chosen such that the direction of the current and ΔE_g in the signal source will be the same. (The change in the current through r_{adj} caused by a change in the input signal may be neglected as the resistance is usually very high.) In consequence, the collector current, Eq. (10.2), will change suddenly too

$$I_{C,st} - \Delta I_C = I_{C,st} - \beta \Delta I_B \tag{10.11}$$

Since the capacitance of the blocking capacitor in the load circuit is always high, a current ΔI_{load} begins to flow in that circuit. The positive direction for this current is usually assumed to be towards the common terminal of the transistor, that is, the emitter.

If the input signal voltage ΔV_s is small, we may take it that the parameters of the transistor remain unchanged for any changes in

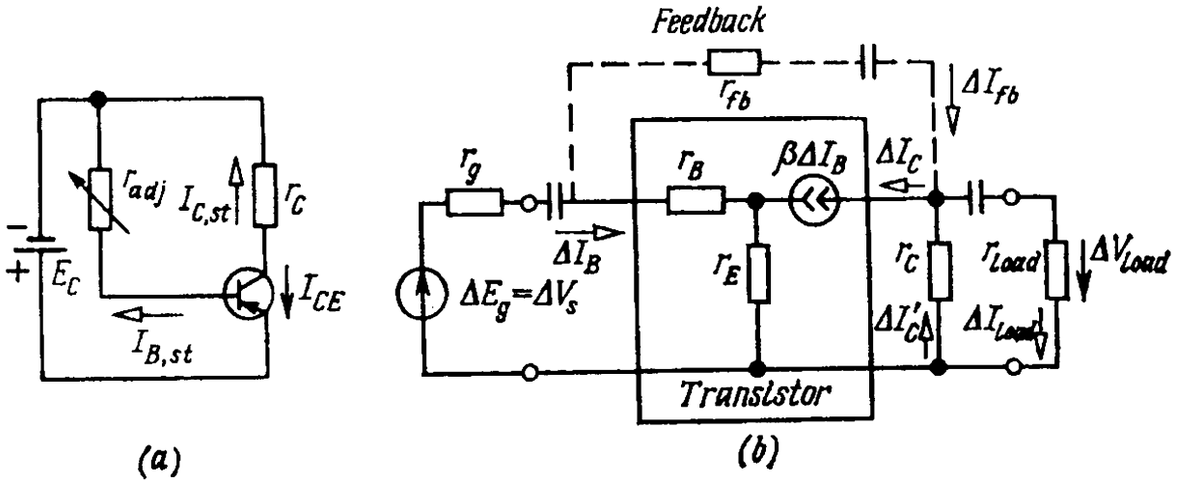


FIGURE 10.20

the signal voltage. Therefore, in the small-signal mode of operation we may treat the amplifier as a linear circuit and represent the total currents existing in it as the superposition of d.c. components due to one emf source E_C (see Fig. 10.20 a) and of a.c. components due to the other emf source ΔE_g (Fig. 10.20 b). In the diagram for the a.c. components the bipolar transistor is replaced with an equivalent circuit which includes the base resistance $r_B = 100\text{-}200\ \Omega$, the emitter resistance $r_E = 15\text{-}25\ \Omega$, and the relation, Eq. (10.2), between changes in the base and collector currents.

The d.c. components in the amplifier circuit (see Fig. 10.20 a) can be found from the known static *collector characteristics* of a transistor connected in a common-emitter circuit, $I_C\ (-V_{CE})_{I_B=\text{const}}$, (see Fig. 10.19 b). Given the direct base current, $I_{B,st}$, the direct col-

lector current, $I_{C,st}$, can be found by the load-line method (see Sec. 1.21), assuming that the operating point is located at A (see Fig. 10.19 b).

The a.c. equivalent circuit of the amplifier is used to find its basic parameters, such as the voltage and current gain, the input and output resistances. If we neglect the effect of the d.c. blocking capacitors (see Fig. 10.20 b), then

$$\Delta V_{load} = -\beta (r_{load} \parallel r_C) \Delta I_B \tag{10.12}$$

$$\Delta V_s = [r_g + r_B + (1 + \beta) r_E] \Delta I_B = (r_{in} + r_g) \Delta I_B \tag{10.13}$$

where

$$r_{in} = r_B + (1 + \beta) r_E \tag{10.14}$$

is the *input resistance of a common-emitter transistor amplifier*, equal in our case to the input resistance of the transistor connected in a common-emitter circuit. For bipolar transistors connected in a common-emitter configuration it is relatively small (1-3 kΩ), which is an important limitation.

The *output resistance of a common-emitter transistor amplifier* is determined on disconnecting the load from its output terminals

$$r_{out} = r_C \tag{10.15}$$

Typically, the output resistance of common-emitter amplifiers is 10-100 kΩ.

As follows from Eqs. (10.12) and (10.13), the small-signal *voltage gain* of a common-emitter transistor amplifier is

$$K_v = \Delta V_{load}/\Delta V_s = -\beta (r_{load} \parallel r_C)/(r_{in} + r_g) \tag{10.16}$$

The "—" sign in Eq. (10.16) indicates that when the signal is positive, the corresponding change in the load voltage is negative, and vice versa.

Similarly, the *current gain* is

$$K_i = \Delta I_{load}/\Delta I_B = -\Delta V_{load}/r_{load} \Delta I_B$$
$$= -\beta r_C/(r_{load} + r_C) \tag{10.17}$$

The typical values of voltage and current gain for common-emitter transistor amplifiers are $K_v = 10\text{-}60$ and $K_i = 15\text{-}80$.

Instead of bipolar transistors, amplifiers may well use FETs (see Fig. 10.9). By analogy with the common-base, common-emitter and common-collector circuit configurations used for bipolar transistors, FETs can be connected in a *common-gate circuit*, a *common-source circuit*, or a *common-drain circuit*. Most often, use is made of the common-source configuration for which the main advantage is a high input impedance (up to 1000 MΩ). A limitation common to all FET amplifiers, as compared with bipolar-transistor amplifiers, is a lower gain.

10.9 Feedback in Amplifiers

The base current gain of a bipolar transistor, β, Eq. (10.2), which occurs in equation (10.16) for voltage gain and equation (10.17) for current gain is anything but stable. In consequence, K_v and K_i for a common-emitter transistor amplifier are not stable, either. Their stability can be improved through feedback.

Feedback refers to the return of a portion of the output signal to the input of an amplifier so as to modify the performance of the amplifier.

In amplifiers, feedback may be series or parallel, in terms of current or voltage, in terms of a d.c. or an a.c. component. We will limit ourselves to parallel negative a.c. voltage feedback and series negative d.c. current feedback as used in a common-emitter transistor amplifier.

Parallel negative a.c. voltage feedback is implemented by placing a feedback resistor, r_{fb} (via a d.c. blocking capacitor) between the collector and the base of the transistor. In the amplifier circuit diagram (Fig. 10.19 a) and in the a.c. (small-signal) equivalent circuit (Fig. 10.20 b), it is shown dashed.

Note that $|\Delta V_{load}| \gg \Delta V_s$. Then, from the a.c. feedback equivalent circuit shown in Fig. 10.20 b it follows that

$$\Delta V_{load} = -\beta (r_{load} \parallel r_C \parallel r_{fb}) \Delta I_B \tag{10.18}$$

$$\Delta I_{fb} = \frac{\Delta V_s - r_{in} \Delta I_B}{r_g} - \Delta I_B \approx -\Delta V_{load}/r_{fb} \tag{10.19}$$

where $r_{in}\Delta I_B$ is the a.c. emitter-to-base voltage, and $(\Delta V_s - r_{in}\Delta I_B)/r_g$ is the input signal current.

In view of Eqs. (10.18) and (10.19), the voltage gain for an amplifier with parallel negative a.c. voltage feedback is given by

$$K_{v,\,fb} = \frac{\Delta V_{load}}{\Delta V_s} = -\frac{\beta (r_{load} \parallel r_C \parallel r_{fb})}{\dfrac{\beta (r_{load} \parallel r_C \parallel r_{fb})\, r_g}{r_{fb}} + r_g + r_{in}} \tag{10.20}$$

Parallel negative a.c. voltage feedback reduces the open-loop small-signal voltage gain, Eq. (10.16), of the amplifier, but improves its stability. This is because the unstable parameter β occurs in Eq. (10.20) so that when it is increased, both the numerator and the denominator increase; when β is decreased, both the numerator and the denominator decrease in the same proportion. Parallel negative a.c. voltage feedback stabilizes the amplifier gain not only in the case of variations in β, but also against variations in other parameters of a bipolar transistor, such as the base resistance r_B and the emitter resistance r_E. Whenever the output voltage of the amplifier goes up (or down) due to any change in the transistor parameters, a

feedback signal is applied to the amplifier input in antiphase with the signal being amplified.

Series negative d.c. current feedback is used to stabilize the direct component of collector current, $I_{C,st}$. This is achieved by placing a feedback resistor in the emitter lead, $r_{E,\,fb}$ (Fig. 10.21 a). The capacitor placed in parallel with this resistor serves to separate the direct com onent from the alternating component of collector current. As

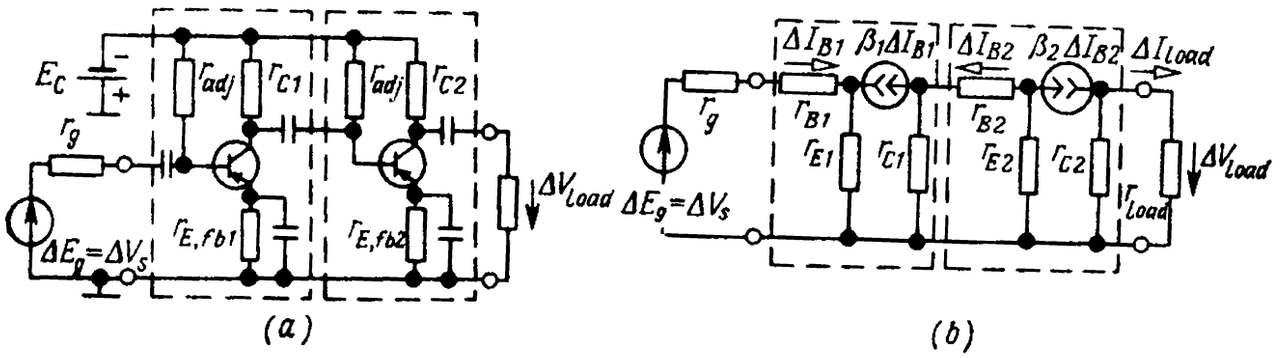


FIGURE 10.21

the direct component of collector current goes up (or down), the voltage drop across $r_{E,fb}$ increases (decreases). In consequence, the potential difference between the emitter and the base is reduced (increased), and the change in base current stabilizes the direct component of collector current.

10.10 The Multistage Amplifier

The purpose of a multistage amplifier is to increase the overall gain. As an example, Fig. 10.21 a shows the circuit of a two-stage transistor amplifier.

On replacing each stage with its a.c. (small-signal) equivalent circuit (see Fig. 10.20 b) and neglecting feedback, we obtain a complete equivalent circuit for our two-stage amplifier (Fig. 10.21 b)*. By analogy with Eqs. (10.12) and (10.13), we find for this circuit that when the input signal is changed by

$$\Delta V_s = (r_g + r_{in1})\,\Delta I_{B1} \tag{10.21}$$

the change in the voltage across the load resistance r_{load} is

$$\Delta V_{load} = (r_{C2} \parallel r_{load})\,\beta_2 \Delta I_{B2} = \frac{\beta_1 \beta_2 r_{C1} r_{C2} r_{load}}{(r_{C2} + r_{load})(r_{C1} + r_{in2})}\,\Delta I_{B1} \tag{10.22}$$

The voltage gain of the two-stage amplifier is

$$K_v = \frac{\Delta V_{load}}{\Delta V_s} = \frac{\beta_1 \beta_2 r_{C1} r_{C2} r_{load}}{(r_g + r_{in1})(r_{C1} + r_{in2})(r_{C2} + r_{load})} \tag{10.23}$$

* Here and elsewhere, d.c. blocking capacitors will not be shown in a.c. equivalent circuits.

Given the same amplified signal, the change in voltage at the output of a two-stage amplifier, Eq. (10.23), takes the sign opposite to that of the voltage change at the output of a single-stage amplifier, Eq. (10.16).

In the general case of an amplifier with N stages, the voltage gain is

$$K_v = (-1)^N \frac{\beta_1 r_{C1}}{r_g + r_{in1}} \frac{r_{load}}{r_{CN} + r_{load}} \times \prod_{n=2}^{N} \frac{\beta_n r_{Cn}}{r_{C(n-1)} + r_{in,\,n}} \qquad (10.24)$$

Since the input resistance, r_{in}, Eq. (10.14), of each stage of a transistor amplifier is small, and it acts as the load resistance, r_{load}, for the previous stage, the stage gain, Eq. (10.16), and the overall gain of a multistage transistor amplifier often turns out to be insufficient. One way to raise the gain is to connect the output of a preceding stage to the input of the next one by an emitter follower.

10.11 The Emitter Follower

An *emitter follower* (Fig. 10.22 a) is an amplifier using a large amount of series negative d.c. current and a.c. voltage feedback. The series negative d.c. current feedback is provided by a series feed-

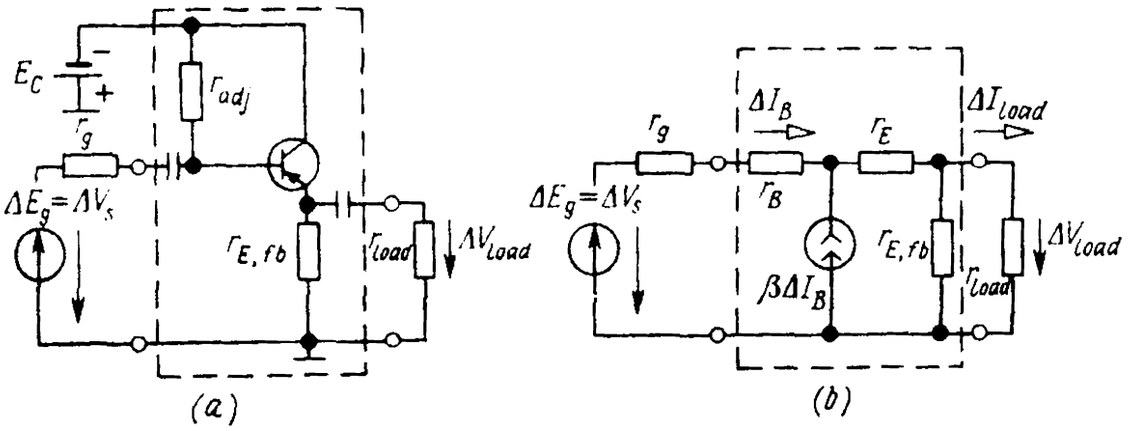


FIGURE 10.22

back resistor placed in the emitter lead, $r_{E,fb}$, and used to set and stabilize the static operating conditions for the emitter follower in the no-signal state, $\Delta V_s = 0$ (see Sec. 10.9). The series negative a.c. voltage feedback is supplied by a parallel combination of the emitter feedback resistor, $r_{E,fb}$, and the load resistance, r_{load}. An a.c. (small-signal) equivalent circuit of an emitter follower is shown in Fig. 10.22 b. As follows from the figure, the change in the output (load) voltage of an emitter follower, ΔV_{load}, for $\Delta V_s \neq 0$ is

$$\Delta V_{load} = (r_{E,\,fb} \parallel r_{load})\,(1+\beta)\,\Delta I_B \qquad (10.25)$$

Hence, by Kirchhoff's voltage law for the outer loop we have

$$\Delta V_s = \Delta V_{load} + [r_g + r_B + r_E (1 + \beta)] \Delta I_B \qquad (10.26)$$

On the basis of Eq. (10.25) and Eq. (10.26), the voltage gain of an emitter follower may be written as

$$K_v = \frac{\Delta V_{load}}{\Delta V_s} = \frac{(1+\beta)(r_{E,\,fb} \parallel r_{load})}{(1+\beta)(r_{E,\,fb} \parallel r_{load}) + r_g + r_B + r_E(1+\beta)} \qquad (10.27)$$

If the internal resistance of the signal source, r_g, is small, and so are the base resistance r_B and the emitter resistance r_E, the voltage gain of an emitter follower is close to unity. Typically, for a signal-source internal resistance of 100-200 Ω the voltage gain of an emitter follower is 0.98-0.99.

The input resistance of an emitter follower (see Fig. 10.22 b) is

$$r_{in} = r_B + (1+\beta)(r_g + r_{E,\,fb} \parallel r_{load}) \approx (1+\beta)(r_{E,\,fb} \parallel r_{load}) \qquad (10.28)$$

Typically, for $r_{E,fb}$ = 5-10 kΩ, r_{load} = 200-400 Ω, and β = = 50-100, the input resistance of an emitter follower is r_{in} = 20-40 kΩ, which is substantially higher than that of a common-emitter amplifier, Eq. (10.14).

The output resistance of an emitter follower (see Fig. 10.22 b), as found with the load disconnected from its output terminals, is given by

$$r_{out} = r_{E,\,fb} \parallel [r_E + (r_g + r_B)/(1+\beta)] \approx r_E \qquad (10.29)$$

Equation (10.29) takes account of the fact that the a.c. component of the collector current, $\beta \Delta I_B$, is proportional to the a.c. component of the base current, ΔI_B, and the a.c. component of the emitter current is $(1 + \beta)\,\Delta I_B$.

Typically, the output resistance of an emitter follower is r_{out} = = 20-30 Ω, which is substantially lower than that of a common-emitter amplifier, Eq. (10.15).

Because of its high input resistance and its low output resistance, the emitter follower is used as a buffer stage between two adjacent stages in a multistage amplifier in order to raise both the stage gain and the overall gain of the amplifier.

10.12 The Power Amplifier

A *power amplifier* is the final stage in the signal amplification chain. Its output is coupled to a high-power load.

Power amplifiers may operate in class A and class B. The circuit of a *class A power amplifier* is shown in Fig. 10.23 a. In order to derive a large output power, it is necessary to reduce the d.c. resistance of the collector circuit as much as practicable and to couple the collector to the load, r_{load}, via a transformer.

By way of proof, we will begin by considering the operation of a power amplifier under static (d.c.) conditions, using as before (see Fig. 10.19 b) the load line method. Since in the static condition the collector resistance of the amplifier is equal to the resistance of the transformer primary, we may take it that $r_C \approx 0$. As a consequence, the d.c. load line runs nearly parallel to the axis of ordinates. Its intersection with the collector (output) characteristic locates the operating point Q of the power amplifier under static (d.c.) conditions (see Fig. 10.23 c).

If we apply a sinusoidal signal to the input of the power amplifier, the a.c. component of the base current, i_B, will likewise be sinusoi-

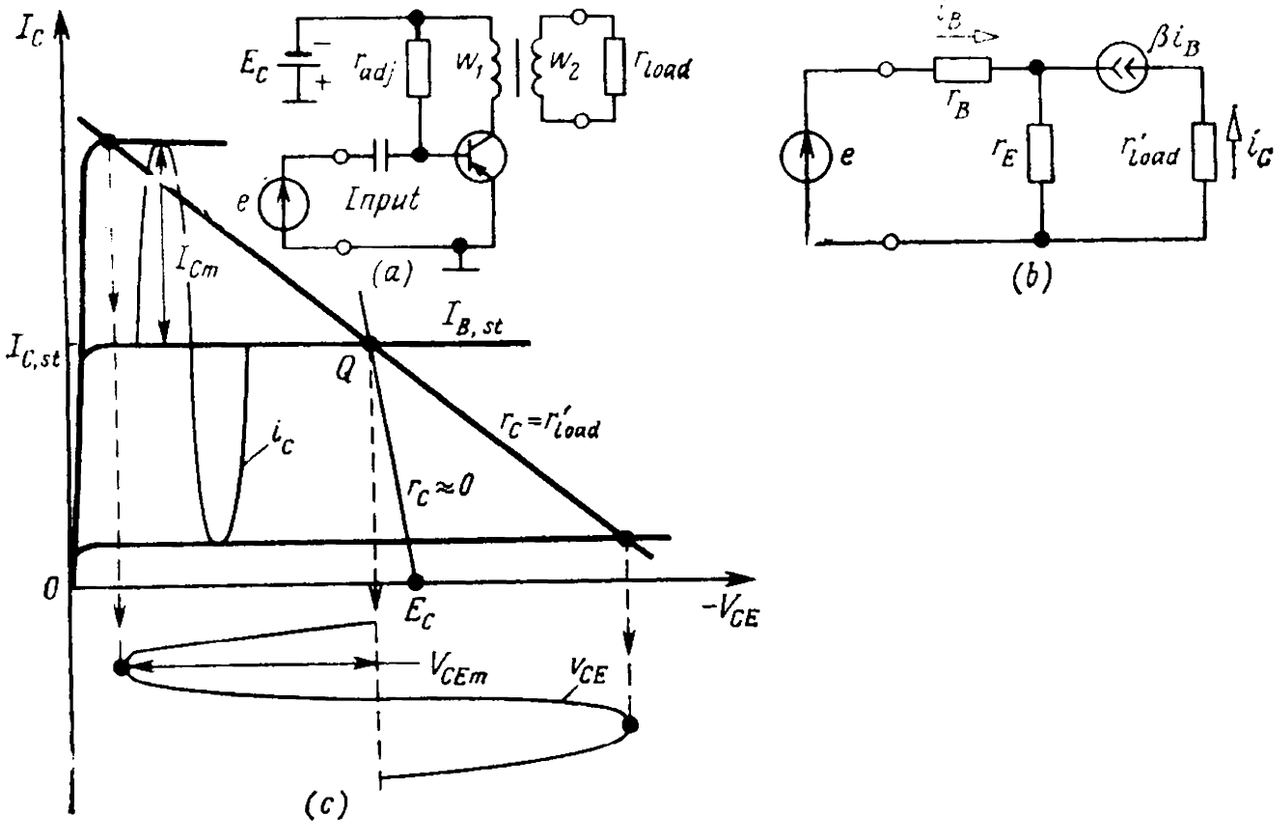


FIGURE 10.23

dal, and so will be the a.c. component of the collector current, $i_C = \beta i_B$, Eq. (10.2). For a.c., however, the collector resistance is equal to the load resistance referred to the primary side of the transformer, Eq. (8.8), that is

$$r_C = r'_{load} = r_{load} (w_1/w_2)^2$$

(Fig. 10.23 b.) Therefore the a.c. (or dynamic) load line differs from the d.c. (or static) load line (when $r_C \approx 0$). Now we pass the a.c. load line through the point Q and construct a plot of v_{CE} as a function of time (see Fig. 10.23 c).

We will determine the load power P_{load} and the signal source power P_s by referring to the a.c. equivalent circuit of the power ampli-

fier (see Fig. 10.23 b) and recalling Eq. (10.14)

$$P_{load} = (r'_{load}/2)\, I^2_{Cm} = r'_{load}\beta^2 I^2_{Bm}/2 \tag{10.30}$$

$$P_s = [\, r_B + (1 + \beta)\, r_E]\, I^2_{Bm}/2 \tag{10.31}$$

Hence, the power gain is

$$K_P = P_{load}/P_s = \beta^2 r'_{load}/[r_B + (1 + \beta)\, r_E] \tag{10.32}$$

Typically, the power gain of a common-emitter stage is 100-200.

A disadvantage common to class A power amplifiers is that they have a low efficiency

$$\eta = (P_{load}/P_{sup}) \times 100 = \frac{1}{2}\,\frac{I_{Cm}V_{CEm}}{E_C I_{C,st}} \times 100 < 50\% \tag{10.33}$$

where $P_{sup} = E_C I_{C,st}$ is the supply power, and $P_{load} = V_{CEm}I_{Cm}/2$ is the power delivered to the load.

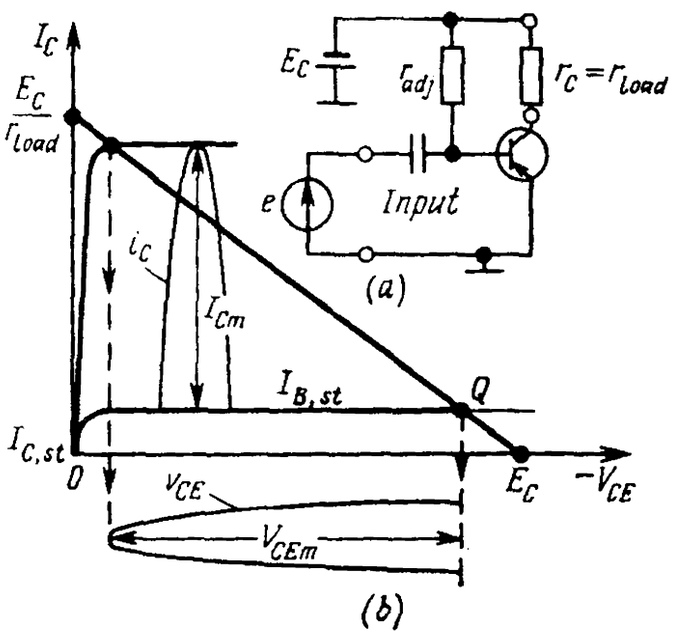


FIGURE 10.24

Class B power amplifiers (Fig. 10.24 a) differ from class A power amplifiers in that the Q (operating) point is chosen such that the a.c. component of collector current, i_C, is limited to half a cycle, as shown in Fig. 10.24 b. During the other half-cycle there is no current flowing in the collector circuit. Connection of the load via a transformer, as in a class A power amplifier (see Fig. 10.23 a), would not give any appreciable advantage in the case of a class B power amplifier. The Q-point is located so that with each type of load connection the supply voltage E_C and, as a consequence, the supply power would be the same. In view of Eq. (10.3),

$$P_{sup} = E_C I_{Cm}/\pi$$

and the power delivered to the load is, in accord with Eq. (10.6), equal to

$$P_{load} = r_C I^2_{Cm}/4 \approx E_C I_{Cm}/4$$

Therefore, the efficiency of a class B power amplifier is

$$\eta = (P_{load}/P_{sup}) \times 100 \approx (\pi/4) \times 100 \approx 75\% \qquad (10.34)$$

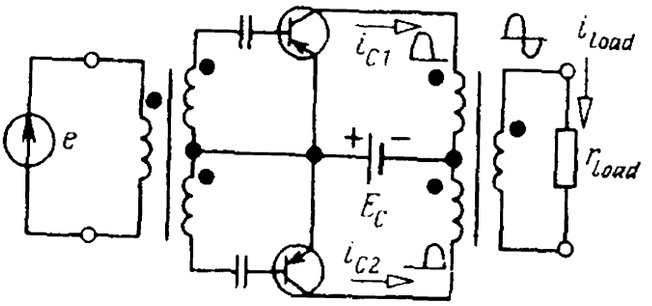


FIGURE 10.25

A still better efficiency of power amplification is achieved with push-pull class B power amplifiers using transformer coupling (Fig. 10.25).

10.13 The D.C. Amplifier

As was noted at the beginning of this chapter, *d.c. amplifiers* are used to amplify unidirectional or slowly varying signals. It was also noted that in a d.c. amplifier one stage is resistively coupled to the next.

The circuit of a three-stage d.c. amplifier is shown in Fig. 10.26. As is seen, the collector of the transistor in a preceding stage is re-

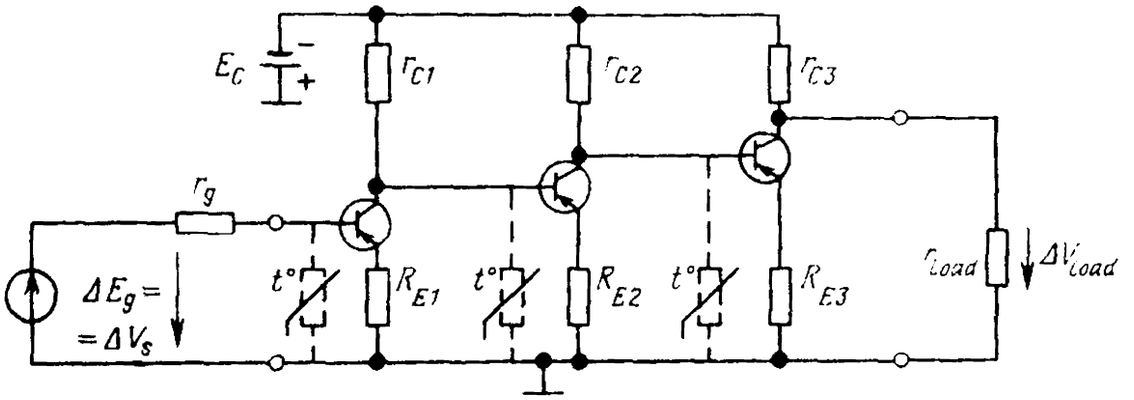


FIGURE 10.26

sistively (or, which is the same, directly) coupled to the base of the transistor in the next stage. For this reason, any change in the collector voltage due to, say, some instability in the transistor parameters or the supply emf, E_C, brings about a change in the output voltage even when the input (signal) voltage is zero. This is known as *drift*,

and its compensation presents a very complicated problem. In part, it can be tackled by placing a resistor, R_E, in the emitter lead of each stage (see Sec. 10.9).

Another approach to the compensation of drift caused by temperature variations in the transistor parameters is to use circuit components which have been matched so that changes in the temperature-dependent parameters of some would balance similar changes in the others. For example, the base current gain β has a positive temperature coefficient, so a rise in temperature would cause the currents in the collector circuits to rise, too. If we place thermistors which have a negative temperature coefficient in the input circuits of the stages (as shown dashed in Fig. 10.26), this would make up for the temperature variations in β.

If we neglect the base resistance ($r_B = 0$) and the emitter resistance ($r_E = 0$) in the equivalent circuit of Fig. 10.20 b, the various potentials (with respect to the ground) in the stages of a d.c. amplifier will be related in the following way

$$|\varphi_{C1}| = |\varphi_{B2}| < |\varphi_{C2}| = |\varphi_{B3}| < |\varphi_{C3}|$$

$$|\varphi_{B1}| = |\varphi_{E1}| < |\varphi_{C1}| = |\varphi_{B2}| = |\varphi_{E2}| < |\varphi_{C2}| = |\varphi_{B3}| = |\varphi_{E3}| \quad (10.35)$$

If all the stages are to operate under the same conditions, the like currents of all the transistors must be the same

$$\frac{E_C - |\varphi_{C1}|}{r_{C1}} = \frac{E_C - |\varphi_{C2}|}{r_{C2}} = \frac{E_C - |\varphi_{C3}|}{r_{C3}}$$

that is,

$$\varphi_{E1}/R_{E1} = \varphi_{E2}/R_{E2} = \varphi_{E3}/R_{E3}$$

and, subject to Eq. (10.35), the following inequalities must be satisfied

$$r_{C1} > r_{C2} > r_{C3}$$

$$R_{E1} < R_{E2} < R_{E3} \quad (10.36)$$

which means that the gain, Eq. (10.16), decreases from stage to stage.

Frequently, drift is compensated by arranging a d.c. amplifier into a *differential circuit* (Fig. 10.27) in which any change in a parameter occurs in both arms at the same time but with a different sign, so this has but a negligible effect on the output voltage.

Let us determine the voltage gain of a differential d.c. amplifier. If it has identical arms, then the input signal ΔV_s will cause the base potentials of the transistors to change by $\Delta\varphi_{B1} = (\Delta V_s - r_g \Delta I_B)/2$ and $\Delta\varphi_{B2} = -(\Delta V_s - r_g \Delta I_B)/2$, where $\Delta I_B = \Delta I_{B1} = -\Delta I_{B2}$ is the change in the base current.

The emitter currents will change in a similar manner, $\Delta I_{E1} = -\Delta I_{E2}$. Since the two currents change in opposite directions, no change takes place in the current flowing through the emitter resistor R_E, and the emitter potentials of the transistors remain unchanged in both arms of the amplifier. In consequence, the voltage gain of each arm is found in the same manner as that of a common-emitter single-stage amplifier loaded into $r_{load}/2$. The voltage gain of a differential d.c. amplifier is found by analogy with Eq. (10.16) and is equal to

$$K_D = \Delta V_{load}/\Delta V_s = -2\beta[(r_{load}/2) \parallel r_C]/(2r_{in} + r_g) \quad (10.37)$$

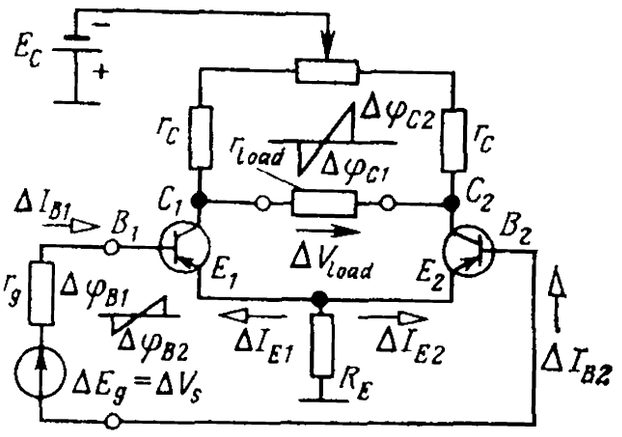


FIGURE 10.27

where r_{in} is the input resistance of a single-stage common-emitter amplifier, Eq. (10.14). The small resistance of the rheostat used to balance the arms of the amplifier is neglected in Eq. (10.37).

Owing to a high voltage gain and a negligible drift, differential d.c. amplifiers are widely used in automatic control, instrumentation, etc. The increase in the number of circuit components used is immaterial because they are mostly miniaturized elements or integrated circuits.

10.14 Pulse Circuits

Semiconductor pulse circuits constitute a large class of devices in which transistors act as static switches.

Consider the operation of a bipolar transistor connected in a common-emitter circuit (Fig. 10.28 a) as a switch. When the input voltage is zero, $V_{BE} = 0$, the collector and base currents are numerically equal

$$I_C = -I_B = I_{C0} \quad (10.38)$$

where I_{C0} is the d.c. collector-to-base leakage current with the emitter open (see Fig. 10.8 a), ranging between 1 and 10 μA from transistor to transistor type and sometimes specified as a parameter. In this condition the transistor is acting as an open switch (the OFF state), so that the collector current is a minimum, $I_{C,min} = I_{C0}$ and $-V_{CE} \approx E_C$ (point M in Fig. 10.28 b). Obviously, at $V_{BE} > 0$ the transistor will likewise be in the OFF state.

If we gradually decrease the voltage at the input to the transistor switch from zero to $V_{BE} < 0$, the base current will cross zero

and become positive, $I_B > 0$. Since the base current is positive, the collector current will also rise, because $I_C = \beta I_B$, which means that the transistor switch is gradually moving into the conducting (ON) state.

The collector current reaches a maximum, $I_{C,max} \approx E_C/r_C$, when the base current is $I_B = I_{B,sat}$, called the base saturation cur-

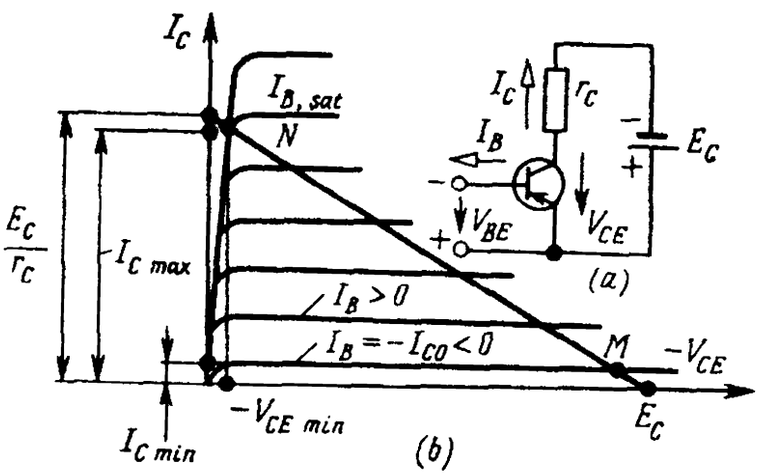


FIGURE 10.28

rent (point N in Fig. 10.28). Any further increase in the base current

$$I_B \geqslant I_{B,sat} = E_C/\beta r_C \tag{10.39}$$

will leave the collector current practically unaffected. In this condition the transistor is acting as a closed switch (the ON state), so that $-V_{CE} \approx -V_{CE,min} \approx 0$.

The use of transistor switches instead of, say, electromechanical switches offers a number of advantages, such as the absence of moving parts likely to wear out, a high speed of response, and a small size. A low-power supply source is required to operate a transistor switch.

Pulse circuits may be classed into those having several stable states and those with several temporarily stable states. In the former case, an external signal must be applied to the circuit each time we want it to change state. In the latter case, the circuit either swings between the ON and the OFF state continuously without any external drive applied, or it is caused to change from one state to the other by an external drive and goes back to the original state spontaneously.

10.15 Bistable Circuits

A bistable circuit, as its name implies, has two stable states. When it is used to produce rectangular pulses or square waves, it is usually called a *bistable multivibrator*. When it is used as a binary memory (storage) cell, a one-bit delay circuit or in other digital ap-

plications, it is called a *flip-flop*. When this distinction is not important, it is usual to refer to the circuit simply as a *bistable*.

Figure 10.29 a shows the diagram of a symmetrical bistable circuit in which the transistors operate as switches. In one stable state, one transistor, say T_1, is conducting (ON), and the other, T_2, is nonconducting (OFF). In the other stable state, the situation is reversed.

To get an idea about how these two states are produced, we will turn to the equivalent circuit in Fig. 10.29 b and assume that transistor T_1 is ON and transistor T_2 is OFF. In the equivalent circuit

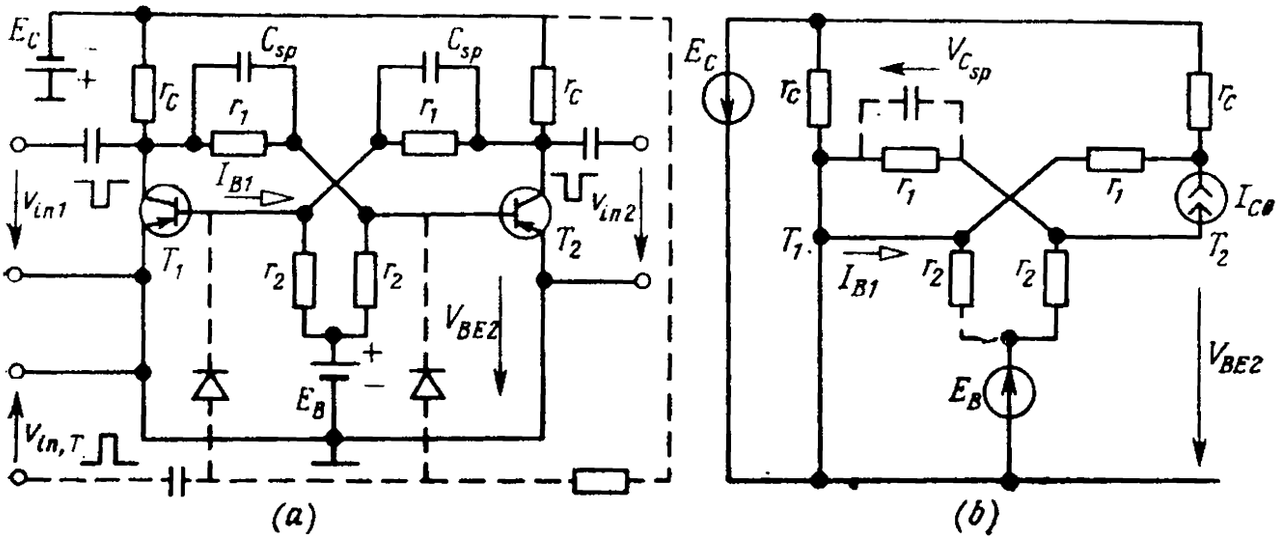


FIGURE 10.29

the "OFF" transistor is replaced with a current source, I_{C0}, Eq. (10.38), and the "ON" transistor is represented by a node where the collector, base and emitter branches meet. This representation is based on the fact that at saturation the residual voltage between collector and emitter is small, $V_{CE} = V_{CE,min} \approx 0$ (point N in Fig. 10.28 b). The base-to-emitter voltage, V_{BE}, may be neglected because the base resistance in this condition is small.

For T_1 to be reliably in the ON state and for T_2 to be reliably in the OFF state, two conditions must be satisfied

$$I_{B1} > I_{B,sat} \tag{10.40a}$$

$$V_{BE2} > 0 \tag{10.40b}$$

where I_{B1} is the base current of T_1 and $V_{BE,2}$ is the base-to-emitter voltage of T_2.

On the basis of the superposition principle, it follows from the equivalent circuit of the bistable circuit (see Fig. 10.29 b) that

$$I_{B1} = (E_C - r_C I_{C0})/(r_1 + r_C) - E_B/r_2 \tag{10.41a}$$

where $E_C/(r_1 + r_C)$, E_B/r_2 and $r_C I_{C0}/(r_1 + r_C)$ are the components of the base current I_{B1} due to E_C, E_B, and I_{C0};

$$V_{BE2} = r_1 E_B/(r_1 + r_2) + (r_1 \| r_2) I_{C0} \tag{10.41b}$$

where $r_1 E_B/(r_1 + r_2)$ and $(r_1 \| r_2) I_{C0}$ are the components of V_{BE2} due to E_B and I_{C0}.

As follows from Eq. (10.41) subject to Eqs. (10.40) and (10.39), the resistances r_1 and r_2 in a symmetrical bistable circuit must be such that

$$r_1 < E_C - r_C I_{C0})/(E_C/\beta r_B + E_B/r_2) - r_C \tag{10.42a}$$

$$r_2 < E_B/I_{C0} \tag{10.42b}$$

As already noted, bistable circuits are often used as flip-flops. According to the way in which it is triggered, it may be a reset-set (or RS) flip-flop and a toggle (or T) flip-flop.

An *RS flip-flop* is triggered into one particular state (that is, set) when a trigger voltage is applied to one input terminal, and triggered into the opposite state (or reset) by a voltage applied to the other input terminal. The inputs may be various pairs of circuit terminals. In Fig. 10.29 a, these are the collector leads of the two transistors and the emitter lead common to both transistors. The sources of trigger voltages, v_{in1} and v_{in2}, are connected in negative polarity to the respective terminals via d.c. blocking capacitors. Transitions between states are accelerated by a speed-up capacitor, C_{sp}, placed in parallel with r_1. For its effect the speed-up capacitor depends on the fact that the voltage across it cannot change abruptly, Eq. (5.2). Therefore, after a trigger pulse v_{in1} has driven T_1 to saturation at time $t = 0$, from the equivalent circuit in Fig. 10.29 b it follows that

$$v_{C_{sp}}(0_-) = r_1 E_C/(r_1 + r_C) - (r_1 \| r_C) I_{C0} = v_{C_{sp}}(0_+) = v_{BE2}(0_+)$$

As a consequence, the base-to-emitter voltage of T_2 is now greater than it would be in the absence of a speed-up capacitor, Eq. (10.41b), and this permits faster turn-on of the transistor in response to a trigger voltage. The speed-up capacitor of the other transistor, T_1, acts in precisely the same way.

A *T flip-flop* is triggered by an input to one terminal. One of the likely arrangements for this form of triggering is shown dashed in Fig. 10.29 a. If initially transistor T_1 is ON and transistor T_2 is OFF, arrival of a trigger voltage, $v_{in,T}$, at the input will cause the flip-flop to change state so that T_1 becomes OFF and T_2 becomes ON. The diodes decouple the base circuits of the transistors when the $v_{in,T}$ pulse ceases. The resistor-containing branch (shown dashed in the figure) serves to turn off the diodes. The next trigger voltage pulse

will cause the flip-flop to change back to its original state, and so on.

When flip-flops are used as binary memory (storage) cells, one state is assigned value 1, and other state, value 0, each value being a digit position of a binary number.

10.16 The Astable Multivibrator. The Monostable Multivibrator

The astable (free-running) multivibrator. This form of multivibrator has two momentarily stable states between which it conti-

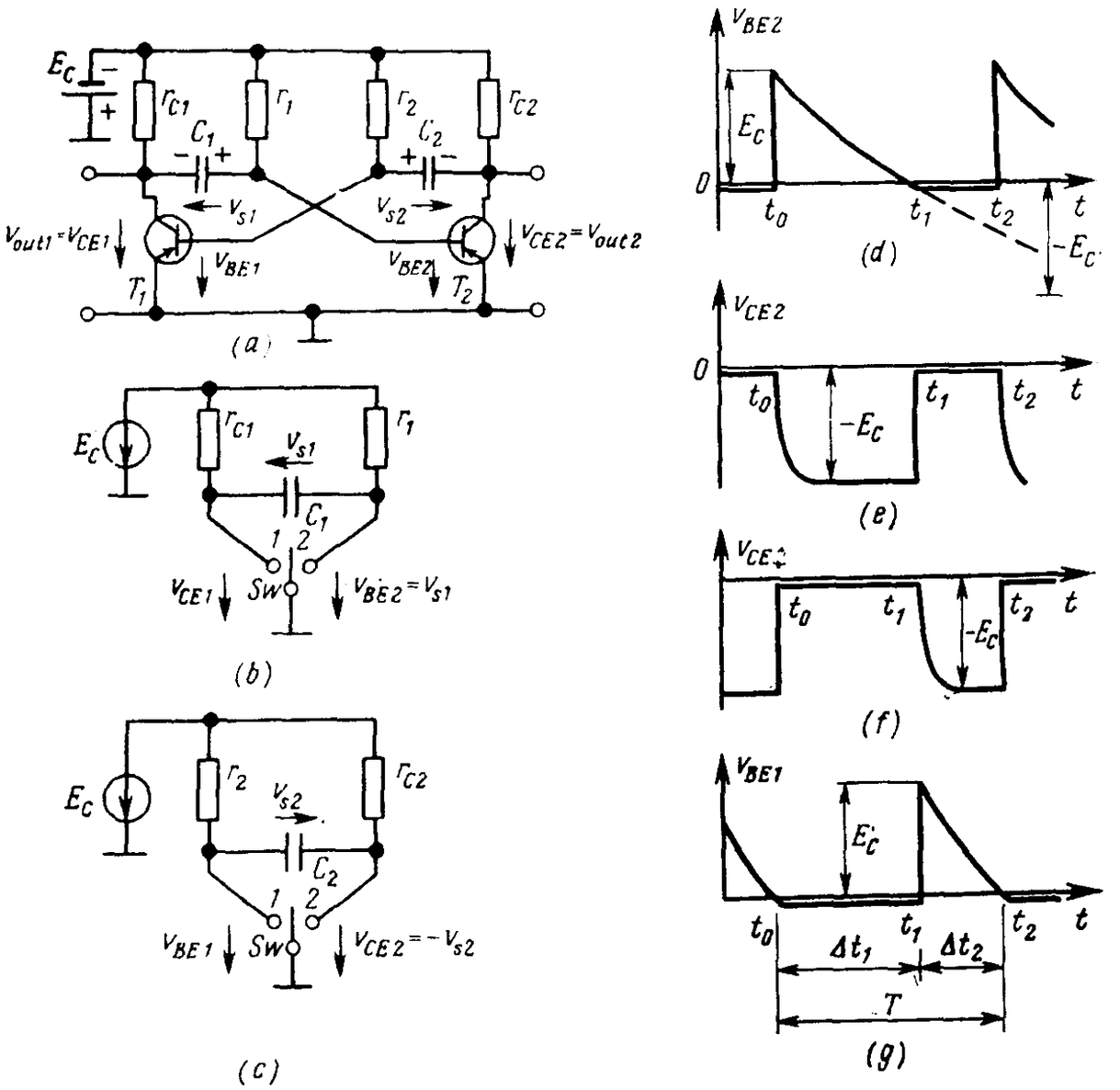


FIGURE 10.30

nuously alternates, remaining in each state for a period controlled by the circuit parameters. Its circuit is shown in the diagram of Fig. 10.30 a. During a part of a period, Δt_1, one transistor is ON and the other is OFF; during the remainder of the same period, Δt_2, the first transistor is OFF and the second is ON. Each transition from

state to state takes place practically instantaneously, whereas during Δt_1 and Δt_2 transients occur in the multivibrator circuit containing capacitors and resistors.

For an astable multivibrator to operate as it should, the following conditions must be satisfied.

With no current in capacitor C_1, resistor r_1 should provide for a saturated state of transistor T_2; similarly, with no current in capacitor C_2, resistor r_2 should provide for a saturated state of transistor T_1. In consequence, $I_{B1} = E_C/r_2 > I_{B,sat}$, and $I_{B2} = E_C/r_1 > I_{B,sat}$. In other words, in accord with Eq. (10.39), the resistances of the two resistors must be such that

$$r_1 \leqslant \beta r_{C2} \tag{10.43a}$$

$$r_2 \leqslant \beta r_{C1} \tag{10.43b}$$

For purposes of analysis, we assume that initially the astable multivibrator is in such a state that T_1 is OFF and T_2 is ON, that is, $v_{CE2} \approx v_{BE2} \approx 0$ and $v_{BE1} \approx v_{C2} \approx 0$.

This state cannot be stable, because in accord with Eq. (10.43b) transistor T_1 must also be ON. The unstable state of the circuit corresponds to a transient process due to which v_{BE1} is decreased. When $v_{BE1} \approx v_{C2} \approx 0$, transistor T_1 begins to turn on (see Sec. 10.14). The increase in the collector current of T_1 brings about a cumulative switchover in the circuit. As a result of this process called relaxation, T_1 turns on, and transistor T_2 turns off.

Let us take a closer look at what happens in an astable multivibrator during one period, beginning at the instant of switchover, $t = t_0$. To simplify the analysis, we replace the transistors with ideal switches and assume that by the time $t = t_0$ capacitor C_1 is charged to the supply voltage, and the voltage across C_2 is zero, that is, $v_{C1}(t_{0-}) = E_C$ and $v_{C2}(t_{0-}) = 0$. The transients occurring in an astable multivibrator at $t \geqslant t_{0+}$ are represented in the equivalent circuits of Fig. 10.30 b and c where switch Sw is moved from position 2 to position 1 at time $t = t_0$. Capacitive element C_1 in the equivalent circuit of Fig. 10.30 b begins to re-charge from the initial voltage $v_{C1}(t_{0-}) = E_C$ to $-E_C$ such that

$$v_{C1} = v_{BE2} = -E_C + 2E_C \exp\left[-(t - t_0)/r_1C_1\right] \tag{10.44}$$

This voltage is equal to the base-to-emitter voltage of transistor T_2 and holds it OFF during the interval $\Delta t_1 = t_1 - t_0$, when $v_{BE2} = v_{C1} \geqslant 0$ (Fig. 10.30 d).

When $v_{BE2} \approx 0$, transistor T_2 begins to conduct, and the subsequent run of the transient process changes. Therefore, assuming v_{C1} in Eq. (10.44) equal to zero, we will determine the duration of the interval Δt_1 during which T_1 is ON and T_2 is OFF

$$\Delta t_1 = r_1C_1 \ln 2 \approx 0.7r_1C_1 \tag{10.45}$$

The capacitive element C_2 in the equivalent circuit of Fig. 10.30 c charges from the initial value $v_{C2}(t_{0-}) = 0$ to E_C such that

$$v_{C2} = -v_{CE2} = E_C\{1 - \exp[-(t - t_0)/r_{C2}C_2]\}$$

At $\Delta t_1 > 3r_{C_2}C_2$, the capacitive element C_2 will have charged practically to E_C (see Fig. 10.30 e). During the entire interval Δt_1 transistor T_1 is ON and $v_{CE_1} = v_{BE_1} = 0$ (see Fig. 10.30 f and g).

At $t = t_1$, a new relaxation process begins due to which transistor T_1 turns off and transistor T_2 turns on. In the equivalent circuits of Fig. 10.30 b and c this corresponds to throwing switch Sw from position 1 to position 2. By a similar reasoning, we can determine the time interval during which the astable multivibrator resides in this state

$$\Delta t_2 = r_2C_2 \ln 2 \approx 0.7r_2C_2 \qquad (10.46)$$

A complete period for an astable multivibrator is

$$T = \Delta t_1 + \Delta t_2 \approx 0.7\,(r_1C_1 + r_2C_2) \qquad (10.47)$$

The ratio of the pulse duration at the multivibrator outputs to the period is called the *pulse duty factor*

$$F_{d1} = \Delta t_1/T$$

$$F_{d2} = \Delta t_2/T$$

Astable (free-running) multivibrators are used as sources of rectangular pulses with adjustable pulse duration and pulse repetition frequency, notably as master oscillators to trigger other pulse circuits, such as flip-flops or bistable multivibrators. If the output of an astable multivibrator is coupled to the complementing input of a T flip-flop, the latter will produce one output pulse for every two input pulses, thereby scaling down the pulse repetition frequency.

The monostable multivibrator. A monostable multivibrator (also called a one-shot or a univibrator) has one stable state and one momentarily stable state. It will reside in its stable state (with transistor T_1 OFF and transistor T_2 ON), if the conditions defined in Eq. (10.43a) are satisfied and

$$E_B r_3/(r_2 + r_3) > 0$$

By treating the transistors as ideal switches, we obtain the equivalent circuit shown in Fig. 10.30 b where the switch Sw is in position 2. As follows from the equivalent circuit, the voltage across the capacitive element C_1 in the stable state of the univibrator is equal to the collector supply voltage E_C.

The univibrator remains in its stable state until a positive-going voltage pulse v_{in} is applied to its input. This pulse initiates a relaxation switchover of the transistors at time $t = t_0$ (transistor T_1 is

ON, transistor T_2 is OFF). In the equivalent circuit of Fig. 10.30 b this corresponds to moving the switch Sw from position 2 to position 1. Capacitor C_1 begins to re-charge from E_C to $-E_C$, Eq. (10.44), thereby maintaining the univibrator in its new state so long as $v_{C_1} = -v_{BE_2} \geqslant 0$, that is, during the interval $\Delta t_1 \approx 0.7 r_1 C_1$, Eq. (10.45).

If we connect the output of a univibrator to a differentiating network, such as an RC network with a time constant $\tau_{RC} \ll \Delta t_1$, as shown dashed in Fig. 10.31, it can be employed as an adjustable

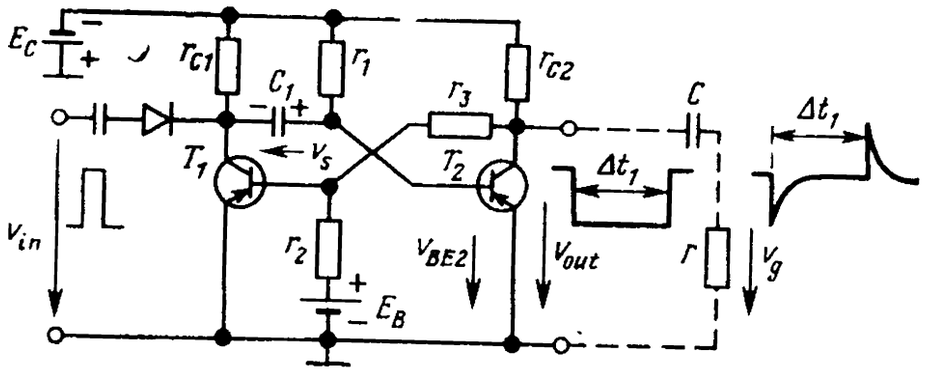


FIGURE 10.31

delay line. Due to the transient process in the differentiating network, similar to that shown in Fig. 5.4 b, the positive-going pulse at the output of the network will be delayed from the positive-going trigger pulse for a time equal to the pulse duration of the univibrator.

The blocking oscillator. This is a form of relaxation oscillator which generates short pulses with a duration of down to 10^{-2} μs with a period-to-pulse duration ratio of over 10 (a pulse duty factor of less than 0.1). It may be made to operate in the free-running mode and the triggered (or synchronized) mode.

Free-running operation. In this mode (see Fig. 10.32 a), switch Sw_2 is in position 1. When, at time $t = 0$, switch Sw_1 brings in the collector supply source E_C between the collector and base of the transistor, the latter is driven into the saturated region. On building up, the collector current i_C induces in the transformer secondary w_2 an emf which speeds up the saturation. Owing to a large amount of positive feedback, the transistor is driven far into the cutoff region almost instantaneously, which constitutes relaxation. Just as the transistor saturates ($v_{CE} \approx v_{BE} \approx 0$), capacitor C_1 in the positive feedback circuit charges. Assuming that the transformer is an ideal one, we can determine the change in the voltage across C_1 by reference to the equivalent circuit of Fig. 10.32 b from which it follows that

$$v_{C_1} = (w_2/w_1)\, v'_{C_1} = (w_2/w_1)\, E_C [1 - \exp(-t/r'_{1b} C'_1)] \qquad (10.48)$$

where

$$r'_{fb} = r_{fb}\,(w_1/w_2)^2$$

and

$$C'_1 = C_1\,(w_2/w_1)^2$$

are the feedback resistor and feedback capacitor referred to the primary side.

Capacitor C_1 charges to a voltage close to $(w_2/w_1)\,E_C$ during a short time interval $\Delta t_1 \approx 3r_{fb}C_1$ which corresponds (in Fig. 10.32 c)

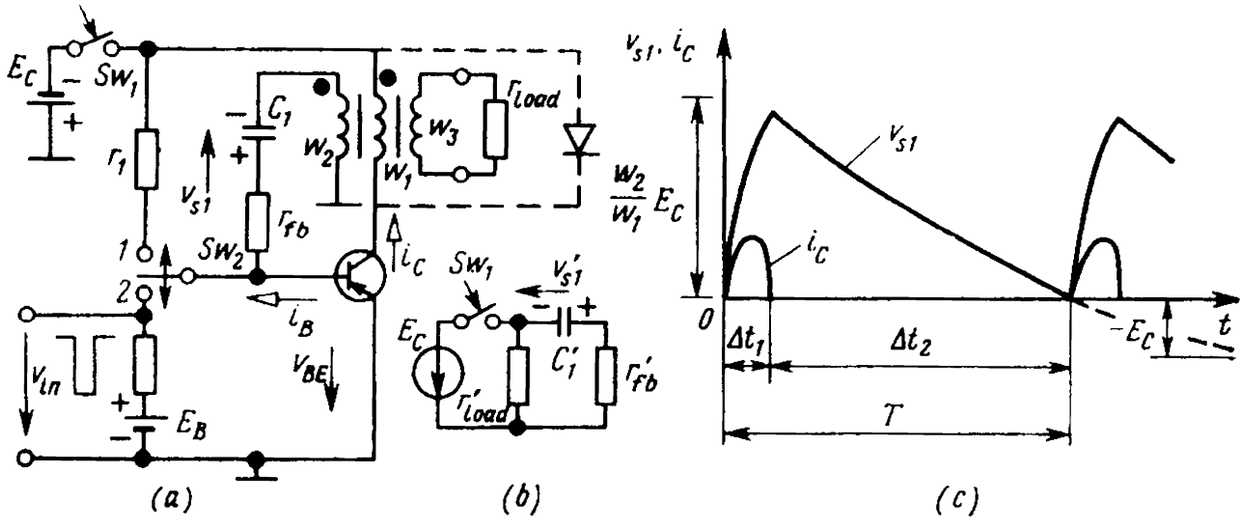


FIGURE 10.32

to the duration of collector-current pulses i_C (and the duration of voltage pulses across the load resistor r_{load}).

On charging, C_1 begins to re-charge from near $(w_2/w_1)\,E_C$ to $-E_C$, with the result that the transistor turns off.

If we neglect the base current i_B of the OFF transistor and assume that the feedback resistor is $r_{fb} \approx 0$, then, by analogy with Eq. (10.44), we find that the voltage across C_1 is

$$v_{C1} = v_{BE} = -E_C + E_C\,(1 + w_2/w_1)\exp\left[-(t - \Delta t_1)/r_1C_1\right] \tag{10.49}$$

On neglecting Δt_1, we can use Eq. (10.49) to determine Δt_2, which is the duration of the OFF state of the blocking oscillator transistor, subject to $v_{BE} \geqslant 0$:

$$\Delta t_2 = r_1C_1 \ln (1 + w_2/w_1) \tag{10.50}$$

A switchover of the transistor from the ON to the OFF state is accompanied by a rapid decrease in the flux linkage with the transformer windings. This results in appreciable voltages across the transformer windings in a polarity which is opposite to that existing when the transistor turns on. To minimize these voltages it is usual

to shunt one of the windings with a diode as shown dashed in the circuit diagram of the blocking oscillator.

Triggered operation. In this mode (see Fig. 10.32 a), switch Sw_2 is in position 2. With switch Sw_1 closed, the blocking oscillator resides in its stable state (the transistor is OFF, because $v_{BE} \approx E_B > 0$) until a negative-going voltage pulse v_{in} is applied to its input. This trigger pulse initiates a relaxation process owing to which the transistor turns on. As in the free-running mode, the transistor remains conducting for a time interval equal to $\Delta t_1 \approx 3r_{fb}C_1$ at the end of which the transistor turns off again, and the blocking oscillator goes back to its initial stable state.

Blocking oscillators can generate pulses much shorter in duration (and with a proportionately smaller pulse duty factor) than multivibrators can do. Also, a blocking oscillator may be transformer-coupled to its load, which is a major advantage in some cases.

10.17 Logic Gates

The proper maintenance of a production process requires that the various pieces of manufacturing plant be started or stopped or otherwise controlled in due time or in a suitable sequence. This involves making decisions that suit the prevailing conditions best.

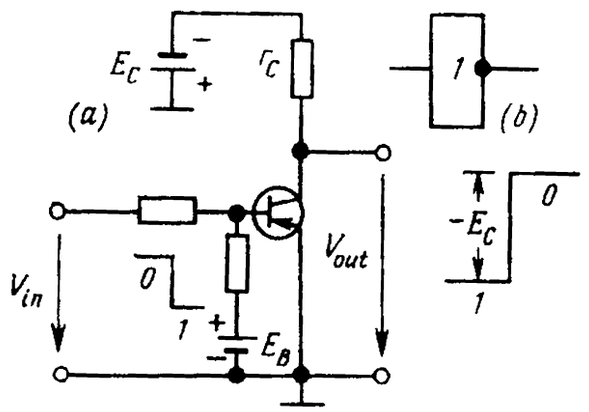


FIGURE 10.33

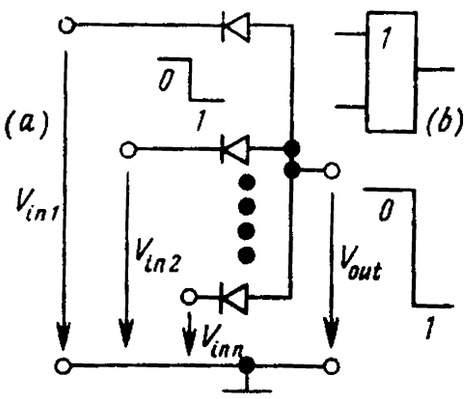


FIGURE 10.34

If we identify the presence or absence of a particular condition with a high or a low level of an electric signal, the necessary decision making can be performed by logic circuits assembled from logic gates. It can be shown that all the basic logic functions can be performed with just three types of elementary logic gates, namely NOT gates (which perform the operation of logic negation or inversion), OR gates (which perform the operation of logic addition), and AND gates (which perform the operation of logic multiplication).

The NOT gate (Fig. 10.33 a). In the circuit of a transistor NOT gate the transistor operates as a switch. In the absence of an input signal, $V_{in} = 0$, the transistor is OFF and the output voltage is

$V_{out} \approx -E_C$ or, which is the same, $V_{out} < 0$. When a sufficiently high negative voltage, $V_{in} < 0$, is applied to the input of a NOT gate, the transistor will turn on, and the output voltage will change to $V_{out} \approx 0$. Thus, in the first case the output voltage goes low (LO); in the second, it goes high (HI). In one form of logic, the LO voltage is assumed to represent a logical 1 and the HI voltage a logical 0.

The OR gate (Fig. 10.34 a). Its output signal is a logical 1 if a logical 1 is present on at least one of its inputs.

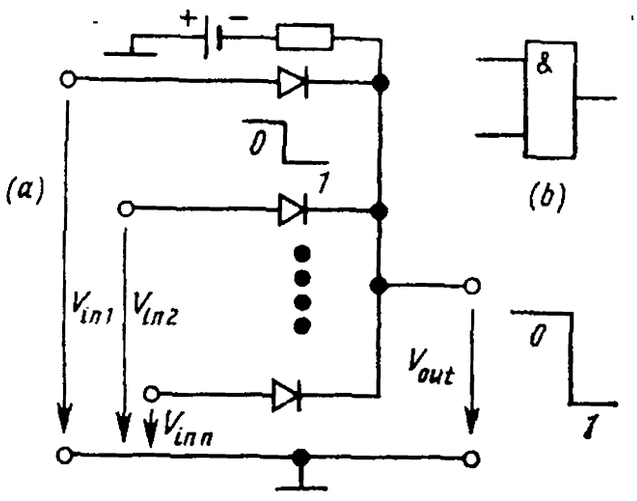


FIGURE 10.35

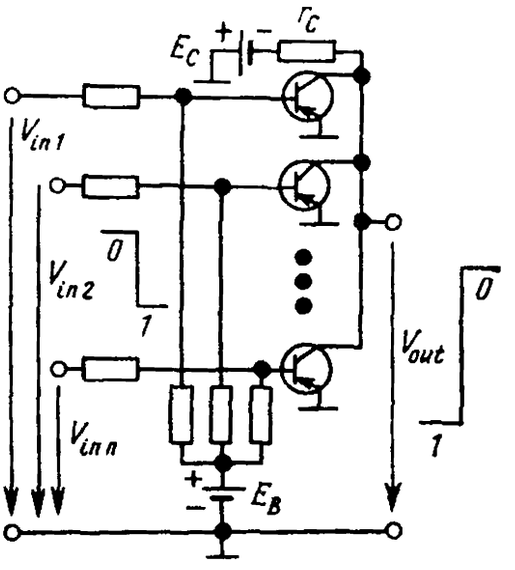


FIGURE 10.36

The AND gate (Fig. 10.35 a). Its output signal is a logical 1 only if a logical 1 signal is present on each input simultaneously, and all the diodes are OFF. If there is a logical 0 signal on at least one input, the diode in the respective lead will be ON, and the output of the AND gate will correspond to a logical 0.

Graphic symbols for the NOT gate (also called the inverter), the OR gate, and the AND gate (as used on logical diagrams in the Soviet Union) are shown in Figs. 10.33 b, 10.34 b and 10.35 b.

A major limitation of diode logic gates is that the output signal strongly depends on the number of series-connected logic gates. This limitation is corrected by using combination logic circuits, such as diode-transistor logic (DTL) circuits and transistor-transistor logic (TTL) circuits.

In a *DTL circuit*, a diode OR gate (see Fig. 10.34 a) or a diode AND gate (see Fig. 10.35 a) is combined with a transistor NOT gate (see Fig. 10.33 a) to give either a NOR (NOT-OR) gate or a NAND (NOT-AND) gate. In the case of a NOR gate, the output of a diode OR gate is connected to the input of a transistor NOT gate. The connections are similar in the case of DTL NAND gate (Fig. 10.35 a). The function of the NOT gate is to restore and invert the output signal.

In a *TTL circuit*, all the three basic logic functions are performed by transistors. There are several modifications of TTL circuits. As an illustration, Fig. 10.36 shows a TTL NOR circuit.

11 Electron Devices and Circuits

11.1 General

Electron devices fall into two broad classes, vacuum and gas-filled.

Vacuum devices (or *tubes*) depend for their operation on the motion of electrons (an electric current) in practically a vacuum (the residual pressure being as low as 10^{-8}-10^{-4} Pa), without collision with gas molecules (at the vacuum stated, there are 10^{8}-10^{9} molecules per cubic centimetre).

Gas-filled devices (or *tubes*), as their name implies, are filled with some gas or metal vapour and depend for their operation on the motion of ions.

Vacuum tubes include the vacuum diode, the vacuum triode, the vacuum pentode, and so on, and also cathode-ray tubes, klystrons, magnetrons and other microwave devices.

In order to produce a stream of electrons in a vacuum tube, one of its electrodes (the cathode) is made to emit electrons. The escape of electrons from the cathode is called *electron emission.*

The minimum amount of energy that must be expended in order that one electron could escape from the cathode material into the surroundings is numerically equal to what is called the *electronic work function* of the material involved. As an example, the electronic work function for platinum is 5.32 eV, for tungsten 4.60 eV, for mercury 4.52 eV, for copper 4.26 eV, for thorium 3.35 eV, and for cesium 1.81 eV, and it is mainly dependent on the material's structure.

The above energy may be imparted to the cathode material in any one of several ways. Accordingly, there are several types of electron emission, namely thermionic, secondary, photoelectric, and autoelectronic (also called field or cold). Most commonly, use is made of thermoelectronic emission.

11.2 The Vacuum Diode

A *vacuum diode* is the simplest of all vacuum tubes. As is shown in Fig. 11.1, it has two electrodes, the anode and the cathode, enclosed in a glass (or metal) envelope evacuated to a high (or hard) vacuum.

The *cathode* may be either directly or indirectly heated. A directly heated cathode is a metal filament shaped according to the design

of the diode. The filament is anch ored to current-conducting supports or stretched taut by one or several springs. The *anode* is a nickel or molybdenum cylinder or box enclosing the cathode.

A directly heated cathode operates on direct current supplied by a low-voltage source (2-3 V). If it were an alternating current at power frequency (50 Hz), the cathode temperature and the diode current would oscillate at twice that frequency (see Fig. 2.30 a), which cannot be tolerated in many cases. This drawback is avoided with

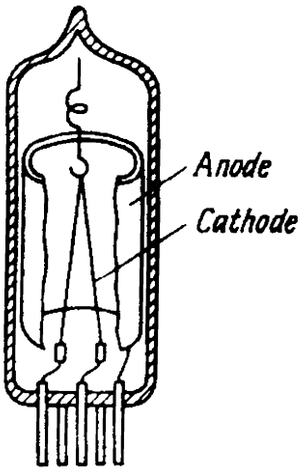


FIGURE 11.1

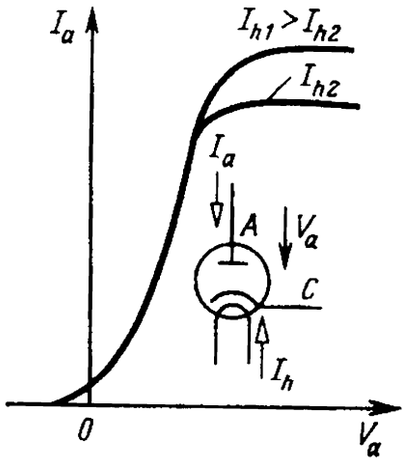


FIGURE 11.2

indirectly heated cathodes. An indirectly heated cathode is the shape of a metal cylinder covered with a heat-resistant material to insulate it from direct contact with a heater.

The leads from the electrodes are terminated in pins which are secured in a plastic or glass base. The anode has one pin, a directly heated cathode has two pins, and an indirectly heated cathode has three pins (Fig. 11.2), one for the cathode proper and two for its heater.

The electrons emitted by the cathode form a negative space charge at the cathode, thereby preventing the travel of electrons towards the anode. To overcome this effect, it is necessary to set up an electric field directed from the anode towards the cathode. In the simplest case, this purpose can be served by a storage battery supplying a voltage of 20-100 V. The negative side of the battery should be connected to the cathode, and the positive side to the anode. Acted upon by this field, the emitted electrons move towards the anode, thus constituting an electric current through the vacuum. The value of anode current, I_a, as a function of anode voltage, V_a, for several values of heater current, I_h, is plotted in Fig. 11.2. This is known as the current-voltage (or volt-ampere) characteristic of a vacuum diode.

As for semiconductor diodes (see Sec. 10.2), the basic parameters for vacuum diodes are the maximum forward voltage V_f, the corresponding forward current I_f, maximum reverse voltage V_{rv}, and the corresponding reverse current I_{rv}.

The principal function of a vacuum diode is to rectify alternating current. For this purpose, vacuum diodes may be arranged in a half-wave or a full-wave rectification circuit, similar to those examined in Sec. 10.6.

It is usual to class vacuum diodes into high-power and low-power devices. The former (often called *kenotrons*) are used in heavy-duty applications to supply direct current from an a.c. power source. The latter are mostly used as signal *detectors*, especially in radio receivers.

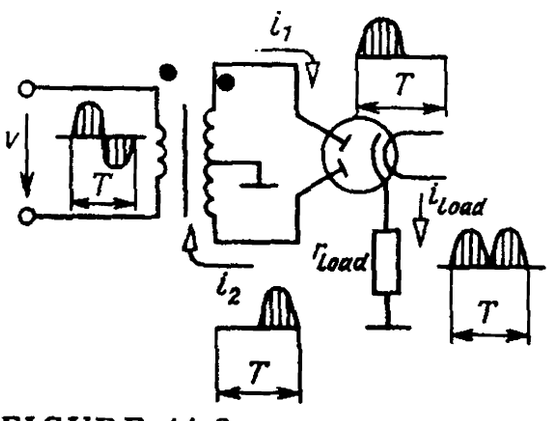


FIGURE 11.3

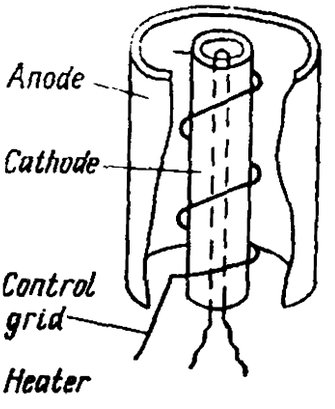


FIGURE 11.4

Full-wave rectifiers often use double diodes in which two single diodes are enclosed in a single envelope. This is done in order to simplify the overall design (Fig. 11.3).

A major disadvantage of vacuum diodes (and, indeed, of all vacuum tubes) is a substantial loss of power within a device. When electrons hit the anode, nearly all of their kinetic energy is converted to heat, so that the anode may rise to an appreciable temperature.

11.3 Vacuum Triodes

A *vacuum triode* has three electrodes: the plate (or anode), the cathode, and a third electrode called the *control grid* and placed between the first two (Fig. 11.4). This electrode owes its name to the fact that it was made in the form of a metal mesh in early vacuum triodes and still is a metal mesh in present-day high-power devices. Ordinarily the grid is a length of wire wound into a helix. By slightly varying the voltage between the control grid and the cathode, we can vary the plate current over a fairly broad range of values. This amplifying effect of the grid arises from the fact that it is located much nearer to the cathode than the plate, and partly shields the cathode against the action of the plate field.

A vacuum triode may be connected in a common-cathode circuit, a common-plate circuit, or a common-grid circuit, similarly to the three circuit configurations used for transistors (see Fig. 10.18).

We will limit ourselves to the common-cathode configuration as the most commonly used in practice.

The operating conditions in a vacuum triode are determined by two mutually independent voltages, plate voltage V_p and grid voltage V_g. The former is applied between the plate and the cathode, and the latter between the grid and the cathode.

The main static characteristics of a vacuum triode are those relating any one current to one of the two voltages, with the other volt-

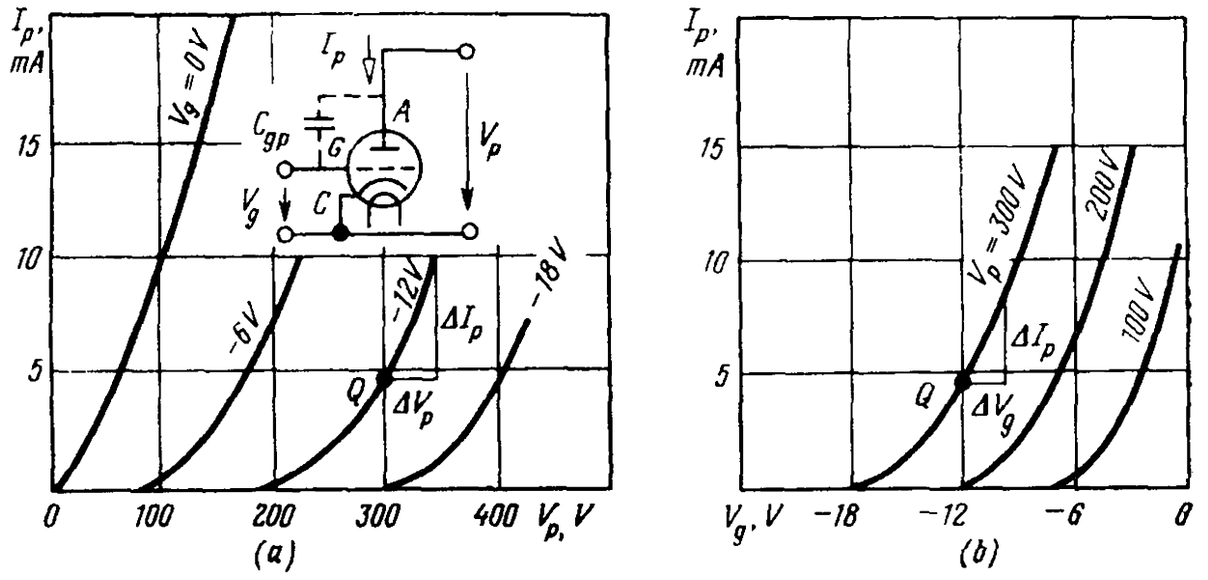


FIGURE 11.5

age held constant. Thus we can plot four sets (or families) of static characteristics for a vacuum triode.

(1) *Plate characteristics* (Fig. 11.5 a)

$$I_p(V_p)|_{V_g=\text{const}}$$

(2) *Plate current-vs-grid voltage* (*mutual* or *transfer*) *characteristics* (Fig. 11.5 b)

$$I_p(V_g)|_{V_p=\text{const}}$$

(3) *Grid current-vs-plate voltage characteristics*

$$I_g(V_p)|_{V_g=\text{const}}$$

(4) *Grid characteristics*

$$I_g(V_g)|_{V_p=\text{const}}$$

In most cases, vacuum triodes draw no grid current in operation ($I_g \approx 0$), so we will not consider their grid characteristics.

In small-signal operation, that is, when I_p, V_p and V_g are changing by a small amount each, the operating conditions in a vacuum triode can be described by a set of two equations

$$\Delta I_p = \frac{\partial I_p}{\partial V_g} \Delta V_g + \frac{\partial I_p}{\partial V_p} \Delta V_p \tag{11.1a}$$

$$\Delta V_p = \frac{\partial V_p}{\partial V_g} \Delta V_g + \frac{\partial V_p}{\partial I_p} \Delta I_p \tag{11.1b}$$

As follows from Eqs. (11.1), in addition to the static characteristics, the properties of a vacuum diode can be stated in terms of three dynamic factors, or parameters.

(1) The *amplification factor of a vacuum triode*

$$\mu = \partial V_p / \partial V_g |_{I_p = \text{const}} \tag{11.2}$$

(2) The *a.c.* (or *dynamic*) *plate resistance*

$$r_p = \partial V_p / \partial I_p |_{V_g = \text{const}} \tag{11.3}$$

(3) The *transconductance* (or *mutual conductance*)

$$g_m = \partial I_p / \partial V_g |_{V_p = \text{const}} \tag{11.4}$$

From the definition of the triode parameters it follows that

$$g_m r_p = \mu \tag{11.5}$$

Approximately the three parameters can be evaluated graphically from the static characteristics as the ratio of changes in the respective currents and voltages at the operating point (Q in Fig. 11.5). Typically, receiving and amplifying triodes have g_m = 2-5 mA V^{-1} and μ = 20-100.

Vacuum triodes suffer from a number of disadvantages which limit their use for signal amplification, above all the decrease in the amplification factor at frequencies of several hundred kilohertz and higher due to the capacitive coupling between the plate and the grid. The grid-plate capacitance, C_{gp}, usually 1 or 2 pF (shown dashed in Fig. 11.5), causes feedback owing to which the a.c. component of plate voltage, which is the output voltage of the triode, may go back to its input and completely upset the operation of the tube.

A major advantage of vacuum triodes (and of all vacuum tubes for that matter) is that their parameters are more stable towards external factors (above all, temperature variations) than those of their semiconductor counterparts.

11.4 Tetrodes and Pentodes

Tetrodes and pentodes have been designed to avoid the disadvantages associated with vacuum triodes.

In the *tetrode* there is one more grid, the *screen grid*, placed between the control grid and the plate (Fig. 11.6). The voltage between the cathode and the screen grid, V_{g2}, is maintained constant anywhere between 15% and 100% of the rated plate voltage, V_p.

The screen grid markedly reduces the electric field between the control grid and the plate, which is equivalent to a reduction in the capacitance between the plate and the control grid ($C_{gp} = 0.01$-0.05 pF). At the same time there is sizeable decrease in the effect of the electric field due to the plate on the electrons near the cathode, whereas the effect of the electric field due to the control grid remains practically unaffected. Therefore, a much smaller change in grid voltage, ΔV_g, is needed in order to maintain the plate current I_p at a constant value against a change in the plate voltage, ΔV_p. Accordingly, the amplification factor μ, Eq. (11.2), of a tetrode is tens or even hundreds of times that of a vacuum triode. Also, since a greater change in plate voltage, ΔV_p, is required in order to bring about the same change in plate current, ΔI_p, in a tetrode than in a triode, the a.c. (dynamic) plate resistance, r_p, Eq. (11.3), of a tetrode is likewise higher than it is in a triode.

The parameters of a tetrode are connected by the same relation, Eq. (11.5), as for a triode.

The plate characteristic, I_p (V_p), of a tetrode should have been the same as that of a triode, perhaps more gradual due to the effect of the screen grid. Actually, the plate characteristic of a tetrode has a kink, as shown in Fig. 11.6, due to secondary emission (sometimes called the dynatron effect) from the plate. At low values of V_p the velocities of the electrons that strike the plate are not high enough to cause secondary emission. When the velocities of the electrons rise high enough, secondary electrons are knocked out of the plate material, and this constitutes secondary emission. Since V_{g2} is held constant, secondary electrons travel at $V_{g2} > V_p$ from the anode towards the screen grid which is opposite to the direction in which primary electrons move. As a result, the plate current (the flow of primary electrons) is reduced by the amount equal to the secondary-electron current which increases as the plate voltage is increased. This goes on so long as $V_{g2} > V_p$ (see Fig. 11.6) and ceases when V_p exceeds V_{g2}. Any further rise in plate voltage brings about a gradual rise in plate current. Secondary emission limits the field of application for tetrodes.

One way to avoid the kink in the plate characteristic and to retain the valuable properties of a vacuum tetrode is to place a third grid, called the *suppressor grid*, between the screen grid and the plate.

It is made of turns of wire spaced wider apart than in the control or screen grid. Such a tube is called a *pentode*. The suppressor grid is held at approximately cathode potential (which, in the simpler circuits, is done by directly connecting it to the cathode as shown in Fig. 11.7), and thus the secondary electrons are repelled back to the

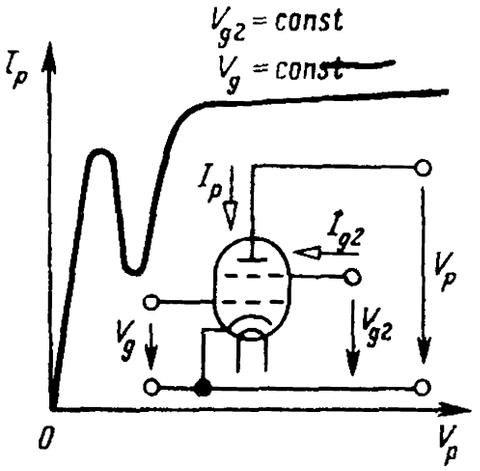


FIGURE 11.6

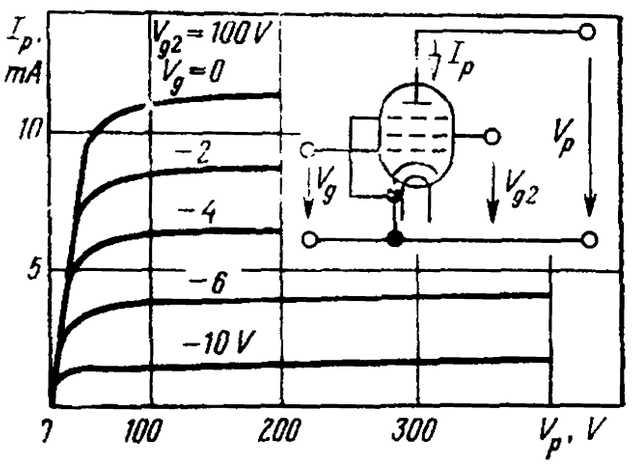


FIGURE 11.7

plate, without giving rise to a secondary-emission current. The plate characteristics of a pentode are shown in Fig. 11.7.

In some electronic circuits, plate current is controlled by varying the potential at the suppressor rather than at the control grid. In such a case, a relatively small negative voltage (about —20 V), when applied to the screen grid, would drive the tube to cutoff.

11.5 The Vacuum Triode Amplifier

The typical circuit of a vacuum-triode amplifier is shown in Fig. 11.8 a ($E_a = E_p$). As is seen, it has much in common with a common-emitter transistor amplifier (see Fig. 10.19 a).

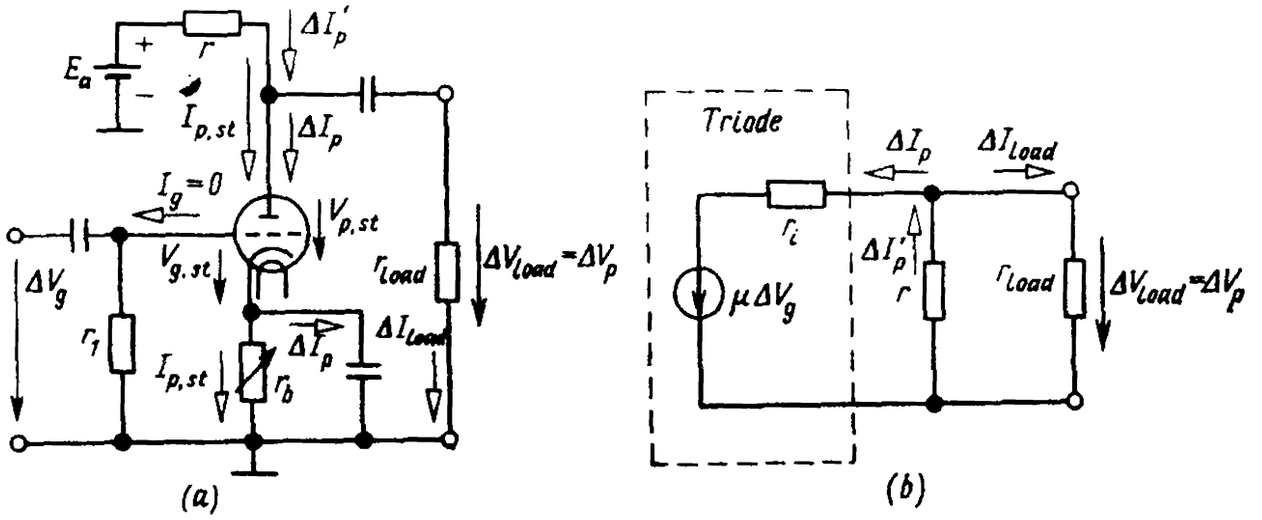


FIGURE 11.8

If we neglect the current drawn by the grid ($I_g \approx 0$), we may say that the plate current is the same as the cathode current. In the ab-

sence of an input signal ($\Delta V_g = 0$) the amplifier is in a static state (Fig. 11.9 a), so that the plate and cathode currents are constant and equal to $I_{p,st}$. The resistor r_b in the cathode lead is provided to maintain a constant voltage between the control grid and the cathode, $V_{g,st} = -r_b I_{p,st} < 0$, called the *self-bias voltage*. By varying the bias voltage with the aid of r_b, we can adjust the operating conditions in the amplifier at will. The required operating conditions (voltages and currents) can be found graphically by the load-line method as illustrated in Fig. 11.9 b (where Q is the operating point).

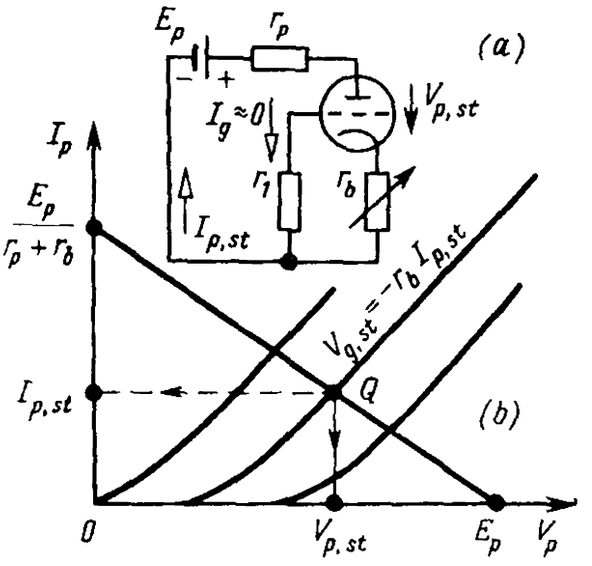


FIGURE 11.9

When the input signal voltage abruptly changes by $\Delta V_g \neq 0$ (see Fig. 11.8 a), an equal change occurs in the grid-to-cathode voltage, leading in turn to a change by ΔI_p in the plate current and by ΔI_{load} in the load current. It is assumed that the positive direction for ΔI_{load} is towards the cathode which is the *common* terminal of the triode for the a.c. component (that is, common to the input and output circuits).

Assuming the small-signal parameters of the triode constant and using Eqs. (11.1b), (11.2) and (11.3), we obtain

$$\Delta V_p = \mu \Delta V_g + r_p \Delta I_p \tag{11.6}$$

Equation (11.6) applies to the a.c. equivalent circuit of the amplifier shown in Fig. 11.8 b, where the equivalent of the vacuum triode is enclosed in a dashed box and r_i stands for r_p. As follows from the equivalent circuit, the *voltage gain* of the amplifier is

$$K_v = \Delta V_{load}/\Delta V_g = -\frac{\mu (r \parallel r_{load})}{r_p + r \parallel r_{load}} \tag{11.7}$$

and its *output resistance* (with the load resistance, r_{load}, disconnected) is

$$r_{out} = r_p \parallel r \tag{11.8}$$

The "—" sign in Eq. (11.7) implies that a change in input voltage is in anti-phase with the related change in output voltage.

Since the current drawn by the grid is $I_g \approx 0$, the *input resistance* of the amplifier shown in Fig. 11.8 a is

$$r_{in} = r_i \tag{11.9}$$

The high input resistance of vacuum-tube amplifiers (up to 10 MΩ) is an advantage over transistor amplifiers in which the input resistance is 1-3 kΩ, Eq. (10.14).

Typically, vacuum-triode amplifiers are used to amplify signals at frequencies up to 500 kHz.

11.6 The Pentode Amplifier

The typical circuit of a pentode amplifier is shown in Fig. 11.10. It differs from that of a vacuum-triode amplifier (see Fig. 11.8 a) in that it has two more grid circuits, with the suppressor grid connected to the cathode, and the screen grid connected to the plate-voltage source E_p via a voltage divider made up of a resistor r_{g2} and a capacitor C_{g2}. With this connection, the potentials at the grids relative to the cathode are maintained at constant values, because for the a.c. components C_{g2} presents an impedance of nearly zero.

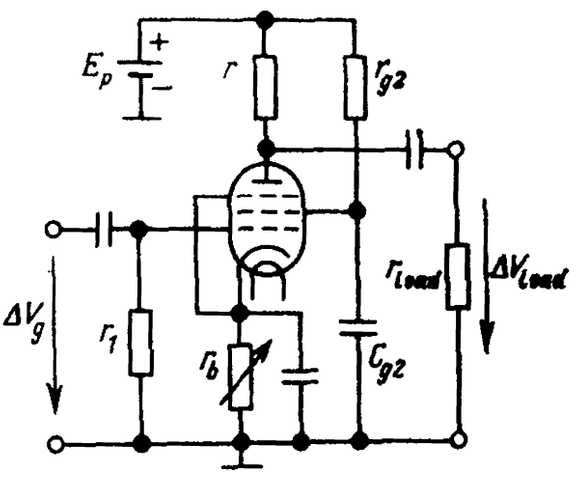


FIGURE 11.10

The operation of a pentode amplifier in terms of d.c. and a.c. components can be analysed similarly to that of a vacuum-triode amplifier, using respectively Eq. (11.9b) and (11.8b) and the static characteristics of a pentode (see Fig. 11.7) along with its values of μ, g_m and r_p.

Pentode amplifiers are basically used to amplify signals at frequencies up to 10 MHz.

11.7 A General Outline of Gas-Filled Tubes

In the manufacture of a gas-filled tube, its envelope is first evacuated to a very hard vacuum and then filled with a suitable gas. On being ionized, this rarefied gas substantially reduces the resistance between the electrodes in the tube envelope. This effect is utilized in gas-filled tubes.

The pressure of the filling gas (or mercury vapour) in the envelope of a gas-filled device should be very low, being 10^{-1}-10^{3} Pa in most cases. This is necessary for the free path length of electrons (the distance travelled from collision to collision) to be sufficiently long. Then even at relatively low electric field intensities the electrons acquire energies required for inelastic collisions with gas or vapour atoms or molecules. In contrast to elastic collisions, inelastic collisions excite and ionize the gas or vapour atoms, thereby producing ever more free electrons and positive ions.

Electrons travel at velocities which are many times that of the relatively heavy positive ions, so the principal charge carriers in gas-filled devices are electrons. The share contributed by positive ions is usually less than 0.1 of the total current across the discharge space. The positive ions are useful in that they neutralize the negative space charge due to electrons.

The medium produced in the discharge space is the plasma in which both negative and positive charge carriers are present in an equal, although a very high number (about 10^9-10^{12} pairs of charge carriers per cubic centimetre). The conductivity of the gas plasma is very close to that of metals, so the current in a gas-filled tube may be very high at a relatively low voltage between the electrodes.

Due to the ionic mechanism of conduction, the formation of a current in gas-filled tubes involves an appreciable time lag, a feature responsible for a good deal of difference between them and vacuum tubes. Once the anode voltage is removed, some of the ions and electrons recombine to form neutral gas atoms at the walls of the envelope, and this entails what is known as deionization delay. This form of delay makes gas-filled tubes unsuitable for use in circuits operating at high frequencies.

The discharge in gas-filled tubes may be *nonself-maintaining* and *self-maintaining*. It takes an additional energy source, called the ionizer, in order to produce and maintain a nonself-maintaining discharge. This may be the thermionic emission from a hot cathode, radiant energy, and so on. The only thing required to initiate a self-maintaining discharge is an electric field within the gas-filled device itself.

Gas-filled devices may use an *arc discharge* or a *glow discharge*.

11.8 Arc-Discharge Tubes. The Thyratron

Given the same size, the power output of an arc-discharge tube is many times that of a vacuum tube. For this reason controlled arc-discharge tubes are widely used for direct control of various circuits in cases where the size has to be kept to a minimum.

Among the devices in this class are gas-filled rectifier diodes and thyratrons which use a nonself-maintaining arc discharge, and also ignitrons and mercury-arc tubes mainly used in high-power rectifiers.

A *thyratron* is essentially a triode whose envelope is filled with a highly rarefied inert gas or mercury vapour. The graphic symbols for the thyratron and its current-voltage characteristics are shown in Fig. 11.11. In its basic modification, a thyratron is a hot-cathode tube as it uses a nonself-maintaining arc discharge. This discharge takes place at what is known as the *firing voltage* V_{fr}. The firing voltage of a thyratron can be varied by varying the voltage between the control grid and the cathode. When the control grid is held negative

with respect to the cathode, the firing voltage is raised; when it is held positive, the firing voltage is reduced. Until the thyratron fires, the current through it is relatively small, because it is produced by the few electrons emitted by the hot cathode that have been able to overcome the opposition of the space charge. The situation changes drastically when the thyratron fires (its filling gas or vapour is being ionized). Now the plate current rises many times rapidly (in a matter of 10^{-8} s) and reaches the value at which an arc discharge takes place (portion 1-2 in Fig. 11.11). Positive ions completely cancel out the space charge, and the greater proportion of the plate-cathode space is filled with a gas plasma.

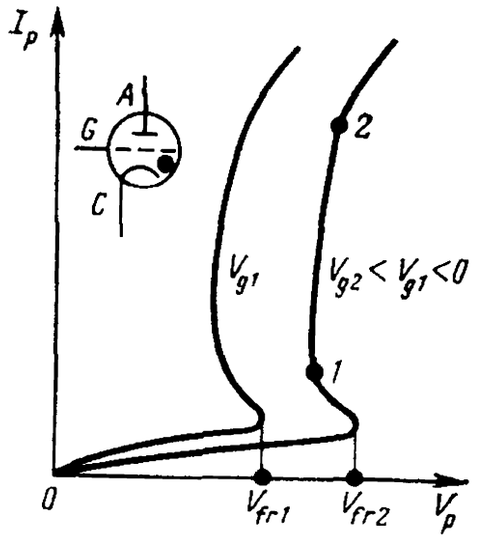


FIGURE 11.11

Once an arc discharge is established, the voltage on the control grid ceases to have any effect on the plate current. Positive gas ions envelope the control grid and cancel out its negative charge, and so electrons are now free to pass through the grid unimpeded.

The current-voltage characteristic of a thyratron is similar to that of a thyristor (see Fig. 10.10 c). For this reason, the two devices have much in common in terms of application as controlled static switches (see Fig. 10.17).

11.9 Glow-Discharge Tubes. The Glow-Discharge VR Tube

A glow discharge is used in low-power gas-filled tubes among which are the glow-discharge VR tube using a nonself-maintaining glow discharge, and the decade-counting tube using a self-maintaining glow discharge.

A *glow-discharge VR tube* is essentially a diode in which a nonself-maintaining glow discharge is initiated in an inert gas by a hot cathode. The graphic symbol and current-voltage characteristic of a typical glow-discharge VR tube are shown in Fig. 11.12.

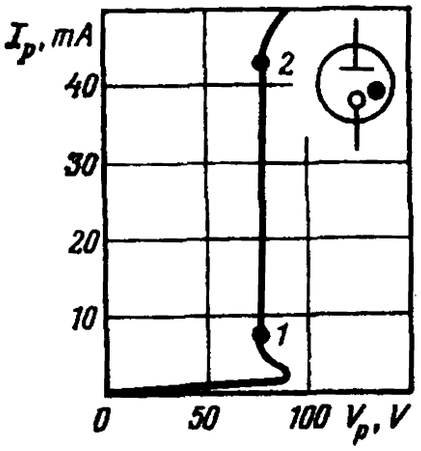


FIGURE 11.12

In order to support a nonself-maintaining glow discharge over a wide range of current values, the hot cathode must have a large surface area. Typically, the internal (plate) resistance of a glow-discharge VR tube within the operating portion of its static current-voltage characteristic

(portion 1-2 in Fig. 11.12) $r_p = dV_p/dI_p$, is very small, being 20-200 Ω. Therefore, glow-discharge VR tubes are used for voltage regulation in electric circuits similarly to Zener diodes (see Fig. 10.7 b).

11.10 Photoelectric Devices

There are three clearly recognized photoeffects. These are (1) the photoemissive effect, (2) the photoconductive effect, and (3) the photovoltaic effect.

The *photoemissive effect* refers to the emission of electrons from a metal or semiconductor surface, caused by the absorption of visible or near-visible radiation.

The *photoconductive effect* refers to the increase in conductivity observed in some materials when they are exposed to light.

The *photovoltaic effect* refers to the voltage generated at metal-semiconductor and semiconductor junctions when they are excited by light.

Our discussion will be limited to the photoemissive effect which is utilized in devices called photocells or phototubes. It is also used in photomultiplier tubes. In each case, the source (or sources) of emitted electrons is a photocathode (or photocathodes).

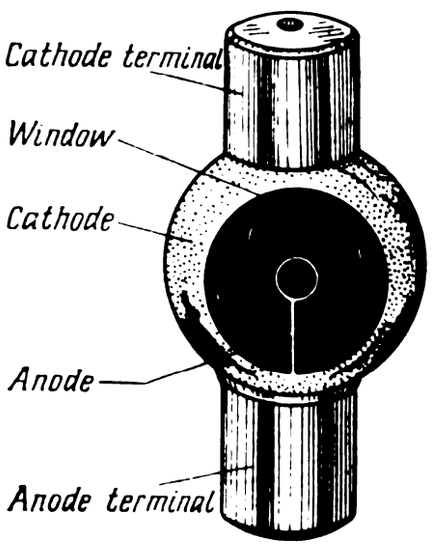


FIGURE 11.13

A phototube is essentially a diode in which the anode and the photocathode are enclosed in a glass envelope. There are vacuum phototubes (in which the envelope is evacuated to a vacuum of 10^{-5}-10^{-6} Pa) and gas phototubes (filled with an appropriate gas under a pressure of 10^{-1}-10^{-3} Pa). The photocathode is a layer of an alkali-earth metal applied to a silver substrate which is deposited directly on the inner side of the glass envelope and is connected to a suitable terminal. On the outside, the silvered envelope produces a mirror surface except for the small window left in the silver deposit to let in incident light. The anode is often made in the form of a wire ring placed in front of the photocathode (Fig. 11.13). For its operation, a phototube needs a d.c. anode voltage source so as to establish an electric field and to steer the emitted electrons towards the anode.

The current-voltage characteristic, I (V_a), of a *vacuum phototube* (Fig. 11.14 a) looks similar to that of a vacuum diode (see Fig. 11.2). At first, when the anode voltage is low, the current I rises with increasing anode voltage. When all of the emitted electrons (available

at a given level of illumination of the cathode) are collected by the anode, the current reaches its saturation value and remains practically unchanged as the anode voltage is raised. The saturation current

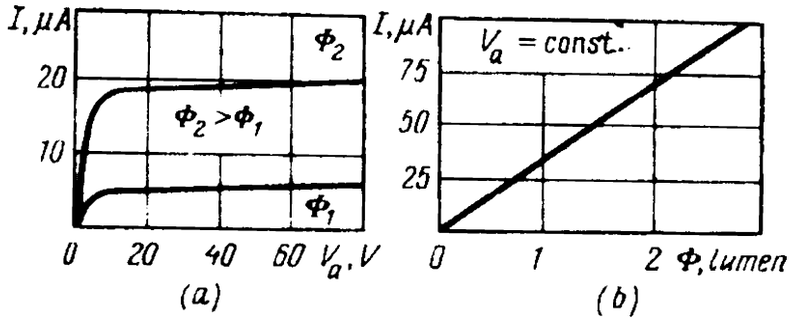


FIGURE 11.14

is a function of only photoelectric emission and, in consequence, of level of illumination. This is a valuable property of vacuum phototubes.

When the saturation current of a phototube is plotted as a function of luminous flux Φ, the result is a linear *light characteristic* (Fig. 11.14 b), in accord with the law of photoelectric emission.

The ratio of a change in phototube current to the change in light flux that has caused it gives the sensitivity of the device, $S_l = \mathrm{d}I/\mathrm{d}\Phi$. It is relatively low, being from 20 to 80 μA lumen^{-1}.

The sensitivity of a phototube can be increased by filling its evacuated envelope with a rarefied inert gas. During the operation

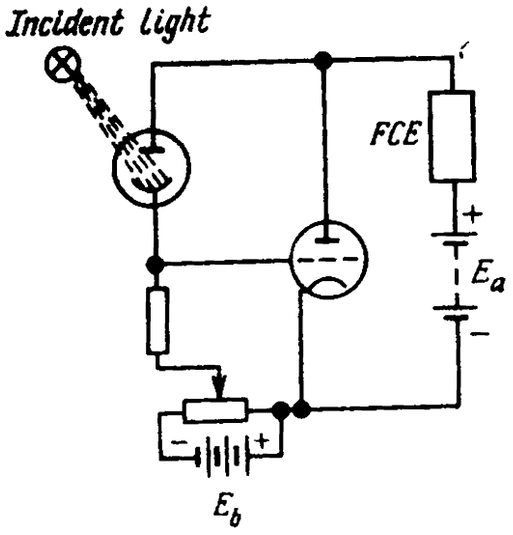


FIGURE 11.15

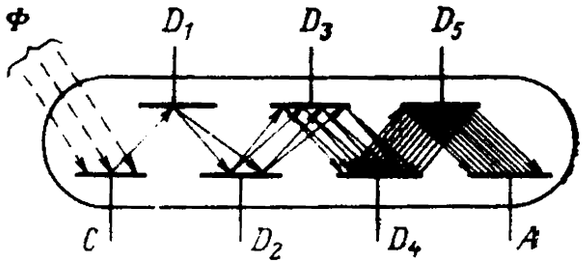


FIGURE 11.16

of this *gas phototube*, the emitted electrons ionize the gas, the stream of electrons moving from the cathode to the anode is amplified (gas amplification), and there appears a stream of positive ions moving in the opposite direction. The sensitivity is thus increased fivefold, being now as high as 100-400 μA lumen^{-1}. Because of the gas, how-

ever, the current-voltage characteristics of gas phototubes are nonlinear. Another disadvantage of gas phototubes is a certain time lag in their response, which is not present in vacuum phototubes.

In most cases the current generated by phototubes (even of the gas-filled variety) is not large enough to drive a final control element. As a way out, phototubes have to be used in conjunction with a tube (Fig. 11.15) or a transistor amplifier. So long as the phototube is not illuminated, the triode is driven to cutoff by the negative potential (with respect to the cathode) supplied by a bias battery E_b. When light strikes the photocathode, the control grid is connected to the positive side of the anode battery E_a, the triode is driven to conduction, its plate current goes up to a value sufficient to cause the operation of the final control element, FCE.

An *electron photomultiplier* is a type of phototube in which a large gain (high sensitivity) is achieved by means of secondary emission without resort to vacuum-tube amplifiers. The basic principle is that the photocathode C emits photoelectrons which are driven by an electric (Fig. 11.16) or a magnetic field to bombard a target electrode or dynode, D, that yields more electrons by secondary emission; these electrons bombard another target electrode or dynode, and so on for several stages. Finally, the electrons are collected by a collector, A, and the net result is a current which is many times the current due solely to the photoemission from the cathode. Owing to this amplification, the sensitivity (gain) of electron photomultipliers is very high, being 1-10 A lumen^{-1}. Unfortunately, their output current is never higher than 10-15 mA. Electron photomultipliers are mainly used to measure light intensity at low levels of illumination.

12 Electrical Measurements

12.1 General

Electrical measurements involve all electric and magnetic quantities, such as current, voltage, power, energy, magnetic flux, and so on. Knowledge of the magnitude of these quantities is essential for proper supervision and/or control of electric devices, circuits, and systems. That is why measurements in electrical engineering are a vital activity.

Electrical measuring instruments are widely used to measure nonelectric quantities as well (such as temperature, pressure, etc.), which are first converted into proportional electric quantities. This is what is known as the measurement of nonelectrical quantities by electric methods. Through the use of electric methods of measurement, we can fairly simply transmit meter indications over large distances (telemetry), control plant and processes (automatic control), record meter indications (on, say, tape, such as in remote monitoring), carry out mathematical operations on the transmitted values, etc.

12.2 Standards, Instruments, and Methods of Measurement

The measurement of a physical quantity is a procedure by which the value of that quantity is determined through an experiment by comparing it with another quantity of the same kind chosen as a unit.

In the general case, such a comparison calls for the use of a standard which is a physical embodiment of the unit, and a *comparator*. For example, in resistance measurements the standard will be a standard resistance coil and the comparator will be a resistance bridge, usually combined into a physically integral instrument.

Measurements are substantially simplified when use is made of *direct-indicating instruments*, often called simply indicators, which present the numerical value of the measurand (the quantity being measured) directly on a scale or a dial. Examples are ammeters, voltmeters, wattmeters, and electricity meters. When such an instrument is available, one needs no standard (such as a standard resistance coil for resistance measurements) for any measurement on hand, but some standard was needed (and, of course, used) when the instrument was first calibrated. As a rule, instruments of the comparator

type offer better accuracy and higher sensitivity, but measurements with direct-reading instruments are simpler, faster and less expensive to carry out.

It is customary (at least in the USSR) to class all forms of measurements into direct, indirect (inferential), and cumulative.

In a direct measurement, the value of the unknown is ascertained directly from experimental data. An example is the measurement of a current with an ammeter.

In an indirect (or inferential) measurement the value of the unknown has to be deduced, or inferred, from direct measurements of other quantities functionally related to the measurand of interest. An example is the determination of the resistance of a circuit by measuring the voltage across and the current in that circuit with a voltmeter and an ammeter, respectively.

It is to be noted that indirect measurements may yield a substantially lower accuracy than it would be if a direct measurement procedure were used. This happens because the errors associated with each of the direct measurements are combined.

In a cumulative measurement, the final result is derived from those of several groups of direct or indirect measurements each involving a separate quantity, and the measurand is a function of the measured quantities. An example is the determination of the temperature coefficient of resistance for a material by measuring its resistance at several values of temperature. This form of measurements is typical of laboratory practice.

According to the manner of using instruments and standards, measurement methods may be divided into three broad classes: direct, null-balance, and differential.

In the *direct measurement method*, the unknown is measured by comparing it directly with an adopted standard (such as in measuring length with a rule) or by noting the indication of the instrument used (such as in measuring a current with an ammeter). In this case, the limit of accuracy is that of the instrument used, which cannot be very high.

In the *null-balance method*, a standard is so adjusted that its magnitude (or the extent of its effect) cancels the magnitude (or the extent of the effect) of the unknown. Now a measuring instrument is only needed to obtain equality (a null balance) between the unknown and the standard. This should be a very accurate instrument, and it is usually called a *null detector* or a *null indicator*. In d.c. measurements, the most commonly used null indicators are moving-coil galvanometers (see Sec. 12.7), and in a.c. measurements, vacuum-tube or transistor null detectors. The accuracy of the null-balance method is very high and mainly depends on the accuracy of the reference standards used and the sensitivity of the null detector or indicator. The null-balance methods most important for electrical

measurements are those based on the use of electric bridges and potentiometers.

A still better accuracy can be achieved with *differential measurement methods*. Now the measurand is balanced with a standard, but a complete balance is never sought. Instead the operator reads the difference thus left between the two. Differential methods are used to compare any two quantities which differ but little from each other.

12.3 Errors of Measurement and Accuracy Classes

The accuracy of a measurement depends on the errors that are likely to creep in during the measuring procedure. For each particular form of measurement, it is usual to set the limit of error.

An error may be stated in any one of three ways as follows. One is to give an *absolute* (or *algebraic*) *error*, ΔA, defined as the difference between the measured value, A_{meas}, and the true value, A, of the measurand

$$\Delta A = A_{meas} - A$$

For example, if an ammeter reads $A_{meas} = 9$ amperes, and we know the true value to be $A = 8.9$ amperes, the absolute error will be $\Delta A = 0.1$ ampere.

Once a measured value is determined, the true value can be obtained by adding to it the correction which is the negative of the absolute error.

The other way is to specify the error as the ratio of the absolute error to the *true value* of the unknown, expressed as a percentage

$$\gamma_0 = (\Delta A/A) \times 100\%$$

Since, however, the difference between the true value and the measured value is small, it may practically always be taken that

$$\gamma_{ind} = (\Delta A/A_{meas}) \times 100\%$$

which is known as the percent-of-indication error. Taking the previous example, the percent-of-indication error is

$$\gamma_0 = (0.1/9) \times 100\% = 1.11\%$$

This form of error presentation is, unfortunately, inconvenient when dealing with the most commonly used pointer-type analogue instruments. The point is that the absolute error ΔA remains usually the same along the entire scale. Therefore, when the measured value, A_{meas}, is progressively reduced, the percent-of-indication error is as progressively increased (Fig. 12.1). That is why it is usually recom-

mended to choose the range of an instrument so that indications will fall within the second half of the scale, that is, nearer to its full value.

Most often, the accuracy of instruments is stated in terms of the percent-of-full scale deflection error defined as the ratio of the absolute error, ΔA, to the full-scale deflection (fsd) value of the instrument's scale, A_{fsd}, expressed as a percentage

$$\gamma_{fsd} = (\Delta A/A_{fsd}) \times 100\% \tag{12.1}$$

If, in our previous example, the full-scale deflection value of the ammeter is $A_{fsd} = 10$ A, the percent-of-fsd error is

$$\gamma_{fsd} = (0.1/10) \times 100\% = 1\%$$

Errors may have their source(s) in an instrument itself and/or environmental factors. In the former case we have what is called the *basic error* of the instrument (usually stated as a percent-of-fsd error). It will not be exceeded so long as the instrument is used under normal conditions, namely at an ambient temperature of 20°C (if not otherwise stated on the scale of the instrument), the normal mounting position (usually indicated by an appropriate symbol on the instrument), absence of ferromagnetic masses and external magnetic fields (except the terrestrial magnetic field) at and around the instrument, the normal voltage, frequency, current or voltage waveform, etc.

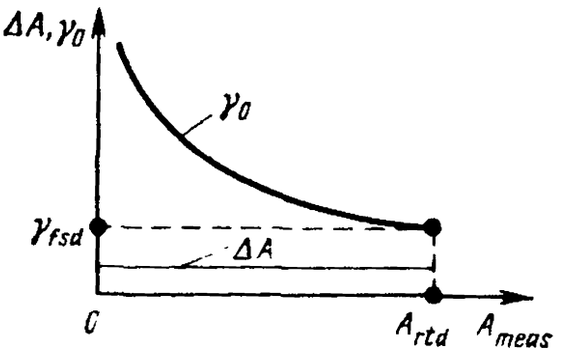


FIGURE 12.1

The limit of basic error determines the accuracy class (or rating) of an instrument. The accuracy class (or rating) of an instrument is designated by a class index which is a number numerically equal to the limit of basic error expressed as a percentage. Examples of the class indices used in the USSR are 0.05, 0.1, 0.2, 0.5, 1, 1.5, 2.5, and 4. For example, class index 1 (or accuracy class 1) indicates that the limit of basic error of the instrument at any scale division is 1%.

Variations in the environmental factors from normal result in *complementary errors*. For example, if the actual ambient temperature is other than 20°C, the instrument will be prone to a temperature error.

Instrument makers ordinarily mark the dials of their instruments with symbols intended to help the user with their operation. Examples of the instrument symbols used in the USSR are given in Table 12.1.

TABLE 12.1 Symbols Marked on the Scales of Electrical Instruments

Particulars	Symbol
Three-phase instrument for an unbalanced load	
Three-phase instrument with a dual-element movement	
Protected against external magnetic fields of, for example, 2 mT intensity	
Protected against external electric fields of, for example, 10 kV m^{-1} intensity	
Class index numerically equal to the limit of error expressed as full-scale deflection (f.s.d.) value, e.g. 1.5	1.5
Same, expressed as a percentage of scale length, e.g. 1.5	
Horizontal scale	
Vertical scale	
Scale inclined at 60° to the horizontal	
Position of the instrument relative to the Earth's magnetic field	
Measuring circuit is insulated from the case and tested at a voltage of, for example, 2 kV	

TABLE 12.1 (continued)

Particulars	Symbol
Not to be tested for insulation strength	
Caution: Insulation strength of the measuring circuit to the case is other than specified by standard. (The caution sign is painted red.)	

12.4 The Effect of an Instrument on the Measured Circuit

When an instrument is connected in the circuit where some electrical quantity is to be measured, this inevitably brings about a change in the current(s) in and the voltage(s) across the circuit. The reason is that the instrument draws some power from the circuit.

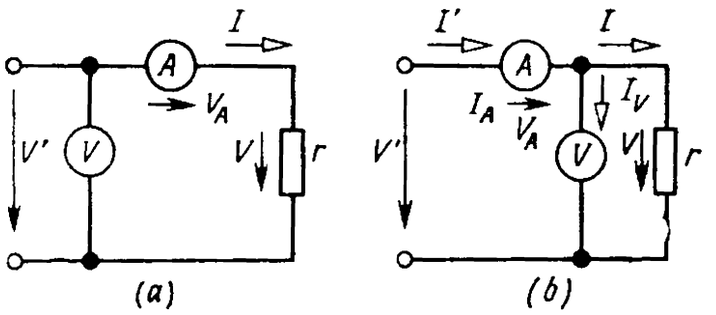


FIGURE 12.2

In the case of low-power circuits, it may strongly distort the results. To avoid this, measures must be taken to keep the instrument's power to a minimum.

An example how an instrument may affect the result of a measurement is the determination of the resistance of a resistor (to direct current) by Ohm's law after the current in and the voltage across the resistor have been measured with an ammeter and a voltmeter respectively. Any one of two set-ups may be used for the purpose (Fig. 12.2), and in each case the power drawn by the instruments must be allowed for in calculating the resistance.

In the set-up of Fig. 12.2 a, the ammeter measures the current I in the resistor of resistance r, and the voltmeter measures the voltage V' $V + r_A I$. Here r_A is the resistance of the ammeter, so the measured voltage is the sum of the voltage across the resistor, V,

and the voltage between the ammeter's terminals. Hence, by Ohm's law, the sum of the resistances of the resistor and of the ammeter is

$$V'/I = r' = r + r_A$$

The actual resistance of the resistor is

$$r = r' (1 - r_A/r')$$

Obviously, the error of measurement can be brought down by reducing the resistance of the ammeter.

In the set-up of Fig. 12.2 b, the voltmeter is connected directly across the terminals of the resistor and reads the voltage V across the resistor. The ammeter measures the sum of the currents in the resistor and in the voltmeter circuit

$$I' = I + I_V$$

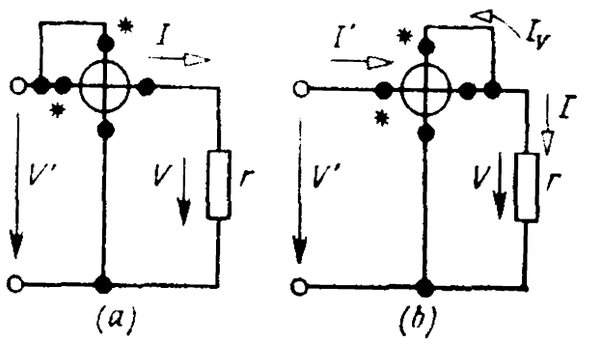


FIGURE 12.3

Hence the total conductance is

$$g' = I'/V = g + g_V = 1/r + 1/r_V$$

where r_V is the resistance of the voltmeter.

The conductance of the unknown resistor can be found by subtracting the conductance of the voltmeter from the total conductance determined earlier

$$g = g' - g_V = g' (1 - g_V/g')$$

that is,

$$r = r'/(1 - r'/r_V)$$

As is seen, we can improve the accuracy of measurement by increasing the resistance of the voltmeter.

Wattmeters, too, affect the circuits in which the power is to be measured. Two basic set-ups used for power measurements with a wattmeter are shown in Fig. 12.3. They respectively correspond to the two set-ups we have used to measure the resistance of a resistor. In the first case, the error is due to the resistance presented by the current (series) circuit of the wattmeter, r_A. In the second case the error is due to the power drawn by the potential (shunt) circuit of the wattmeter.

In the set-up of Fig. 12.3 a the wattmeter measures the power delivered to the load and also the power dissipated in the resistance of its own current (series) circuit

$$P_{meas} = P + r_A I^2$$

When power is measured using the set-up shown in Fig. 12.3 b, the wattmeter indicates the power delivered to the load and also the power dissipated in its own potential (shunt) circuit

$$P_{meas} = P + g_V V^2$$

In the case of a.c. circuits, the errors are more difficult to evaluate (and the corrections are more difficult to apply), because the impedance of an a.c. circuit is a complex quantity.

The effect caused by the power (energy) drawn by instruments on the circuits they measure is aggravated when the circuits themselves are in a low-power category. This, for example, is true of automatic-control and electronic circuits.

12.5 The Movements of Indicating Instruments

In direct-acting indicators, there is a moving part which is caused to move relative to a stationary part by the quantity being measured. Indication can be effected by a pointer (pointer-type instruments) or by a light beam reflected from a moving mirror (mirror-

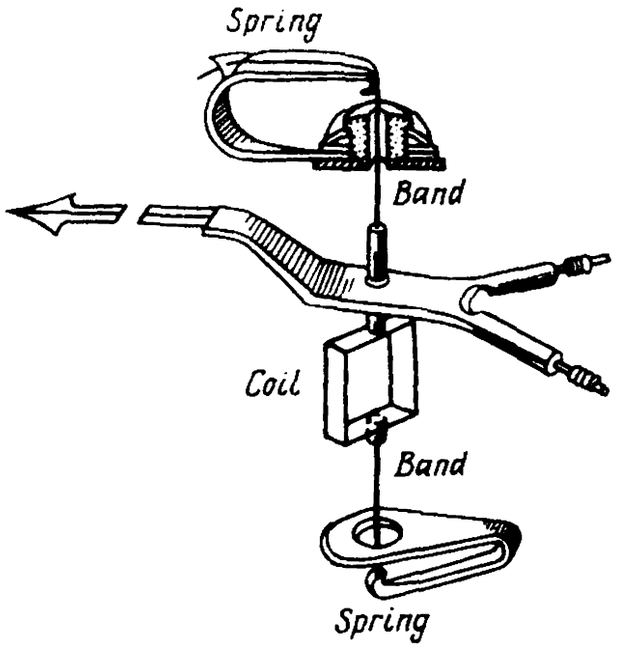


FIGURE 12.4

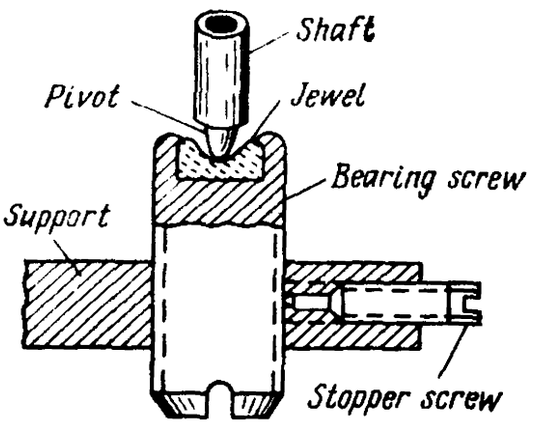


FIGURE 12.5

type or reflecting instruments). A feature common to both types is that the moving element (or movement) can be mounted on taut bands, on pivots, or a quartz (or metal) ligament.

Taut bands are currently used in most indicating instruments. The bands are made of beryllium or tin-lead bronze. The movement is mounted between two bands (Fig. 12.4) whose ends are made fast to two leaf springs. The taut bands can be used to convey electric current to the moving element. Also, when they are twisted, they supply a restoring torque.

Pivots were used in nearly all direct-acting indicators until quite recently. The movement in this case is mounted on a shaft (Fig. 12.5), usually a light-weight aluminium tube. Secured in the ends of the shaft are two pivots made of steel wire. The pivots are in turn set in the craters made in jewel bearings (at one time made of ruby, but now mostly of synthetic agate, corundum and the like). The jewel bearings are rolled into bearing screws with which the position of the jewel relative to the pivot can be adjusted continuously. When the shaft is upright, the friction between the pivots and the jewels is smaller than it is when the shaft is horizontal. For this reason, most precision instruments have their scales arranged horizontally so that the shaft is vertical. Yet, this friction is a major drawback of pivot mounting; because of it the power drawn by the instrument itself is increased many times, leading to a complementary error (known as the pivot-friction error) which increases as the pivots and jewels wear out.

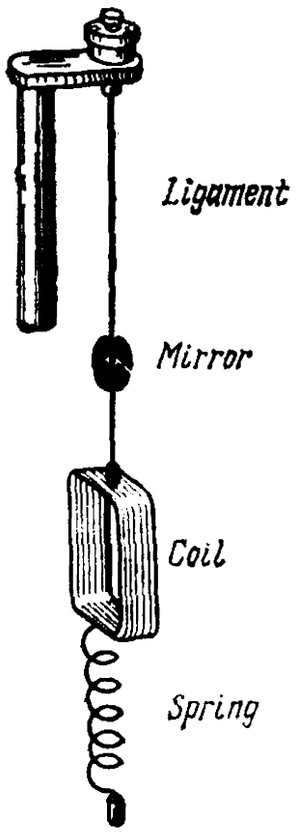


FIGURE 12.6

A single quartz or metal ligament (Fig. 12.6) is used in especially sensitive instruments, such as mirror galvanometers. An instrument with its moving element suspended on a single ligament has to be supplied with a level of its own, because the position of the moving

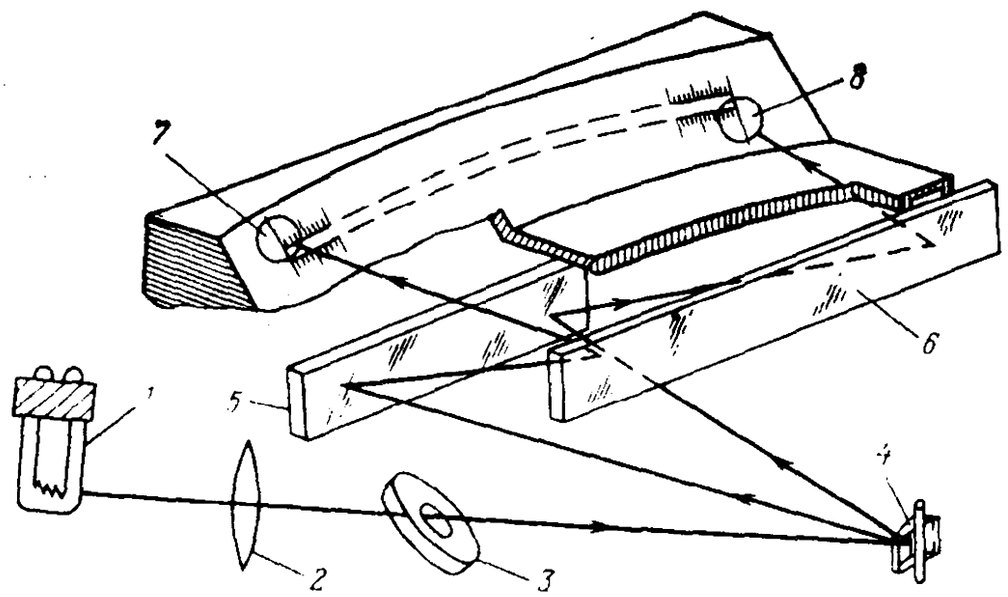


FIGURE 12.7

element relative to the stationary part depends on whether the instrument is properly levelled or set upright.

Indication by a light beam is used in most highly sensitive instruments. The arrangement of a reflecting instrument is shown in Fig. 12.7. Inside the instrument there is a lamp, 1, from which the light beam passes through a focusing lens, 2, and a diaphragm, 3, and strikes a mirror, 4, attached to the moving element of the instrument. On being reflected from the moving mirror, it is then reflected from stationary mirrors, 5 and 6, so that it forms a light spot on a scale which has its beginning at point 7 and its end at point 8. This extends the distance travelled by the light spot and, in consequence, improves the sensitivity of the instrument.

Owing to the multiple reflection of light within the instrument, the mirror galvanometer shown in Fig. 12.6 which calls for the use of a separate illuminator and an external scale (all which must be carefully aligned and preserved in a stationary set-up) may in some cases be replaced with a portable reflecting instrument. Unfortunately, even a portable instrument requires a power supply for its lamp and its internal arrangement is rather complicated.

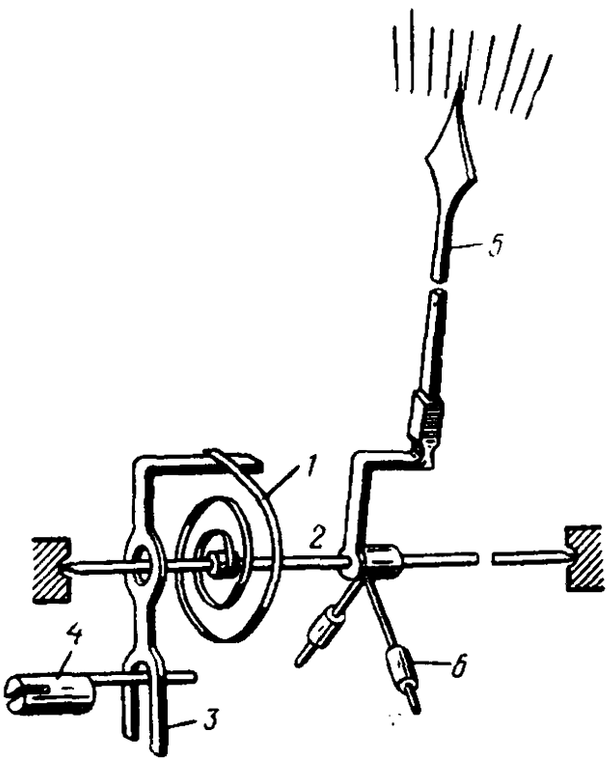


FIGURE 12.8

In the case of taut-band and single-ligament suspension, the necessary restoring (or controlling) torque is supplied as the bands or ligament are twisted. In pivoted instrument the restoring torque is supplied by additional spiral springs made of phosphorus bronze (at 1 in Fig. 12.8). The restoring (or controlling) torque must be directed opposite to the deflecting torque so that the deflection of the moving element is a unique function of the quantity being measured. In the absence of a controlling torque, the pointer of the indicator would be swept off scale at any value of the deflecting torque.

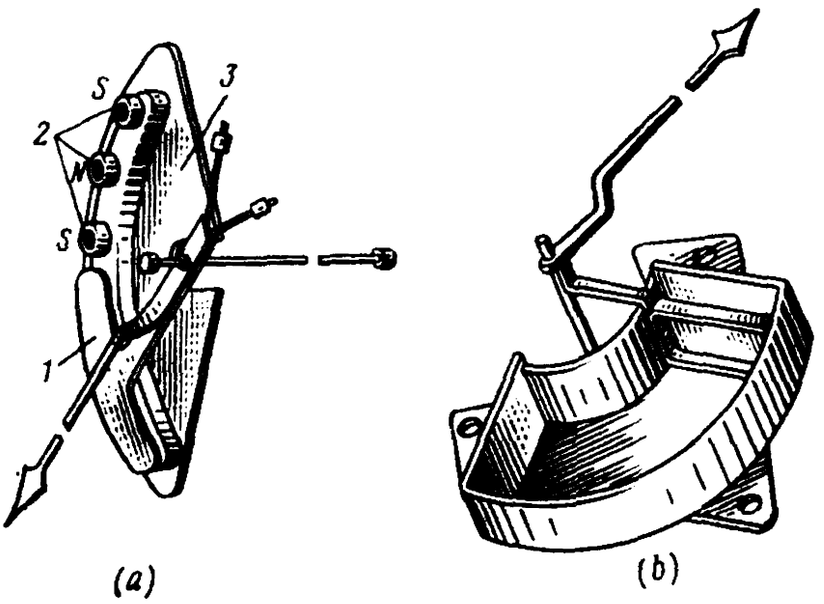
Variations in ambient temperature, permanent set or, more frequently, overloads may cause the pointer of an instrument to move and stay away from its zero mark even when it is not in use. If so, the instrument must be set to zero prior to a measurement. This is performed with the aid of a zero adjuster. When the restoring torque is supplied by a spiral spring (at 1 in Fig. 12.8), one of its ends is made fast to the shaft 2 of the moving element and its other end is secured to the arm of a fork 3 which straddles a zero-adjuster

screw 4. When the pointer 5 is off zero, the operator rotates the zero-adjuster screw to move the arm, thereby winding up or unwinding the opposing spring, and sets the pointer against the zero mark.

The moving element of an indicator is usually balanced with counterweights 6. When properly balanced, the instrument indication will be independent of its position.

12.6 Dampers

The moving element and the opposing spring of an indicator make up a mechanical system which has both mass and elasticity and is therefore capable of mechanical vibrations. That is why when the quantity being measured changes in value, the moving element does not reach a new position of equilibrium until after a few cycles of



(a) (b)

FIGURE 12.9

oscillation about it. This transition can be speeded up through the use of dampers which act by absorbing the kinetic energy of the moving system. For most pointer-type instruments the damping time ought not to exceed 4 s. (The damping time is defined as the time between the instant when the instrument is turned on and the instant when the pointer deflects not more than 1% of scale length from the position of equilibrium.)

Present-day instruments mostly use eddy-current dampers and air dash-pots.

An eddy-current damper (Fig. 12.9 a) is essentially an aluminium sector 1 mounted on the shaft of the moving element so that it cuts the field set up by several permanent magnets, 2, attached to a baseplate 3. When the sector cuts the magnetic field, eddy currents are induced in it. Their interaction with the field of the per-

manent magnets produces a force which (by Lenz's law) tends to brake the moving element.

An air dash-pot utilizes the difference in air pressure which is produced on either side of a light-weight aluminium vane moving in a closed chamber as the vane is made to move (Fig. 12.9 b). The vane is mounted on the shaft of the moving element. Air dash-pots are less effective than eddy-current dampers. They have to be used in cases where the presence of a permanent magnet inside an instrument might cause complementary errors.

In some instruments, use is made of liquid dash-pots. In a liquid dash-pot the vane attached to the moving element moves in a non-drying liquid.

12.7 Types of Indicating Instruments

Any direct-acting instrument consists of a measuring mechanism and a measuring circuit.

The purpose of the measuring mechanism is to convert the applied electric energy into mechanical energy, that is, the travel of the moving element and the associated pointer. The measuring circuit transduces the unknown electric quantity (voltage, power, current, and so on) into a proportional quantity which directly acts on the measuring mechanism. For example, the measuring circuit in a voltmeter consists of the coil of the measuring mechanism and a series resistor. Since the resistance of the measuring circuit is held constant, the current flowing in the measuring mechanism of the voltmeter is proportional to the voltage being measured. The same measuring mechanism may be combined with a variety of measuring circuits, each combination being adapted to measuring a particular quantity.

Indicating instruments are available in several types each utilizing a different principle of operation. The most commonly used types of indicating instruments along with their graphical symbols (as used in the USSR) are listed in Table 12.2.

Moving-coil instruments. In a moving-coil instrument the deflecting torque results from the interaction of the unknown current, which energizes a moving coil, with the field set up by a permanent magnet. As a result of this interaction, the coil is forced to move on its taut bands or pivots in the air gap of the permanent magnet.

The permanent magnet may be placed outside the coil (an external-magnet moving-coil instrument) or inside the coil (an internal-magnet moving-coil instrument).

When the permanent magnet (at 1 in Fig. 12.10) is external to the moving coil, the magnetic circuit of the instrument is formed by a yoke 2, pole-pieces 3, and a cylindrical core 4, all made of a magnetically soft metal which is relatively easy to work. The pole

TABLE 12.2 Types of and Symbols for Indicating Instruments

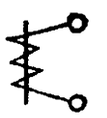
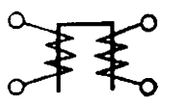
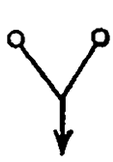
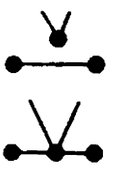
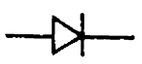
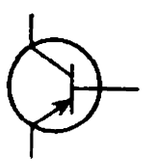
Type and description	Symbol	
	conventional type	ratiometer type
Moving-coil instrument		
Permanent-magnet moving-coil instrument		
Moving-iron instrument		
Polarized moving-iron instrument		
Electrodynamic instrument		
Iron-cored electrodynamic instrument		
Induction instrument		
Electrostatic instrument		
Vibrating-reed instrument		
Hot-wire (thermal) instrument		
Bimetal instrument		

TABLE 12.2 (continued)

Type and description	Symbol	
	conventional type	ratiometer type
Thermal converter: contact-couple, insulated-couple		
Crystal-diode transducer		
Mechanical-rectifier transducer		
Electronic transducer		
Vibration-type pulse transducer		

pieces and the core are carefully machined and precisely aligned so that a practically uniform, radial magnetic field is set up in the air gap over nearly all of the angle that the pieces enclose the core.

When the permanent magnet (at 1 in Fig. 12.11) is internal with respect to the moving coil, the core is the magnet itself, and it is enclosed in a cylindrical yoke 3 made of a magnetically soft material. With this magnet design, the instrument can be made very small in size, but the magnetic field in the air gap is anything but uniform because the magnetic lines of force differ in length. To make the magnetic field more uniform, the internal magnet is fitted with magnetically soft straps 2.

In a moving-coil instrument with a uniform magnetic field the angle between the magnetic induction vector **B** in the air gap and the moving-coil turns carrying a current I is 90°. Therefore, each of the turns is acted upon by an electromagnetic force given by

$$F = BIl$$

where l is the length of an active turn.

The coil has w turns and is wound on a former whose diameter is d. Therefore the torque acting on the moving element is

$$T_{defl} = 2Fwd/2 = wldBI = wABI \tag{12.2}$$

where $A = ld$ is the cross-sectional area of the coil.

Thus, the deflecting torque in a moving-coil instrument is directly proportional to magnetic induction and current.

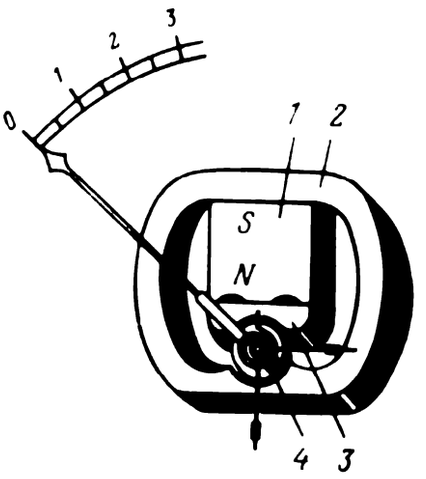


FIGURE 12.10

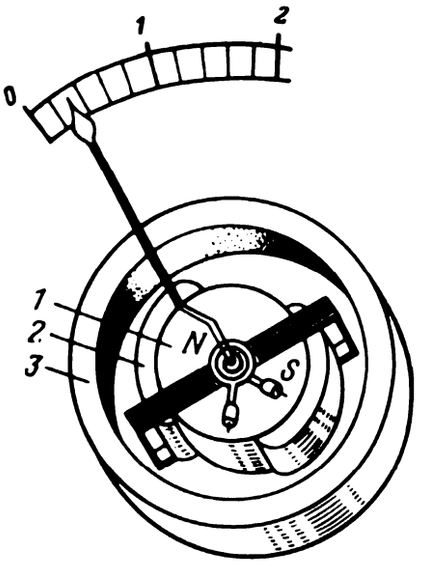


FIGURE 12.11

In the narrow air gap of the magnetic circuit the magnetic induction is usually 0.2-0.3 T and is nearly uniform. In consequence, the deflecting torque in a moving-coil instrument may be taken as being directly proportional to the current

$$T_{defl} = k_{defl}I$$

Since the restoring torque produced by the opposing springs is directly proportional to the twisting angle

$$T_{rest} = k_{rest}\alpha$$

it follows that the angle through which the coil turns is directly proportional to the current being measured

$$I = k_{rest}\alpha/k_{defl} = C\alpha$$

where C is the constant (sensitivity) of the instrument.

The direction of the deflecting torque in a moving-coil instrument can be determined by the left-hand rule. By applying this rule, it can be readily ascertained that the moving coil tends to take up a position in which its field is in the same direction as that due to the permanent magnet. The aluminium former on which the coil is wound doubles as an eddy-current damper.

When the current in the coil is reversed, the deflecting torque is also reversed. If the coil were energized with an alternating current, the moving element would be acted upon by rapidly reversing (alternating) deflecting torques. On combining, these torques would cancel one another, and the moving element would remain undeflected, and so would the pointer.

The permanent magnet produces a strong magnetic field (0.2-0.3 T) in the air gap, so even at low values of the unknown current an ample deflecting torque is developed. For this reason moving-coil instruments are fairly sensitive. Notably, most galvanometers are made of the moving-coil type.

The sensitivity of the instrument enables the designer to give it a substantial margin of strength by keeping the current density in the current-carrying parts to a reasonable minimum. Therefore, when properly designed, a moving-coil instrument is capable of standing up to appreciable overloads. This capability is further enhanced by the fact that the deflecting torque is a linear function of the unknown current, and not a square one typical of most of the other types of instruments.

Since a moving-coil instrument has a strong magnetic field of its own, external (or stray) magnetic fields have but a negligible effect on its readings. The power drawn by the instrument itself is low owing to the high sensitivity.

Moving-magnet instruments. In this type of instrument, the deflecting torque acts on a movable permanent magnet, and the coil carrying the unknown current remains stationary. Instead of opposing springs, the control force is supplied by an additional permanent magnet located on the instrument case next to the movable permanent magnet. Moving-magnet instruments are simple in design, stand up well to mechanical overloads, and are inexpensive. Unfortunately, they are low in accuracy because the magnetic field is nonuniform, and their use is largely restricted to that of indicator, especially to show battery charge and discharge current, as for vehicle use.

Rectifier-type instruments. A rectifier instrument is essentially a combination of a moving-coil instrument and a crystal diode (or diodes) connected in a half-wave (Fig. 12.12 a) or a full-wave rectifier circuit (Fig. 12.12 b).

In the case of half-wave rectification, the alternating current i is allowed to pass through the instrument only during the positive or negative half-cycles. As a rule, the moving element, ME, and the series-connected diode D_1 (see Fig. 12.12 a) are shunted by a second diode, D_2, connected back-to-back (in the reverse direction) with D_1. This is done in order to isolate the instrument from the circuit being measured, and also to prevent the full voltage from being impressed on D_1 as this might cause its breakdown.

In the case of full-wave rectification (Fig. 12.12 b), the average (rectified) current i flowing through the moving element is twice as large as it is in the previous case, so the sensitivity of the instrument is enhanced. Among the disadvantages of full-wave (or bridge) rectifier instruments is the fact that the unknown voltage is divided between two diodes, so the rectification is substantially impaired at low values of voltage.

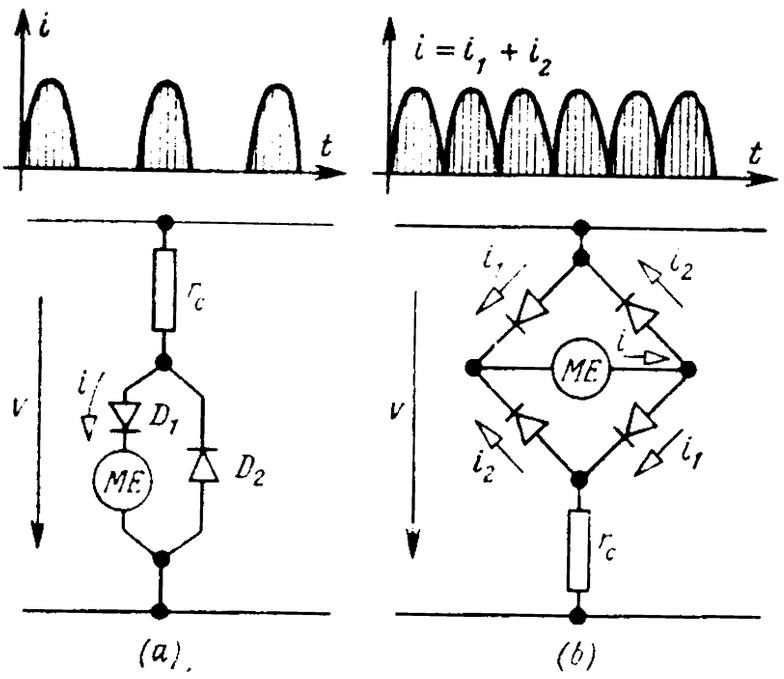


FIGURE 12.12

The current flowing through the moving element of a rectifier-type instrument is a pulsating one. Since the moving element has a noticeable inertia, its deflection is proportional to the deflecting torque averaged over a period

$$T_{\text{defl}} = \frac{1}{T}\int_0^T wABi\,\mathrm{d}t = wAB\,\frac{1}{T}\int_0^T i\,\mathrm{d}t = wABI_{\text{av}}$$

Thus, a rectifier-type instrument measures an average current in a.c. circuits, but their scales are calibrated to read the rms values of current. In the case of sinusoidal currents, the rms value is 1.11 times the average value, Eq. (2.16a) and Eq. (2.18a). When the unknown current is other than sinusoidal, there occurs a complementary error known as the waveform error.

The temperature coefficient of forward and reverse resistance of crystal diodes is negative and relatively large, so the need arises to counteract the effect of temperature variations. To this end, the measuring circuit of a rectifier-type instrument contains a compen-

sating resistor r_c which has a positive temperature coefficient of resistance (see Fig. 12.12 a and b).

Rectifier-type instruments use a uniform scale except for the initial crowded part (accounting for 10-15% of the total scale length) which has a square-law shape. This is done to make up for the impairment in rectifying action at low values of voltage. Their accuracy is relatively low (being not better than 1.5% f.s.d.) owing to a number of factors, such as nonlinearity and instability of their current-voltage characteristic of the diode, its dependence on frequency, and so on.

Rectifier-type instruments are often used as multirange voltammeters. Fitted with switchable shunts and multipliers, a rectifier-type instrument may have as many as 14 ranges on a.c. and

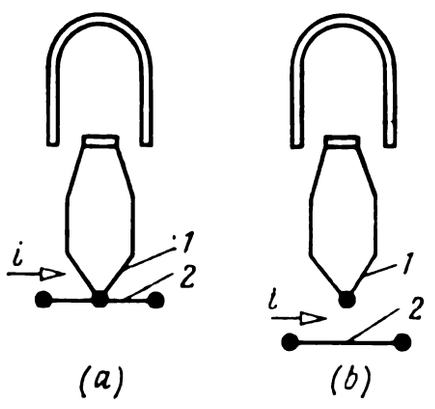


FIGURE 12.13

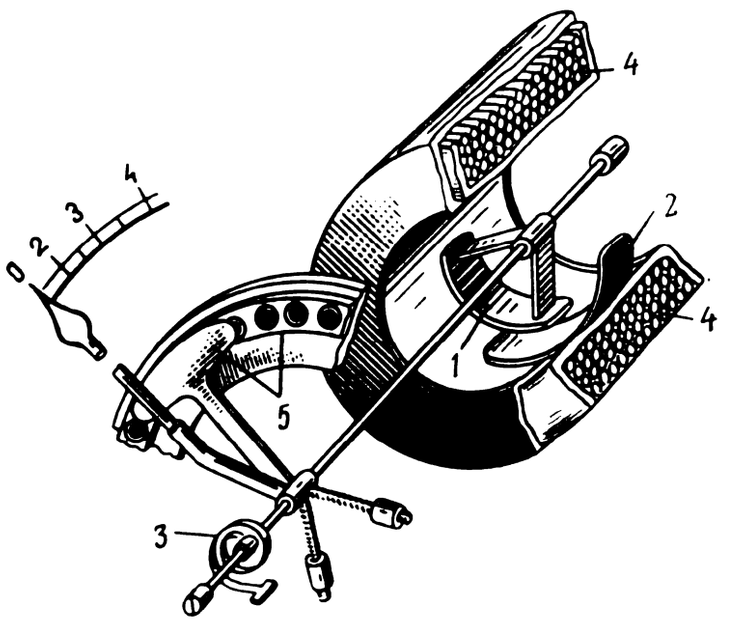


FIGURE 12.14

(with the diodes brought out of the circuit) 14 ranges on d.c. The crystal diodes are small enough to be built into a miniature instrument case.

Thermo-emf type instruments. An instrument in this category is a combination of a moving-coil movement and one (Fig. 12.13) or several thermal elements (or converters). A thermal element (or converter) usually consists of one or several thermocouples 1 placed by or on a heater wire 2 carrying the current being measured. The thermo-emf may be assumed to be proportional to the difference in temperature between the hot and cold junctions of a thermocouple. Owing to the thermal lag associated with thermocouples, the temperature difference is proportional to the square of the rms value I of a.c. current. Therefore, the deflecting torque acting on the moving

element is proportional to I^2, so the instrument is fitted with a square-law scale.

Thermal elements (or converters) are classed into the contact-couple type (Fig. 12.13 a) and the insulated-couple type (Fig. 12.13 b). In the former case the hot junction of the thermocouple is welded to or held against the heater wire. In the latter case the hot junction is held insulated from the hot wire by an insulating material. With the latter arrangement, a larger thermo-emf can be generated by connecting several thermocouples in series, but the thermal lag of the instrument is inevitably increased. Because of the low efficiency of thermal elements, thermo-emf instruments are of low sensitivity and have to operate with the heater raised to a high temperature. Because of this they draw a relatively large amount of power and are liable to break down when the overload is in excess of 50%, as the temperature of the heater wire is proportional to the square of current. Thermo-emf instruments are especially efficient in a.c. measurements at radio frequencies.

Moving-iron instruments. In this type of instrument the deflecting torque is produced by the interaction of the magnetic field set up by the current being measured and flowing in a fixed coil with a movable ferromagnetic armature (a moving iron). The mechanical forces existing in such an arrangement tend to move the armature so that the energy of the magnetic field is a maximum.

At present, two basic types of moving-iron instruments are in use. They are the repulsion type and the attraction type.

The arrangement of a *repulsion-type moving-iron instrument* is shown in Fig. 12.14. It has a round coil 4 which encloses a pivoted iron vane 1 (mounted on the shaft of the moving element) and a fixed iron 2, both fabricated from a magnetically soft material. The current flowing in the coil magnetizes the pivoted and the fixed irons, and their like poles repel each other, thereby producing a torque which causes the pivoted vane to deflect. The restoring (or controlling) torque is produced by a spiral spring 3. Damping is provided by a multipole induction-type damper 5.

The arrangement of an *attraction-type moving-iron instrument* is shown in Fig. 12.15. Here, a single pivoted iron 1 is attracted into the coil 2 carrying the current being measured. There is also a fixed iron 3 which is intended to augment the magnetic field and to adjust the deflecting torque. The restoring (controlling) torque is produced by a spiral spring, and damping is effected by a vane-type air dashpot. The movement is enclosed on all sides in a ferromagnetic shield 4 (in Fig. 12.15 it is shown with its top cover removed) in order to protect it against the action of external (stray) magnetic fields.

In the general case the electromagnetic force developed by the movement of a moving-iron instrument can be expressed in terms of the derivative of the magnetic-field energy with respect to the

travel of the moving element, Eq. (6.16):

$$F = \partial W_M/\partial x$$

If we assume that the moving element turns through an angle $d\alpha$ and that the arm of the applied force is R, we get

$$dx = R\, d\alpha$$

Therefore, the deflecting torque is

$$T_{defl} = FR = \partial W_M/\partial\alpha$$

The energy stored by the magnetic field of the coil in a moving-iron instrument is $LI^2/2$. As the pivoted vane turns, the inductance L is changed, and this brings about a change in the energy stored by the magnetic field of the coil. Therefore,

$$T_{defl} = \partial W_M/\partial\alpha = (I^2/2)\, \partial L/\partial\alpha$$

As is seen, the deflecting torque is proportional to the square of the current and the rate of change of the movement's inductance as the moving element rotates. The relation $dL/d\alpha = f(\alpha)$ can be given

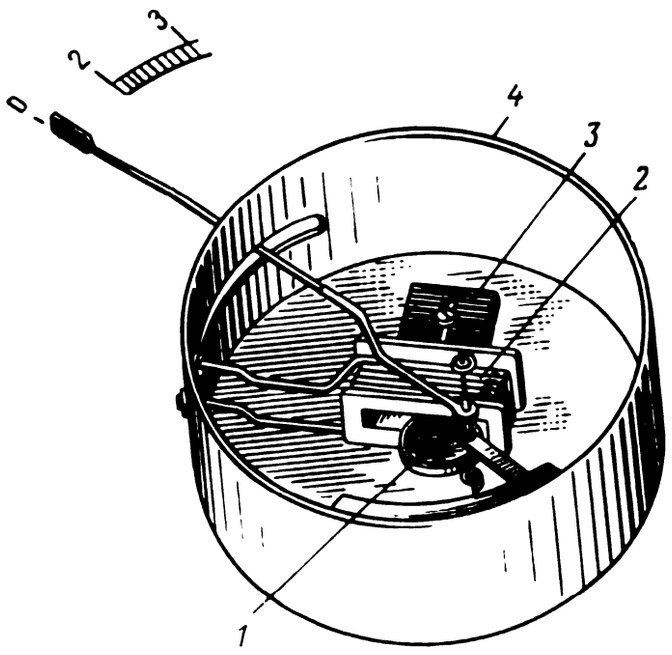


FIGURE 12.15

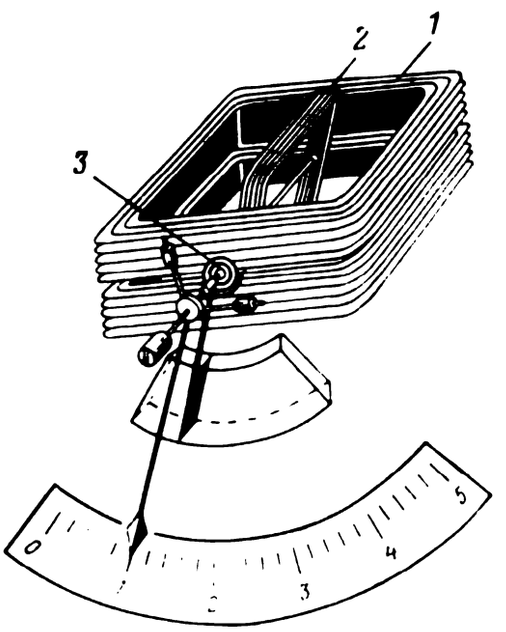


FIGURE 12.16

the desired form by suitably shaping the pivoted and fixed irons, their position in the coil, etc. The scale of a moving-iron instrument will be uniform, if $T_{defl} = IC$, where C is a constant, that is, if the deflecting torque is directly proportional to the unknown current and, in consequence, $I\partial L/\partial\alpha = \text{const}$. This condition cannot be satisfied within the initial portion of the scale, because, as follows from it, with I tending to zero, the ratio $\partial L/\partial\alpha$ must tend to infinity,

which is unfeasible. Yet, the scale is arranged to be linear except for the initial 10-20%, which is always contracted.

A reversal of the current in the coil does not cause a reversal of the deflecting torque in a moving-iron instrument. It follows, therefore, that moving-iron instruments may be used on both d.c. and a.c. On a.c., however, instrument readings may be affected by hysteresis and eddy-current losses. In present-day moving-iron instruments which use irons fabricated from special-purpose ferromagnetic alloys the difference in readings on d.c. and a.c. is insignificant. Therefore, one and the same scale may be used to read both d.c. and a.c.

The magnetic field in a moving-iron instrument is excited by the unknown current itself and is relatively small, because the larger part of the magnetic path is in air. For this reason moving-iron instruments are of low sensitivity and it is difficult to build a moving-iron ammeter for low values of current (less than, say, 0.5 A) or a moving-iron voltmeter for low values of voltage (say, less than 10 V). Since the instrument's own magnetic field is weak, it has to be protected against stray magnetic fields. This can be done either as shown in Fig. 12.15 with the aid of a ferromagnetic shield, or by making the instrument astatic.

In general terms, the *astatic feature* consists in the following. Two coils, instead of one, are used, and both contribute to the deflecting torque, but their magnetic fields are in opposite senses. Any stray uniform magnetic field, while boosting the magnetic field of one coil, bucks that of the other by the same amount. As a result, stray magnetic fields do not change the overall deflecting torque developed by the moving element.

Losses in moving-iron instruments are relatively large, being 0.3-1.0 W in a moving-iron ammeter and 3-6 W in moving-iron voltmeters where an additional loss occurs in the series (multiplier) resistor.

The accuracy of moving-iron instruments is usually no better than 1.5, mainly due to the effect of hysteresis (remanent magnetization), especially noticeable in work on d.c. This error can however be minimized by making the pivoted and fixed irons from a ferromagnetic material having a negligible coercive force, such as Permalloy. The accuracy of portable moving-iron instruments using Permalloy irons is 0.5 or even 0.2 on both d.c. and a.c. (provided the frequency does not exceed 1500 Hz).

Moving-iron instruments offer a number of advantages. For one thing, they use stationary coils which can readily be fabricated with a sufficient margin of safety to cater for likely overloads. Therefore overloads may be appreciable. Also, these instruments are inexpensive and simple in design.

Moving-iron instruments are mainly used to measure alternating

voltages and currents at moderate frequencies. In industrial low-frequency a.c. circuits, the ammeters and voltmeters are mostly of the moving-iron type.

Electrodynamic instruments. For their operation electrodynamic instruments depend on the interaction of current-carrying conductors: Two parallel conductors will attract each other if the currents they carry are in the same direction, or they will repel each other if the currents they carry are in the opposite directions.

The movement of an electrodynamic instrument consists essentially of a fixed coil 1 and a moving coil 2 (Fig. 12.16). The restoring (controlling) torque is produced by spiral springs 3 which also serve to conduct current to the moving coil. Acted upon by electromagnetic forces, the moving coil tends to take up a position in which its magnetic field is in the same direction as the magnetic field of the fixed coil.

As in the case of a moving-iron instrument, the deflecting torque developed by an electrodynamic instrument is given by

$$T_{\text{defl}} = \mathrm{d}W_{\text{M}}/\mathrm{d}\alpha$$

In an electrodynamic instrument the energy stored by the magnetic field changes in response to the rotation of the moving coil, that is, as a result of a change in the mutual inductance M between the coils. In accord with Eq. (2.80), the energy stored in a magnetic field common to two coils is given by

$$W_{\text{M}} = MI_1I_2$$

where I_1 is the current in the moving coil and I_2 is the current in the fixed coil. Therefore the deflecting torque is

$$T_{\text{defl}} = I_1I_2\partial M/\partial\alpha \tag{12.3}$$

The dependence of M on the position of the moving coil can be varied at will by suitably shaping the coils, adjusting their relative position, and so on.

Since an electrodynamic instrument has two coils, its field of application may be expanded considerably. Depending on the connection of the moving and fixed coils, either a wattmeter, voltmeter or ammeter results. The shape of the scale is likewise varied to suit the application.

To make an electrodynamic voltmeter, the two coils having a large number of turns are usually connected in series with each other (Fig. 12.17 a) and with a multiplier resistor r_{m}. Thus, in an electrodynamic voltmeter

$$I_1 = I_2 = V/r_{\text{V}}$$

where V is the voltage being measured, and $r_V = r_c + r_m$ is the total resistance of the measuring circuit equal to the sum of the resistances of the two coils, r_c, and the multiplier resistor, r_m.

On substituting for I_1 and I_2 in Eq. (12.3), we get

$$T_{\text{defl,V}} = (V^2/r_V^2)\, \partial M / \partial \alpha \qquad (12.4)$$

It follows then that the square-law scale of an electrodynamic voltmeter may be changed into a linear one by adjusting the relation $M = f(\alpha)$. The initial portion of the instrument scale is inevitably

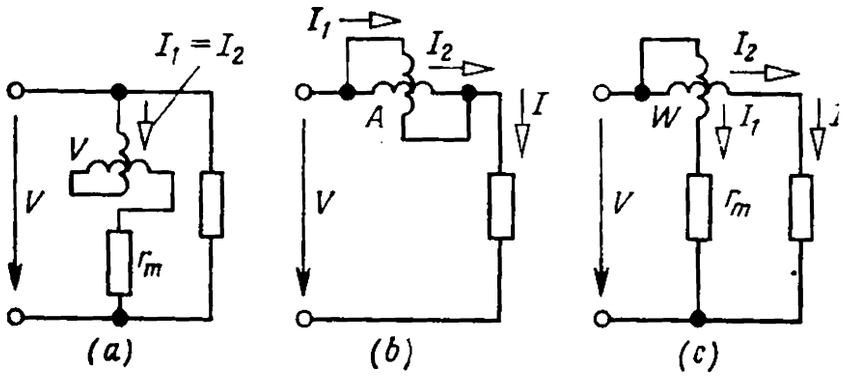


FIGURE 12.17

crowded for the same reasons as apply to the scale of a moving-iron instrument.

In electrodynamic ammeters for currents up to 0.5 A the fixed and moving coils are likewise connected in series. For higher values of the unknown current I, the moving and fixed coils are connected in parallel (Fig. 12.17 b). Now the moving coil takes only a fraction of the current taken by the fixed coil, such that $I_1 = k_1 I$ and $I_2 = k_2 I$. Therefore

$$T_{\text{defl,A}} = kI^2\, \partial M/\partial \alpha$$

That is, the deflecting torque developed by an electrodynamic ammeter is a quadratic function of current, whereas in an electrodynamic voltmeter it is a quadratic function of voltage.

In an electrodynamic wattmeter (Fig. 12.17 c), the current in the fixed coil is equal to the current in the circuit being measured, that is, $I_2 = I$. The voltage across the measured circuit is impressed on the moving coil, so that $I_1 = V/r_V$, where r_V is the overall resistance of the potential (shunt) circuit of the wattmeter.

On substituting for the coil currents in Eq. (12.3), we get

$$T_{\text{defl,W}} = (1/r_V)\, P\, \partial M/\partial \alpha \qquad (12.5)$$

where $P = VI$ is the power in the circuit being measured.

Thus (in contrast to an electrodynamic voltmeter or ammeter) the deflecting torque of an electrodynamic wattmeter is proportional

to the power being measured. For the scale of an electrodynamic wattmeter to be linear, it suffices to satisfy the condition

$$M = k_M \alpha$$

where k_M is a constant, so

$$\mathrm{d}M/\mathrm{d}\alpha = \text{const}$$

This condition is relatively easy to satisfy through a proper design, so most electrodynamic wattmeters have a linear scale.

Equation (12.5) has been developed for d.c. In work on a.c. the deflection of the moving element is determined owing to its sluggishness by the deflecting torque averaged over a period

$$T_{\text{defl. av}} = \frac{1}{T} \int_0^T (1/r_V)\, p\, (\partial M/\partial \alpha)\, \mathrm{d}t = (1/r_V)\, (\partial M/\partial \alpha)\, (1/T) \int_0^T p\, \mathrm{d}t$$

where p is the instantaneous power, and

$$\frac{1}{T} \int_0^T p\, \mathrm{d}t = P$$

is the active power in an a.c. circuit. Thus, in work on a.c., the average deflecting torque of an electrodynamic wattmeter is

$$T_{\text{defl, av}} = (1/r_V)\, P\, \partial M/\partial \alpha \qquad (12.6)$$

which is similar to expression (12.5) for the deflecting torque developed by an electrodynamic wattmeter on d.c.

Thus, electrodynamic instruments may be used on both d.c. and a.c. with the same scale.

In electrodynamic instruments the unknown current sets up a magnetic field in air, and it is relatively weak. To boost the magnetic field, one has to use coils with a large number of turns, and this leads to an increase in the power drawn by the instrument itself. Because the own magnetic field is weak, electrodynamic instruments are sensitive to stray magnetic fields, and the latter have to be kept out by means of shields. Since the moving element always carries a current and cooling is anything but adequate (heat is transferred through air), electrodynamic instruments do not permit any appreciable overloads (especially ammeters). Last but not least, electrodynamic instruments are expensive. Since, however, there are no ferromagnetic parts inside the magnetic field, electrodynamic instruments may have an accuracy of as high as 0.2 or even 0.1.

The advantages and disadvantages of electrodynamic instruments make them suitable for laboratory and test work on a.c., mainly for power measurements. In most cases portable wattmeters are of the electrodynamic type.

The disadvantages of electrodynamic instruments, such as sensitivity towards stray magnetic fields, low deflecting torque and low stability towards overloads may be corrected by using ferromagnetic cores for the coils. In the simplest case, the moving coil may have a core similar to that used in moving-coil instruments (see Fig. 12.10). Unfortunately, iron cores substantially impair the accuracy of measurement due to the effect of hysteresis and eddy currents. This is the reason why iron-cored electrodynamic instruments are a poor choice for precision measurements. They are mainly used as switchboard-mounted wattmeters and recorders.

Induction-type instruments. For their operation induction instruments utilize a rotating magnetic field. A rotating magnetic field can be produced by using a pair of coils so positioned relative to each other that a phase difference exists between their currents. If we put a body made from a low-reluctance material on the axis of rotation of the magnetic field, eddy currents will be induced in that body. The interaction of the eddy currents with the rotating magnetic field produces a torque causing the body to rotate.

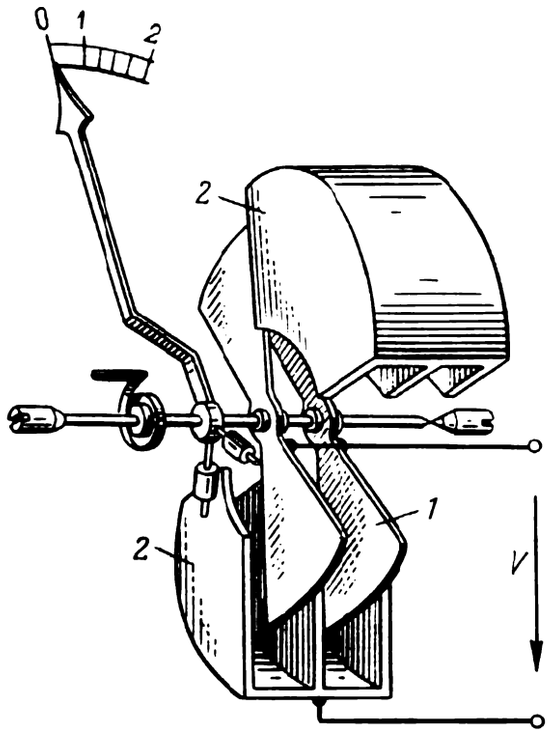


FIGURE 12.18

In an induction instrument, the deflecting torque is produced by the action of the resultant magnetic field of two a.c. solenoids on the moving element, an aluminium disc, in which the field induces eddy currents. The solenoids are energized by the alternating current being measured, so the deflecting torque depends on the value of current in both coils and the phase angle between them. This valuable property of induction instruments is utilized in making instruments which measure a.c. power and energy. In more detail the theory of induction instruments is examined in Sec. 12.9.

Electrostatic instruments. In this type of instrument, the deflecting torque is produced by the forces of electrostatic interaction between two charged electrodes.

The most common form of electrostatic instruments is the voltmeter. It comprises a variable capacitor with one vane moving, 1, and one fixed, 2 (Fig. 12.18). This capacitor is charged by the voltage being measured. The resultant force of electrostatic attraction produces a deflecting torque, and the necessary restoring (or controlling) torque is supplied by a spring.

A reversal of voltage polarity does not cause the deflecting torque to reverse, so electrostatic voltmeters can measure both d.c. and a.c. voltages. In some cases it is important that an electrostatic instrument take practically no power from the circuit being measured. Unfortunately, at low voltages the deflecting torque is too small, and one has to use more than two capacitor vanes and suspend them by a ligament instead of pivoting them on jewels.

Electrostatic voltmeters are mainly used in the laboratory to measure voltages in low-power circuits and also for direct measurements of high voltages.

Hot-wire (thermal) instruments. In this type of instrument, the moving element is caused to deflect by the thermal extension of a conductor carrying the current being measured. Among the advantages of hot-wire instruments is the independence of their readings from the frequency and waveform of the unknown current. However, this is also true of thermo-emf instruments. Hot-wire instruments are of low overload capacity and of low accuracy (due to temperature instability), and take a sizeable amount of power from the circuit being measured, so the result is that this type is now largely out of use, being employed mostly at radio frequencies.

Other types of instruments serve to measure very few quantities (for example, a vibrating-reed instrument is intended to measure frequency only), and they will be described in part later on.

The choice of a particular instrument type depends on the metrological, operating and service requirements to be met. Where service conditions are such that one needs rugged and inexpensive instruments to measure a.c. voltages and currents, the natural choice would be moving-iron instruments. Where accuracy is the primary consideration, it should be remembered that an accuracy of 0.1 and 0.2 on d.c. can be obtained only with moving-coil instruments, and an accuracy of 0.2 on a.c. only with electrodynamic instruments. Permalloy-cored moving-iron instruments may have an accuracy of as high as 0.5 (or 0.2 when mounted on taut bands). Iron-cored electrodynamic instruments can have an accuracy of 1.0 at most, and rectifier instruments a maximum of 1.5. Power should best be measured with electrodynamic and iron-cored electrodynamic instruments.

At frequencies up to 1000 Hz, use may be made of electrodynamic, Permalloy-cored moving-iron and iron-cored electrodynamic instruments. At frequencies up to 10-20 kHz preference should be given to rectifier-type instruments. At still higher frequencies (into the megahertz region), the choice must be thermo-emf instruments.

The range of an instrument can be extended through the use of switchable resistors. A resistor connected in series with the coil of the moving element is called a *multiplier*, and a resistor connected in parallel with the coil (or the arm containing the coil and a multiplier) is called a *shunt*.

12.8 Ratiometers

In most electrical instruments, the deflection of the moving element is a function of the current(s) in the coil(s). In cases where the unknown quantity (resistance, inductance, capacitance, phase angle, frequency, and so on) is not a direct function of current, the resultant deflecting torque must be made dependent on the quantity being measured and independent of the supply voltage.

This is achieved in systems in which the deflection of the moving element is determined solely by the ratio of the currents in its two coils and is independent of their values. Such instruments are quite aptly called ratiometers. The ratiometer principle can in fact be

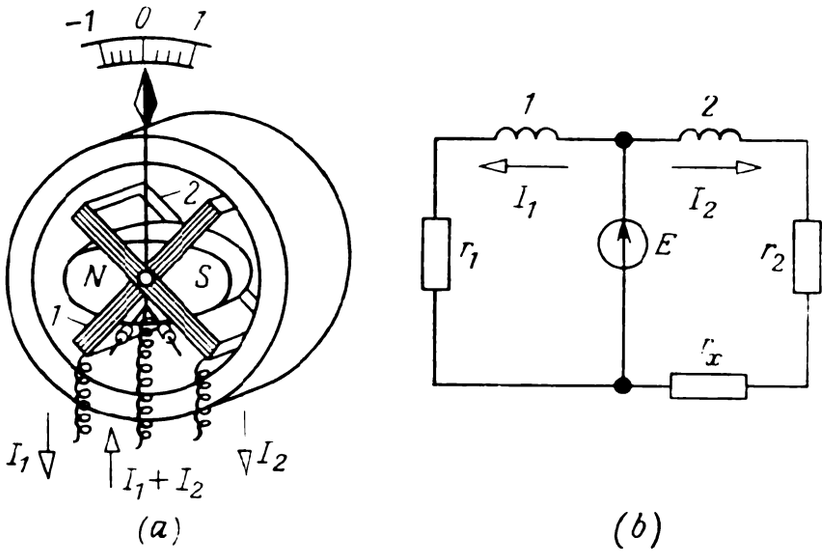


FIGURE 12.19

applied to any type of instrument we have discussed. A special feature about ratiometers is the absence of any restoring (controlling) torque supplied by springs or taut bands.

As a simple example, we will examine the operation of a moving-coil ratiometer using a core with an ellipsoid cross-section. In contrast to the regular moving-coil instrument with a restoring torque, the moving element of a ratiometer consists of two coils (at 1 and 2 in Fig. 12.19 a) mounted on a common shaft and made fast to each other at an angle. Current to the coils is conducted over three soft silver spirals which produce no torque when twisted (one of the spirals is common to both coils). If the moving element is well balanced, it will take up a position of indifferent equilibrium when not energized, so that the pointer can be set at any scale division.

When the coils are carrying currents, I_1 and I_2, the moving element is acted upon by two torques in opposite directions. These torques are produced by the interaction of a magnetic field with flux

densities B_1 and B_2, with the coil currents in accord with Eq. (12.2)

$$T_{\text{defl.1}} = w_1 A B_1 I_1$$

and

$$T_{\text{defl.2}} = w_2 A B_2 I_2$$

where w_1 and w_2 are the numbers of turns in the respective coils, and A is the cross-sectional area of each coil, which is the same for both coils.

Suppose that $T_{\text{defl.1}} > T_{\text{defl.2}}$. The greater torque causes the moving element to turn, but as it does so, the first coil acted upon by the greater torque moves into the weaker magnetic field (that is, one with a lower flux density, B_1, owing to the greater air gap in the magnetic circuit). At the same time the second coil acted upon by the smaller torque moves into the stronger magnetic field (the air gap is smaller and the flux density B_2 is higher). In the process, the greater torque decreases and the smaller torque increases until the two torques balance each other in some definite position of the moving element so that $T_{\text{defl.1}} = T_{\text{defl.2}}$.

In the diagram of Fig. 12.19 b, a ratiometer is shown connected to measure resistances. Here E stands for an emf source, r_1 is the constant resistance of the first coil, r_2 is the constant term of the resistance of the second coil, and r_x is the unknown resistance (the resistance to be measured), located outside the instrument. The currents in the coils are given by

$$I_1 = E/r_1$$

and

$$I_2 = E/(r_2 + r_x)$$

When the two torques are the same (the moving element has come up to a steady-state position)

$$w_1 B_1/r_1 = w_2 B_2/(r_2 + r_x)$$

Hence

$$r_x = r_1 (w_2/w_1) (B_2/B_1) - r_2$$

The ratio B_2/B_1 depends on the form of the instrument's magnetic circuit and depends on the position of the moving element, that is, the angle α. Thus, for every value of r_x the moving element of the ratiometer takes up a certain definite position independent of the supply emf E.

In reality, the moving element is never perfectly balanced and the current-conducting spirals are inevitably twisted—both factors serve to produce some mechanical torque.

Let us assume that the torques developed by the coils are proportional to voltage. We also assume that one of the torques is a function of the unknown quantity x and that both depend on the angle of deflection α of the moving element, but do so differently so that $f_1(\alpha)$ and $f_2(\alpha)$. Then

$$T_{\text{defl},1} = k_1 V f_1(\alpha)$$

$$T_{\text{defl},2} = k_2 V f_2(\alpha) f(x)$$

Hence, the equation of torque balance, including the mechanical torque, is

$$T_{\text{defl},1} \pm T_{\text{mech}} = T_{\text{defl},2}$$

or

$$k_1 f_1(\alpha) \pm T_{\text{mech}}/V = k_2 f_2(\alpha) f(x) \tag{12.7}$$

In other words, the mechanical torque makes the indications of a ratiometer dependent on the voltage V so that this dependence grows stronger as the voltage decreases. It is usual to mark the dial of a ratiometer with the lower and upper voltage limits. The lower limit is specified in view of the likely effect of mechanical factors, and the upper limit is set on the basis of the likely temperature rise of the measuring circuits.

In some cases it is desired that the instrument should not take up a position of indifferent equilibrium as it might erroneously be taken for an indication although the instrument is de-energized. For example, on an airplane a ratiometer may serve as the indicator of a fuel quantity gauge. If the instrument were allowed to take up a position of indifferent equilibrium when de-energized, the pilot might erroneously take its indication for the quantity of fuel still available. This is avoided by causing the pointer to swing off scale when the instrument is de-energized. The simplest way to do this is to use an incompletely balanced moving element capable of producing a mechanical torque.

12.9 Electricity Meters

The energy delivered by an electric station to, or taken by loads from, a power system is measured and registered by electricity meters.

As will be recalled, the moving element of an indicating instrument turns through an angle in response to a change in the quantity being measured. In an electricity meter the moving element, or the rotor, is rotating continuously. In most electricity meters the torque developed by the rotor is proportional to the power in the external circuit and is opposed by a retarding (or braking) torque (which does the job of the controlling torque of an indicating instrument). The

retarding torque is usually produced by an eddy-current brake in the form of an aluminium disc mounted on the spindle (or shaft) of the rotor and allowed to rotate in the field of a permanent magnet (or magnets) (Fig. 12.20). This retarding torque can be determined as follows. The emf induced in the disc is proportional to the flux Φ set up by the permanent magnet and the average peripheral speed v of the disc's portion located between the magnet poles

$$E = k_1 \Phi v$$
$$= k_1 \Phi 2\pi Rn/60 = k_2 \Phi n$$

where n is the rpm of the disc and R is the mean radius of the disc's part located between the magnet poles. The angular speed of the disc is $\omega = v/R$.

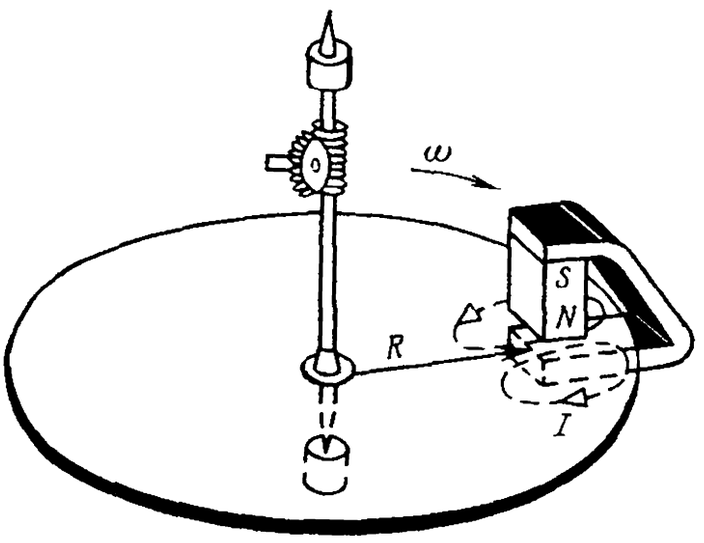


FIGURE 12.20

The induced emf gives rise to a current in the disc. This current is proportional to the volumetric conductivity of the disc material γ_v

$$I = k_3 E \gamma_v$$

The interaction of the field set up by the permanent magnet with the current induced in the disc produces the retarding torque directly proportional to the current and flux

$$T_{retard} = k_4 I \Phi$$

On substituting for the current and emf, we get

$$T_{retard} = k_5 R \gamma_v \Phi^2 n = k_{retard} n \qquad (12.8)$$

Thus, the retarding torque is directly proportional to the rpm n of the disc, which is what is required for the operation of an electricity meter.

The deflecting torque developed by the rotor of an electricity meter must be proportional to the power in the circuit

$$T_{defl} = k_{defl} P$$

At a steady rpm, this deflecting torque must be equal to the retarding torque. Therefore,

$$P = k_{retard} n / k_{defl}$$

The time integral of power is

$$\int_{t_1}^{t_2} P\,dt = \frac{k_{retard}}{k_{defl}} \int_{t_1}^{t_2} n\,dt = \frac{k_{retard}}{k_{defl}} N$$

where N is the total number of revolutions completed by the rotor during the time interval $t_2 - t_1$. The quantity

$$\int_{t_1}^{t_2} P\,dt = W$$

is the electric energy expended in the measured circuit during the same time interval $t_2 - t_1$. Thus

$$W = k_{retard} N / k_{defl} = C_m N$$

where C_m is the constant of the electricity meter.

The number of revolutions, N, completed by the rotor is totalized by a register coupled to the meter shaft by a worm gearing. The gear ratio between the shaft and the register is chosen such that the register reads directly kilowatthours rather than the number of revolutions.

Present-day electricity meters may be the induction, electrodynamic, iron-cored electrodynamic, and moving-coil types. For a.c., only induction meters are used, while electrodynamic and iron-cored electrodynamic types are used only for d.c. (such as on railway transport), and moving-coil meters only serve special purposes (notably as amperehour meters for storage batteries).

The properties common to all types of meters can be examined using an *electrodynamic meter* (Fig. 12.21) as an example. Like a wattmeter, it has a current (or series) circuit and a voltage (or shunt) circuit. The current circuit is formed by two stator (or field) coils, 1, in which the current is equal to the direct current I in the external circuit. This current sets up the main magnetic field in the meter. So the main magnetic flux in the rotor is

$$\Phi = k_\Phi I$$

where k_Φ is a constant.

The rotor shaft mounts three coils, 2, which make up the armature of the meter and whose leads are connected to commutator bars 3 as shown in Fig. 12.22. The commutator bars, mounted on the shaft and insulated from one another, are held in contact with brushes 4 through which the armature current I_{arm} enters the armature coils. The main magnetic field interacts with the current in the armature coils and causes them to rotate about their axis. Owing to provision

of a commutator and brushes, the rotor always rotates in the same direction, because the direction of the current in the coil is reversed when the coil approaches the position in which its magnetic flux is in the same direction as the main flux.

The armature circuit contains a high-value series resistor, r_s (at 6 in Fig. 12.21), and since the armature circuit is energized with the voltage V of the external circuit, the armature current is given by

$$I_{arm} = (V - E_{arm})/(r_{arm} + r_s)$$

where E_{arm} is the emf induced in the armature coils as they cut the main magnetic field, and r_{arm} is the armature resistance. Electricity

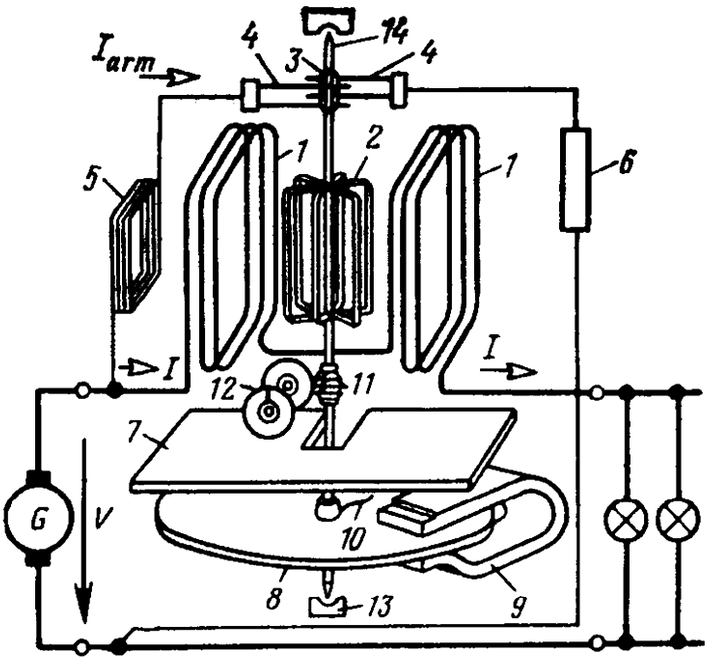


FIGURE 12.21

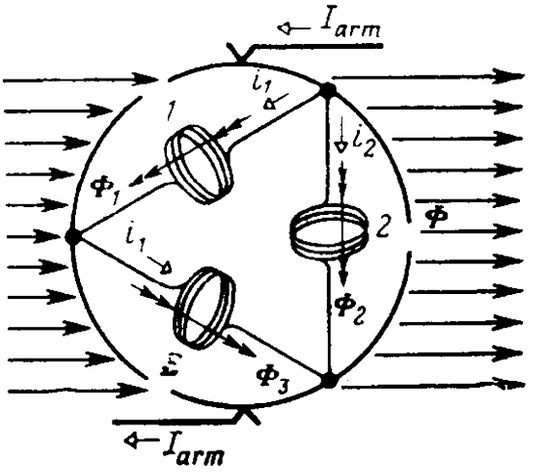


FIGURE 12.22

meters are designed so that $E_{arm} < V$ and it may be neglected, so that

$$I_{arm} = k_V V$$

Since the meter is of the electrodynamic type, its deflecting torque, Eq. (12.3), is proportional to the product of the currents in the moving and fixed coils, $I \times I_{arm} \partial M/\partial\alpha$ or

$$T_{defl} = k_{defl} VI = k_{defl} P$$

where k_{defl} is a constant coefficient of proportionality. It is to be noted that as the armature rotates, the deflecting torque pulsates because of the relative displacement of the coils; in the above equation we have written its value averaged over a complete revolution of the rotor.

As already noted, the retarding torque is produced by an eddy-current brake in the form of an aluminium disc 8 mounted on the rotor shaft and rotating in the field set up by a permanent retarding magnet 9 (see Fig. 12.21). Since the coils are screened from the permanent-magnet field by a shield 7, the retarding torque, Eq. (12.8), is directly proportional to the rpm n of the armature.

As is seen, the above design satisfies the two conditions that assure a direct proportionality between the speed of the rotor and the energy taken or delivered.

The sensitivity and accuracy of a meter can be markedly affected by friction in the register (11 and 12), between the brushes and commutator, and in the jewel bearings 13 and 14. The reduction in sensitivity caused by friction is especially troublesome. If friction is excessive, the meter may fail to rotate altogether, thus failing to measure and register the energy consumed. In an electrodynamic meter, friction is compensated by a small coil 5 connected in series with the armature circuit. The magnetic field of the compensating coil interacts with the armature current I_{arm} in coils 2 and produces a torque proportional to the product of the currents in the compensating and armature coils. Since it is the same current, $I_{arm} = k_V V$, the compensating torque is

$$T_{comp} = k_{comp} V^2$$

The compensating torque is independent of the load current. It must cancel out the frictional torque T_{fric}, that is, it is desired that $T_{comp} = T_{fric}$.

Thus, the overall torque equation of an electricity meter takes the form

$$T_{defl} + T_{comp} = T_{retard} + T_{fric}$$

A major drawback is that because of the presence of a constant compensating torque, T_{comp}, the meter may run away, that is, rotate continuously at no load. This may happen as a result of an increase in T_{comp} due to a rise in the voltage V, or as a result of a decrease in T_{fric} due to vibrations (dithering) in the wall on which the meter is mounted. To avoid this, the meter is fitted with a brake hook, 10, soldered to the meter shaft or disc and attracted by the retarding magnet 9. The compensating torque alone cannot overcome the force of attraction acting on the hook. When, however, the disc is rotating, the attraction of the brake hook does not affect the average deflecting torque — it slows down the disc during one half-revolution and speeds it up during the other.

The speed of the meter can be adjusted by altering the position of the retarding magnet, and its sensitivity can be adjusted by altering the position of the friction compensating coil.

In principle, electrodynamic meters may be used for a.c. as well, but a better choice for a.c. service is the simpler, stronger and less expensive induction meters.

The arrangement of a single-phase *induction meter* is shown in Fig. 12.23. An induction meter has a rotor consisting of a light aluminium disc and a driving element. The driving element consists of two electromagnets, one energized by a voltage coil, w_V, and the other by a current coil, w_I. So the disc cuts two alternating magnetic

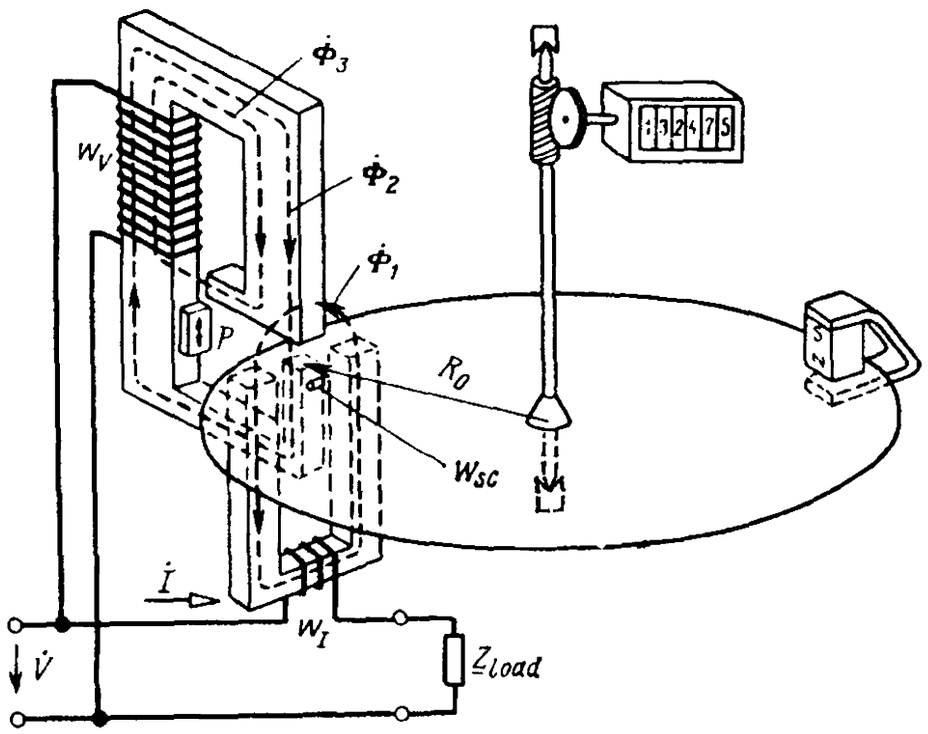


FIGURE 12.23

fluxes, and the deflecting torque is produced by the interaction of one alternating flux with the current induced in the disc by the other alternating flux. With only one alternating flux present, the disc cannot rotate continuously. The two alternating fluxes are

$$\Phi_1 = \Phi_{1m} \sin (\omega t + \psi)$$

and

$$\Phi_2 = \Phi_{2m} \sin \omega t$$

(Fig. 12.24). The two fluxes induce in the respective loops of the disc two emfs, each of which is in quadrature lagging with (90 electric degrees behind) the respective flux

$$e_1 = -d\Phi_1/dt = -\omega\Phi_{1m} \cos (\omega t + \psi)$$

$$e_2 = -d\Phi_2/dt = -\omega\Phi_{2m} \cos \omega t$$

These emfs induce eddy currents, i_1 and i_2, in the disc.

In analysing the operation of an induction meter at power frequency (50 Hz), we may neglect the inductive reactances of the eddy-current loops in the disc and assume that the eddy currents are in phase with the respective emfs

$$i_1 = e_1/r_{s1} = -(\omega/r_{s1})\,\Phi_{1m}\cos(\omega t + \psi)$$

$$i_2 = e_2/r_{s2} = -(\omega/r_{s2})\,\Phi_{2m}\cos\omega t$$

where r_{s1} and r_{s2} are the resistances of the loops. The forces acting on the moving element are proportional to the respective fluxes and currents, that is

$$F_1 = k_1\Phi_1 i_2$$

$$F_2 = k_2\Phi_2 i_1$$

where k_1 and k_2 are constant coefficients taking care of the geometry of the meter movement.

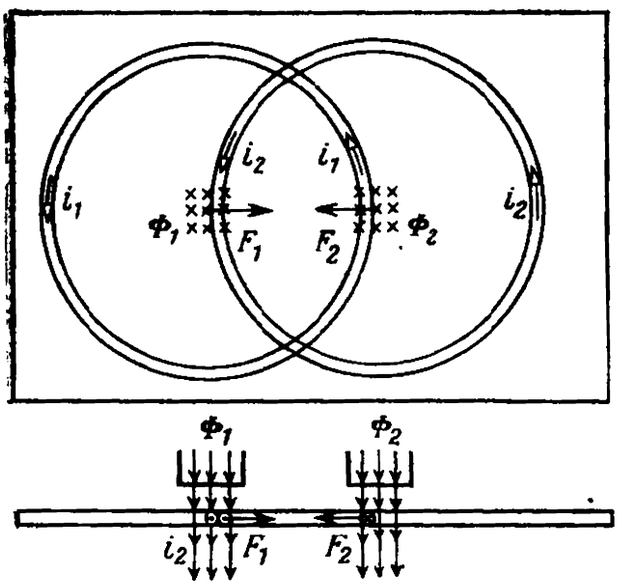


FIGURE 12.24

Since both the magnetic fluxes and the currents are respectively in the same direction (see Fig. 12.24), the above forces oppose each other. Therefore, the average deflecting torque can be found by taking the integral of the product of $(F_2 - F_1)$ by the lever arm R_0 (see Fig. 12.23) over a period T

$$T_{defl} = \frac{1}{T}\int_0^T (F_2 - F_1)\,R_0\,dt$$

$$= \frac{\omega R_0}{T}\,\Phi_{1m}\Phi_{2m}\left[-\frac{k_2}{r_{s2}}\int_0^T \sin(\omega t + \psi)\cos\omega t\,dt\right.$$

$$\left. + \frac{k_1}{r_{s1}}\int_0^T \sin\omega t\cos(\omega t + \psi)\,dt\right]$$

Noting that

$$\int_0^T \sin\omega t\cos\omega t\,dt = \int_0^T \frac{\sin 2\omega t}{2}\,dt = 0$$

$$\int_0^T \cos^2\omega t\,dt = \frac{T}{2}$$

and

$$\int_0^T \sin^2 \omega t \, dt = \frac{T}{2}$$

we obtain after simple manipulations

$$T_{defl} = (\omega R_0/2)\ (k_1/r_{s1} + k_2/r_{s2})\ \Phi_{1m}\Phi_{2m} \sin \psi$$

where r_{s1} and r_{s2} are inversely proportional to the volumetric conductivity, γ_v, of the disc material (aluminium).

On writing the constant coefficient in the expression for the deflecting torque as k_{defl}, we get

$$T_{defl} = k_{defl}\omega\gamma_v\Phi_{1m}\Phi_{2m} \sin \psi \qquad (12.9)$$

Since the deflecting torque of an induction meter is proportional to the angular frequency ω, it follows that the instrument can be used in an a.c. circuit at one particular frequency only. It is also seen that the deflecting torque is proportional to the volumetric conductivity γ_v of the disc material. This material is aluminium which has a fairly large temperature coefficient of resistance, around 0.004 $°C^{-1}$ (see Table 1.2), which means that a change of 10 deg C in the disc's temperature will change the deflecting torque by as much as 4%. Fortunately, the deflecting and retarding torques in an electricity meter are equally dependent on the disc resistance, so temperature variations have but a negligible effect on its indication.

In an induction meter, the deflecting torque, Eq. (12.9), should be proportional to the active power, $P = VI \cos \varphi$, in the circuit. In order to satisfy this condition, one of the two magnetic fluxes, say Φ_{2m}, should be proportional to the voltage V of the supply line, and the other, Φ_{1m}, to the current I in the main circuit.

The voltage electromagnet is shaped so that the greater proportion of the magnetic flux Φ_2 has its path confined within the ferromagnetic core (see Fig. 12.23). Therefore, we may, on the basis of Eq. (7.2), take it that

$$V = 4.44 f w_V \Phi_{2m} = \Phi_{2m} \times \text{const}$$

The current electromagnet usually has a U-shaped iron core, so that its magnetic flux has about half its path through air. Since the reluctance of the ferromagnetic portion is negligible in comparison with that of the air gap, we may, by virtue of Eq. (6.7), express the instantaneous value of the flux set up by the current electromagnet as

$$\Phi_1 = i w_I / R_M \qquad (12.10)$$

where R_M is the reluctance of the air gap. The quantities on the right-hand side, except the current, are constant, so
$\Phi_1 = i \times \text{const}$
and Φ_{1m} is proportional to I.

A further requirement is that the deflecting torque of an induction meter should be proportional to $\cos\varphi$. Since Eq. (12.9) contains only one function of phase angle, $\sin\psi$, it is necessary to obtain $\sin\psi = \cos\varphi$ or $\psi = \pi/2 - \varphi$.

Approximately, the complex flux $\dot{\Phi}_1$ is in phase with the complex current $\dot{I}$ (Fig. 12.25). Therefore the phase difference between the phasors $\dot{V}$ and $\dot{\Phi}_2$ must be equal to $\pi/2$ in order to yield $\psi = \pi/2 - \varphi$. As follows from the phasor diagram in Fig. 7.8 drawn up for an iron-cored coil (see Fig. 7.7 a), the phase difference between the phasor of the voltage across the coil and the phasor of the flux in the core can be made equal to $\pi/2$ by adjusting the leakage reactances while holding the other quantities unchanged. An induction meter has a provision for such an adjustment. The leakage flux linkage with the voltage coil w_V has a component $w_V\Phi_3$, where Φ_3 is the flux passing outside the disc. This component can be adjusted by altering the width of the air gap in the path of $\dot{\Phi}_3$ with the aid of an iron plate, P (see Fig. 12.23).

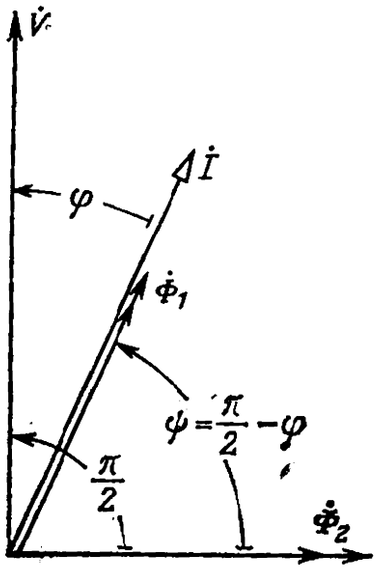


FIGURE 12.25

The friction torque in an induction meter is substantially smaller than it is in an electrodynamic meter, because the rotor in an induction meter is lighter in weight. However, there is an appreciable friction in the register in either type of meter, so induction meters need compensation for friction as well. In designs of the induction meter the friction compensating torque is produced by the same principle—by allowing a lack of symmetry in the flux Φ_2 (which is proportional to V). Thus, the friction compensating torque is produced by an induction mechanism. To this end a short-circuited turn of copper wire, w_{sc}, is fitted partway around the cross-section of the voltage electromagnet core near the disc. The magnetic field set up by the current flowing in the short-circuited turn produces under the turn a small magnetic flux which links the disc. Acting together with the main flux, it produces the torque that cancels out the friction torque.

In three-phase systems, energy is measured and registered by three-phase meters. In a three-phase meter, two or three induction-type moving elements are coupled together mechanically (by a com-

mon shaft) and drive a common register. The internal connections in a three-phase meter may be similar to those used for power measurements by the two-wattmeter method (see Fig. 3.13 a) or by the three-wattmeter method (see Fig. 3.14) or by some special method.

12.10 Resistance Measurements

Very often resistance is measured by the direct method using specially designed instruments which are called *ohmmeters*. Since they are used for direct measurement, their scales are laid out in units of resistance (ohms or multiples).

Ohmmeters may be the moving-coil type (used for less critical measurements) or the ratiometer type (used for more critical measurements). In turn, the moving-coil type may be a series ohmmeter or a shunt ohmmeter.

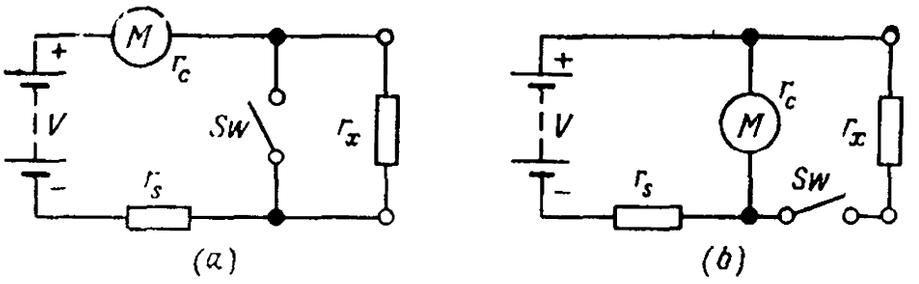


FIGURE 12.26

Series ohmmeters are used to measure high values of resistance, and their movement, M, is connected in series with the resistance being measured (Fig. 12.26 a). By Ohm's law, the current in the circuit is given by

$$I = V/(r_x + r_c + r_s)$$

where V is the voltage supplied by a dry cell (or a storage battery), r_x is the unknown resistance, r_c is the resistance of the moving coil, and r_s is a fixed multiplier (or series) resistor built into the instrument case.

Provided that the supply voltage is constant in value, the scale of the instrument may be calibrated to give the values of r_x directly. When the current in the circuit is zero, the pointer reads $r_x = \infty$. When r_x is zero, the current is

$$I = V/(r_c + r_s)$$

When an ohmmeter is in use, it should be regularly zeroed (adjusted for zero) and its voltage supply tested. To this end, the instrument has a switch, Sw, to short together the leads to which the unknown resistance is connected. In this position of the test switch, the pointer should remain at the zero mark ($r_x = 0$). If so, this is an indication

that the instrument is properly zeroed and its voltage supply is in working order.

A disadvantage of series ohmmeters is that their scale is highly nonuniform.

Low values of resistance (down to a few tenths of an ohm) are measured with a shunt ohmmeter. Now the unknown resistance r_x is shunted across the instrument (Fig. 12.26 b). In this case the current flowing through the movement is

$$I = Vr_x/[r_x (r_c + r_s) + r_c r_s]$$

In a shunt ohmmeter, the current through the movement is zero when the unknown resistance is zero. Conversely, the current is a maximum, $I = V/(r_c + r_s)$, when the unknown resistance is infinitely large, $r_x = \infty$. The scale of a shunt ohmmeter is likewise very nonuniform. When the test switch *Sw* is open, the pointer should read $r_x = \infty$.

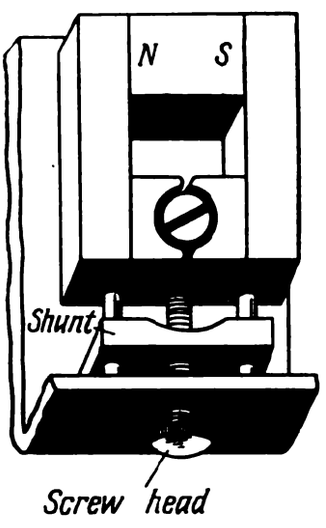


FIGURE 12.27

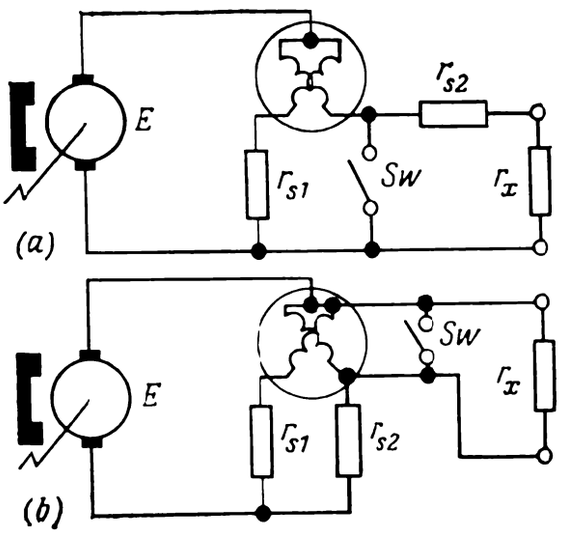


FIGURE 12.28

With time the supply voltage V decreases, which fact makes some adjustment necessary. To this end, both series and shunt ohmmeters are fitted with adjustable magnetic shunts. Such a shunt (Fig. 12.27) is a steel plate intended to divert a portion of the magnetic flux set up by the permanent magnet of the moving element. The position of the shunt with respect to the permanent magnet can be altered with a screw provided on the instrument case. A change in the shunt's position brings about a change in the magnetic flux density in the air gap and, as a consequence, in the deflecting torque, Eq. (12.2). For adjustment, the test switch *Sw* is closed in a series ohmmeter (see Fig. 12.26 a) and opened in a shunt ohmmeter (see Fig. 12.26 b) and the position of the shunt is varied until the pointer reads $r_x = 0$ in the former case and $r_x = \infty$ in the latter case.

A major disadvantage of moving-coil ohmmeters (both series and shunt), apart from the dependence of readings on supply voltage, is the fact that this voltage is too small, especially for measuring insulation resistance. Since insulation resistance strongly depends on the voltage applied, the latter must of necessity be as high as the operating voltage of the circuit being tested. Voltage of such magnitude can be supplied by a hand-driven d.c. generator. The rated emf supplied by such a generator is from 100 to 2500 V. Ohmmeters fitted with hand-driven d.c. generators are designed to measure very high resistances, so they are usually called *megohmmeters*. (There are also teraohmmeters which measure resistances of the order of 10^{14} Ω.)

Since the emf of a hand-driven d.c. generator is anything but constant (it is a function of the speed at which the generator is driven), the usual form of movement in megohmmeters is a ratiometer.

Some makes of megohmmeters can operate of an a.c. power line as well.

Like moving-coil ohmmeters, there may be series megohmmeters and shunt megohmmeters, as is shown in Fig. 12.28 (which are similar to Fig. 12.26). In either type, one of the ratiometer coils is connected in series with a fixed series (multiplier) resistor r_{s1}. For the measurement of high resistances, the second coil of the ratiometer movement is connected in series with the unknown resistance via a second multiplier resistor r_{s2}, as shown in Fig. 12.28 a. For the measurement of low resistances, the second ratiometer coil is shunted across the unknown resistance as shown in Fig. 12.28 b. A change in the generator emf E brings about an equal change in the deflecting torques of the two coils and, since the torques oppose each other, the change in emf does not affect the deflection of the moving element — it solely depends on the resistance ratio of the two coil circuits, Eq. (12.7). Therefore, the scale of a megohmmeter may be calibrated in units of resistance directly.

In some ohmmeters the movement can be connected with the aid of a switch either in series or in parallel with the unknown resistance. Accordingly they have two scales, one for high values of resistance (megohms) and the other for relatively low values of resistance (kilohms).

12.11 Frequency Measurements

Frequencies at 50 Hz and up to about 1000 Hz are mostly measured with direct-reading frequency indicators built around a ratiometer-type movement. The circuit of one ratiometer coil is practically a pure resistance, and the current in it depends solely on the applied voltage and is independent of frequency. The circuit of the other ratiometer coil contains an inductive and a capacitive element, so the current in it depends on both the applied voltage (as in the

former case) and the frequency. In the final analysis, the ratio of the currents in the two coils depends only on frequency and is independent of voltage.

One of the likely arrangements of this type of deflectional frequency indicator is shown in Fig. 12.29. The ratiometer coils W_1 and W_2 are connected across bridge rectifiers B_1 and B_2. The circuit energizing B_1 contains only a purely resistive element r, so the current in W_1 is independent of frequency. The circuit of B_2 contains a series combination of an inductive element L and a capacitive element C. Over the working frequency range of the instrument, the impedance of this circuit corresponds to the up-sloping portion of the current-vs-frequency characteristic $I(\omega)$ (see Fig. 2.47) and even a small change in the unknown frequency brings about a marked change in the current in the coil W_2. Therefore, the deflection of the movement strongly depends on the unknown frequency and is independent of the voltage in the measured circuit. This assures a high sensitivity towards changes in frequency. Another LC network, L_f and C_f, make up a filter which keeps harmonics out of the measuring circuit, should the applied voltage be other than sinusoidal.

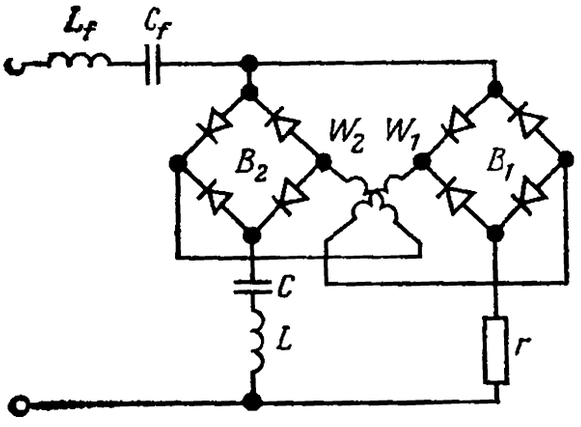


FIGURE 12.29

12.12 Measurements with Bridges

Electric bridges are used to measure resistances, inductances, and capacitances. Resistances are mostly measured with d.c. bridges, while inductances and capacitances are measured with a.c. bridges.

Figure 12.30 a shows a *four-arm d.c. bridge*. A d.c. source is connected across one pair of opposite terminals, and a galvanometer G can be connected across the other pair with the aid of switches Sw_1 and Sw_2. The unknown resistance is placed between points d and b in one arm of the bridge, and the remaining three arms contain variable resistors of known values.

Figure 12.30 b shows a potential diagram (see Sec. 1.7) for the right- and left-hand sides of the bridge with switch Sw_1 open and with the potential at point a assumed to be equal to zero ($\varphi_a = 0$). When switch Sw_1 is closed, the pointer of the galvanometer deflects, and the direction in which it does so gives the sign of the potential difference $\varphi_b - \varphi_c$ and whether it is necessary to adjust the value of the variable resistors (in our case r_1 or r_3 must be increased or r_2 de-

creased) so as to adjust the bridge for balance. Just as the bridge approaches balance, the second switch Sw_2 is closed so as to enhance the sensitivity of the galvanometer.

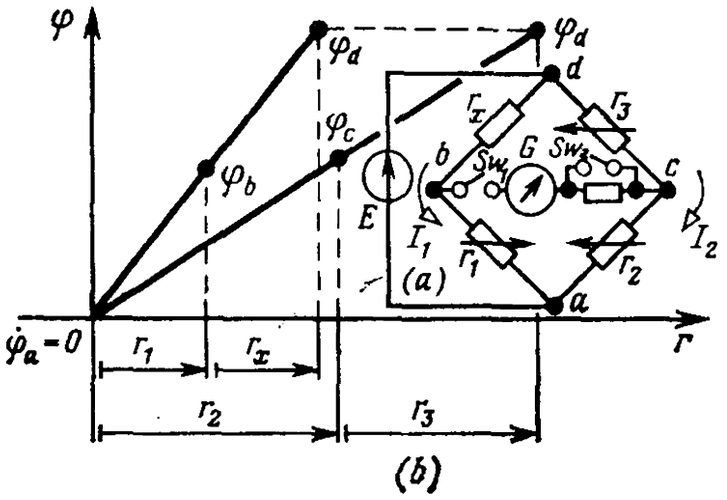


FIGURE 12.30

At balance, $\varphi_b = \varphi_c$, so

$$r_1 I_1 = r_2 I_2 \quad \text{and} \quad r_x I_1 = r_3 I_2$$

On dividing the first equality termwise into the second, we obtain the unknown resistance

$$r_x = r_1 r_3 / r_2 \qquad (12.11)$$

Alternating-current bridges are many and diverse. We will only consider the simplest of them.

An example of a four-arm a.c. bridge is shown in Fig. 12.31 a along with its potential diagram in a complex plane in Fig. 12.31 *b* (see Sec. 2.23), with the galvanometer circuit open. Since the right- and left-hand sides of the bridge have right voltage triangles, the tips of the phasors representing the potentials at points *b* and *c* are located on a semi-circle whose diameter is equal to the source emf $\dot{E} = E$. The values of the variable elements are then adjusted until the potentials at points *b* and *c* are the same, which corresponds to a condition of balance. The fact that the bridge is at balance

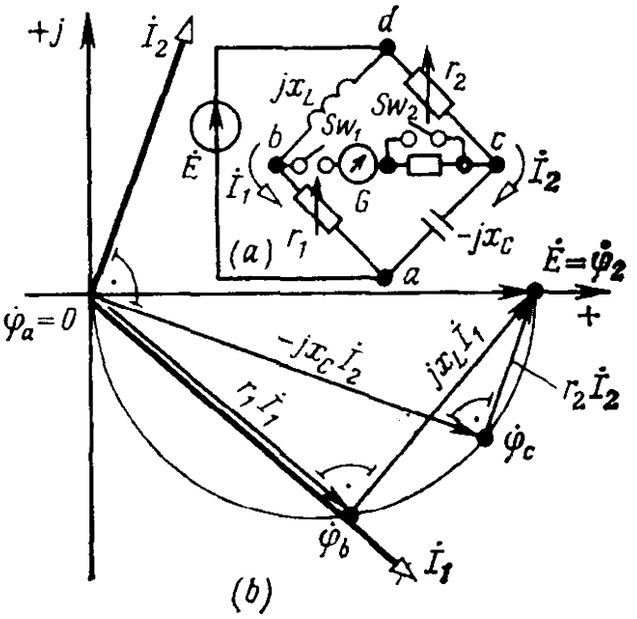


FIGURE 12.31

($\dot{\varphi}_b = \dot{\varphi}_c$) will be indicated by the galvanometer. Switches Sw_1 and Sw_2 serve the same purposes as they do in a d.c. bridge.

When the bridge is at balance, it follows from the equality of the voltage triangles that

$$r_1\dot{I}_1 = -jx_C\dot{I}_2$$
$$jx_L\dot{I}_1 = r_2\dot{I}_2 \qquad (12.12)$$

On dividing one equation into the other termwise and noting that $x_L = \omega L$ and $x_C = 1/\omega C$, we get

$$L = r_1 r_2 C \qquad (12.13)$$

Equation (12.13) shows that an a.c. bridge can be used to measure an inductance (or a capacitance), if the parameters of the other bridge elements are known.

In the case of real inductors, one needs to know the element values of the respective equivalent circuit (see Fig. 7.7 a) which con-

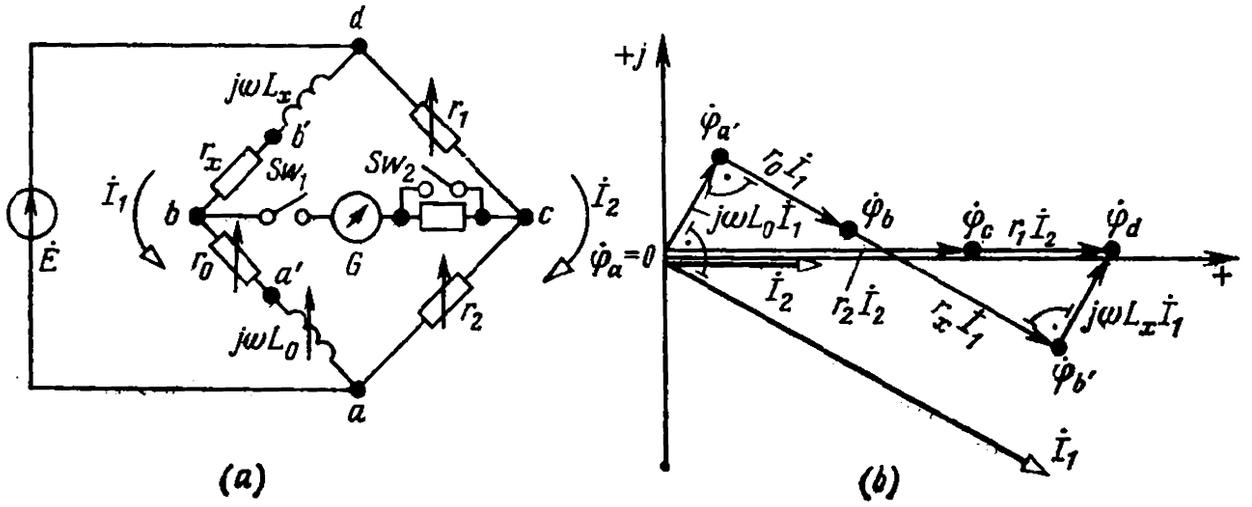


FIGURE 12.32

sists of a series combination of a resistive element and an inductive element. The parameters L_x and r_x making up the equivalent circuit of a real inductor can be measured with the bridge whose circuit is shown in Fig. 12.32. Its potential diagram is shown in Fig. 12.32 b. At balance, the potentials at points b and c are the same ($\dot{\varphi}_b = \dot{\varphi}_c$), so

$$L_x = L_0 r_1/r_2, \quad r_x = r_0 r_1/r_2$$

12.13 Measurements with Potentiometers

Potentiometers are mainly used to measure small emfs (such as generated by thermocouples) and to calibrate electrical measuring instruments.

The circuit of the simplest potentiometer used to measure small emfs is shown in Fig. 12.33. There is an auxiliary supply of direct current I (which can be adjusted with a rheostat r_{adj}), which is connected to a potentiometer of resistance r_p. The difference in potential

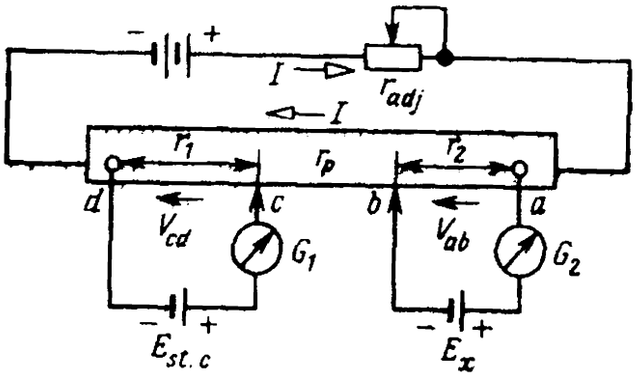


FIGURE 12.33

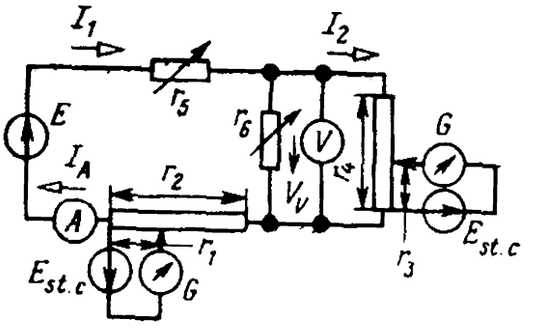


FIGURE 12.34

between points a and b balances the unknown emf, $E_x = V_{ab}$, and the difference in potential between points c and d balances the emf supplied by a standard cell, $E_{st.c} = V_{cd}$.

In order to balance E_x and $E_{st.c}$, the movable contacts b and c must be shifted until the galvanometers G_2 and G_1 read zero.

The value of $E_{st.c}$ is known exactly. Therefore, after $E_{st.c}$ is balanced, we can readily determine the exact value of the current in the potentiometer

$$I = E_{st.c}/r_1$$

from the noted setting on the resistance r_1. The unknown emf is found such that

$$E_x = V_{ab} = r_2 I = (r_2/r_1)\, E_{st.c}$$

where r_1 and r_2 are read off the scale of the potentiometer.

An important advantage of measurements with a potentiometer (which is in effect the null-balance method of measurement) is that one can measure very small emfs or utilize it for purposes of control.

Figure 12.34 shows a set-up used to calibrate ammeters and voltmeters with the aid of potentiometers. As the first step, the wipers of the potentiometers are positioned so as to reduce to zero the currents supplied by the standard cells. Since the resistances of the potentiometers are known, the second step is to establish the rela-

tion between the deflections of the ammeter and voltmeter and the measured currents

$$I_A = I_1 = E_{st.c}/r_1$$
$$V_V = r_4 I_2 = E_{st.c} r_4/r_3$$

Finally, by altering the settings of the variable resistors r_5 and r_6, we can calibrate the scales of the ammeter and voltmeter.

12.14 The Electronic Voltmeter

A major disadvantage of indicating voltmeters using meter movements is the low input resistance (5-10 kΩ). With a low input resistance the current through the coils of the movement (see Fig. 2.11, 2.16 and so on) is large, and so is the torque acting on the moving element, which is the objective sought for this type of instrument. However, such instruments are useless in measuring voltages across resistances comparable in value with the input resistance of the voltmeter (see Fig. 12.2).

This drawback is avoided in *electronic voltmeters* which have a high input resistance (10 MΩ and even more). This materially extends their field of application and provides for a high sensitivity.

There are two main classes of electronic voltmeters: electronic voltmeters proper (EVM) and digital voltmeter (DVM). Electronic voltmeters proper are normally taken as instruments which use a linear amplifier to bring the input signal electronically to a level sufficient to be displayed on a meter movement readout. Of late, there has been a trend to call them analog voltmeters.

Although analog voltmeters are used mostly to measure voltages, many versions are adapted also to measure currents (by measuring the voltage across an internal parallel monitoring resistor), or resistance (by measuring the voltage drop across an internal series resistor connected between the source and an internal fixed test voltage). Even when currents rather than voltages are being measured, the instrument is still often referred to as an analog voltmeter, but again there is trend to call them analog multimeters.

Initially electronic voltmeters proper employed tubes and were called *vacuum tube voltmeters* (VTVM) and, in Britain, *valve voltmeters* (VVM). The more recent makes of instruments use bipolar or field-effect transistors, rather than tubes (valves).

Electronic analog voltmeters come in a variety of circuit designs and arrangements. We will limit ourselves to one (Fig. 12.35) set up as a d.c. bridge. Two arms of the bridge contain identical vacuum triodes, T_1 and T_2. Although presumably identical, the tubes would usually differ in parameters, and this calls for provision of a zero balance control, which is the potentiometer Pot_1. When the bridge

is at balance, the meter (usually a moving-coil galvanometer) connected across one pair of opposite junctions of the bridge will read zero with the input terminals shorted together. In this condition, the grid-to-cathode voltage in the two tubes is the same

$$V_g = Er_1/r - r_2 I_p$$

where $I_p = I_{p1} = I_{p2}$ is the plate current which is the same in both tubes. The tube voltages and currents can be adjusted with another potentiometer, Pot_2.

When the voltage to be measured, V_x, is applied to the input of the voltmeter, the balance of the bridge is upset, and a current begins to flow through the galvanometer. The deflection of the galvanometer is proportional to the unknown voltage.

The voltmeter can be calibrated and its range switched with a variable resistor r_{adj}.

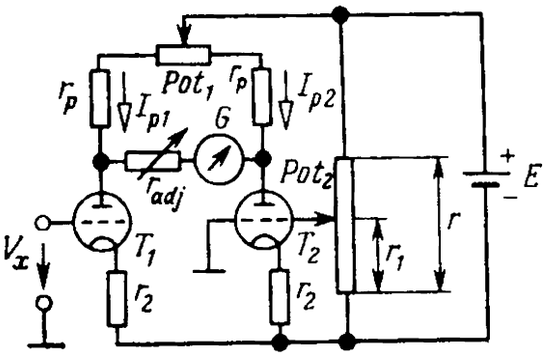


FIGURE 12.35

Since the readout of an electronic analog voltmeter is a meter movement, the accuracy of an electronic voltmeter cannot be better than that of its meter-movement readout. In fact, its accuracy may even be worse due to the difference between the tubes.

The high input resistance of an analog voltmeter results from the low value of the grid current drawn by tube T_1. Its high sensitivity is explained by the fact that even a small change in the unknown voltage strongly upsets the balance of the bridge and leads to a large current through the galvanometer.

Transistorized analog voltmeters have a lower input resistance.

12.15 Digital Instruments. The Digital Voltmeter

Instruments using a deflectional readout suffer from a common limitation which is the human error, for the position of the pointer on the dial can be ascertained no better than the experimentor can do his job. This limitation is avoided in *digital instruments* which present the value of the unknown quantity as a string of digits. Digital instruments are widely used to measure frequency, time intervals, voltage, etc.

We will limit ourselves to a d.c. digital voltmeter of the ramp type. Its block diagram is shown in Fig. 12.36 and a combined timing diagram of the waveforms present in the instrument is shown in Fig. 12.37.

Referring to the diagram, the ramp-type digital voltmeter consists of a clock generator 1, a ramp voltage generator 2, a comparator 3, a count pulse generator 4, a counter 5, and a gate. The clock generator supplies negative-going voltage pulses v_1 at a repetition rate f_1,

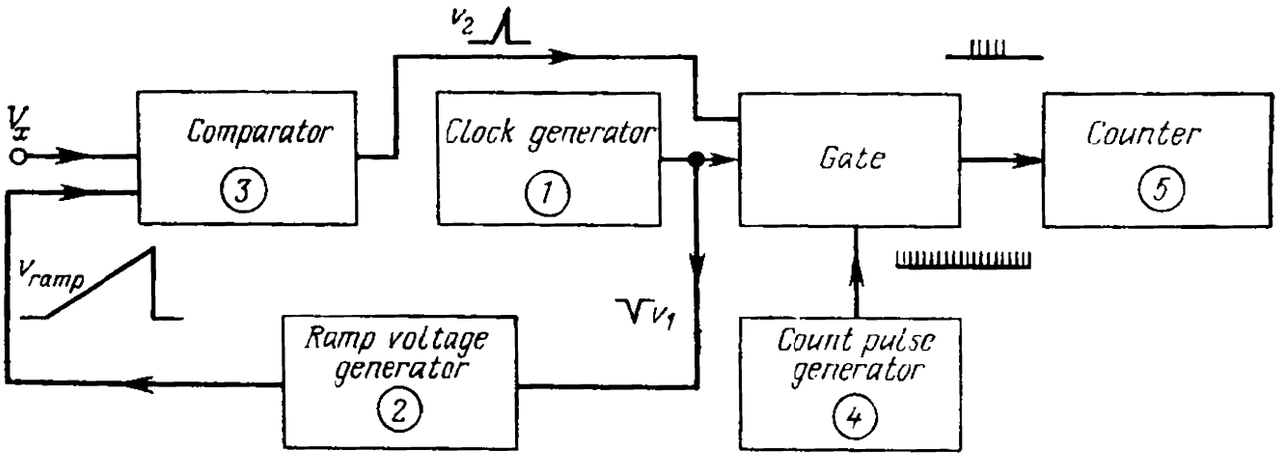


FIGURE 12.36

which simultaneously turn on the ramp generator and the gate. The ramp generator supplies a linearly rising voltage such that $v_{ramp} = g_c t$, where g_c is the conversion transconductance. The ramp voltage is applied to the comparator which also accepts the voltage to be measured, V_x, and the two are compared. During the time inter-

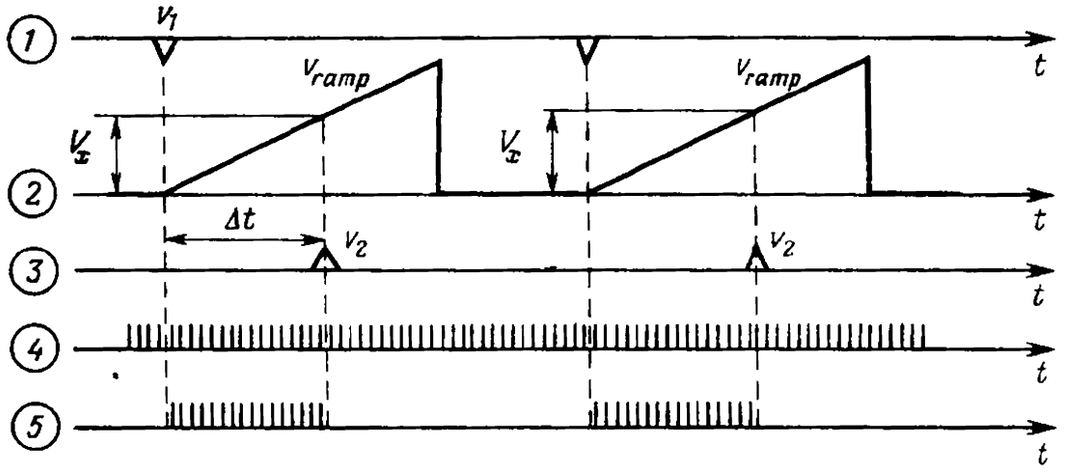


FIGURE 12.37

val Δt that the ramp rises from zero to the value of the unknown voltage V_x, the gate remains open and let the pulses supplied by the count pulse generator at f_2 pass onto the counter. At the end of the time interval Δt the ramp voltage becomes equal to V_x, and the comparator produces a positive-going pulse v_2 which turns off the gate, thus cutting off the stream of count pulses from the count pulse generator to the counter. In this way, the count of pulses represents the duration of the sampling interval.

Thus, in a ramp-type d.c. digital voltmeter the unknown voltage is first converted to a proportional time interval $\Delta t = V_x/g_c$, then this time interval is converted to a proportional count of pulses

$$n = \Delta t f_2 = f_2 V_x / g_c$$

which is displayed by the digital readout already converted to units of voltage.

Since f_2 is high and g_c is small, even a low value of V_x corresponds to a large pulse count n, so the instrument has a high sensitivity and a high accuracy. The direct numerical readout eliminates the human error associated with conventional indicating instruments.

12.16 The Mirror-Galvanometer Oscillograph

Mirror-galvanometer oscillographs were at one time widely used to observe and record periodic variations (oscillations) in electric and nonelectric quantities at frequencies up to about 5 kHz. In

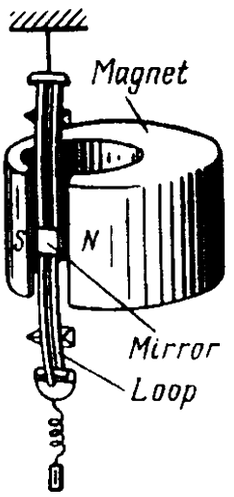


FIGURE 12.38

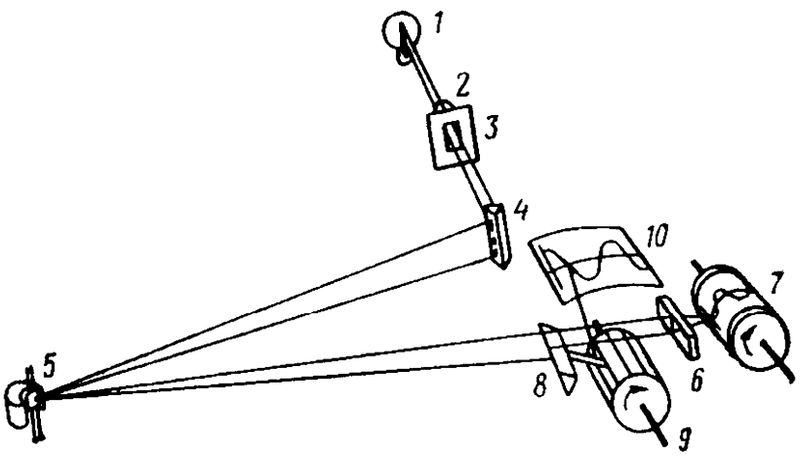


FIGURE 12.39

the case of nonelectric quantities, they have first to be converted to electric form by suitable transducers.

A mirror-galvanometer oscillograph consists essentially of a mirror galvanometer (or galvanometers), an optical train, and devices for visual observation and recording.

The galvanometers used for oscillographic work are made to have as little inertia as practicable. Their moving element, or vibrator, may use either a single-turn coil (as in Fig. 12.38) or a multiturn coil (which is a more recent development). In the latter case, the sensitivity of the instrument is markedly improved.

The optical train of a mirror-galvanometer oscillograph is shown in Fig. 12.39. A narrow beam of light from a bright light source 1 passes through a lens 2, a slit 3, a prism 4, and strikes the mirror

5 mounted on the moving loop or coil. (Present-day mirror-galvanometer oscillographs use mercury-vapour lamps as light sources, as their light produces a better image on photographic emulsion.) On being reflected from the mirror, the beam is focused by prisms 8 and 6 into a small light spot on a strip of photographic paper or film 7. When the moving coil is energized, it reacts with the electromagnetic field and turns through an angle proportional to the unknown quantity, thus moving the light spot across the paper (or film) strip. As the unknown quantities oscillate, the light spot moves to and fro and produces a straight line. Since the record strip is rotated at a uniform speed in a direction perpendicular to the travel of the light spot, the line is spread out and a time-swept trace, or oscillogram, is produced on the recording medium.

Most mirror-galvanometer oscillographs have provisions for visual observation of periodic variations (in which case it would be more appropriate to call them oscilloscopes). To this end, a part of the light beam reflected from the galvanometer mirror is directed onto a rotating polyhedral mirror 9 which throws the light beam onto a frosted screen 10 where a light trace is produced. If the polyhedral mirror is rotated in step with the periodic process being observed, the same angles of incidence and reflection of the light beam on and from the polyhedral mirror will correspond to a particular phase of the process during each period. As a result, the light spot will trace out one and the same wave, and this will appear stationary to the observer so that it can be conveniently viewed.

Some mirror-galvanometer oscillographs are adapted to view and record several quantities at a time. For this purpose a single oscillograph may incorporate several galvanometers (as many as 24 or even more). The mirror of each galvanometer is illuminated by a separate beam of light extracted from a common beam by a system of prisms and lenses. In most oscillographs, the galvanometers can be changed at will to suit the prevailing conditions and the objective of a study.

The principal parameters of a galvanometer are the natural frequency of oscillation and current sensitivity usually called the ampere constant. A mirror galvanometer can record fairly accurately the processes whose frequency is not more than 0.2 the natural frequency of the vibrator (from 500 Hz to 20 kHz).

The *ampere constant* of a galvanometer is defined as the deflection (in mm) produced by a current of 1 mA on a scale (or viewing screen) 1 m distant. Its magnitude varies from 1 to 50 mm/mA, depending on the type of vibrator. The higher the ampere constant, the lower the natural frequency of the vibrator.

12.17 The Cathode-Ray Oscilloscope

The *cathode-ray oscilloscope* differs from the mirror-galvanometer type in that the waveform or other pattern to be examined is traced out by a beam of electrons. It is primarily used to observe (on a fluorescent screen) and/or to record (with the aid of an adjunct photographic camera) the waveforms of fast changing processes. However, it can be readily adapted to measure frequency or to plot various dynamic characteristics such as the hysteresis loop (see Fig. 7.10), the characteristics of vacuum tubes or semiconductor devices, and the like.

The heart of a cathode-ray oscilloscope is the cathode-ray tube (CRT), which consists of an electron gun, a deflecting system, and

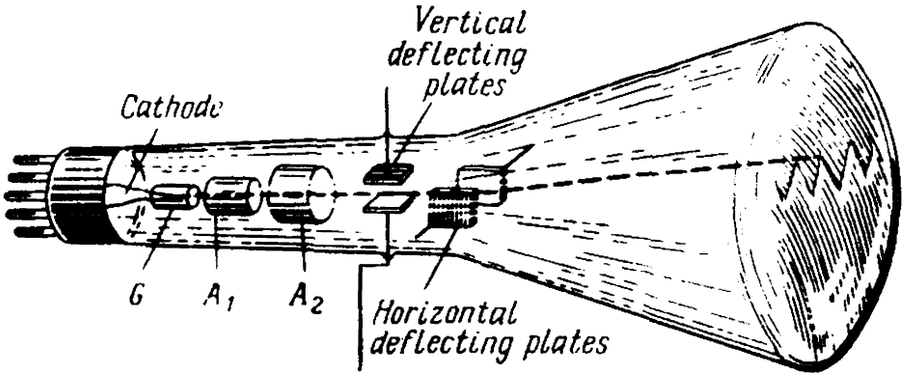


FIGURE 12.40

a screen. The electron gun produces a narrow beam of electrons. Acting through the deflecting system, the quantity being observed controls the motion of the electron beam which acts as the practically inertia-free moving element of the oscilloscope. The screen is given a coat of phosphor which fluoresces as the emitted electrons strike it. When zero voltage is applied to the deflecting system, the electron beam strikes the centre of the CRT, causing a small luminous spot to appear. When a voltage is applied to the deflecting system, the electron beam is deflected horizontally and vertically, thus producing an image of the process waveform.

The glass envelope of the CRT is the shape of a cone which changes into a cylinder called the tube's neck (Fig. 12.40). The electron beam is produced and shaped in the neck which encloses the electron gun consisting of an indirectly heated cathode, a control grid G (or modulator), a focusing anode A_1, and accelerating anode A_2.

The cathode is enclosed in the cylindrical control grid which has in the centre of its bottom a hole (a diaphragm) to pass the electrons emitted by the cathode. Since the control grid is held at a negative potential of 20-50 V, the electrons passing through are repelled by its negative charge and this serves to focus the stream

of electrons roughly into a narrow beam. At the same time, the negative potential on the control grid repels some of the electrons back towards the cathode. Thus, by varying the potential at the control grid, the user can vary the number of electrons in the beam and control the brightness of the spot on the screen.

The electric field required to accelerate the electrons is produced by the high positive potentials at the two anodes which are hollow cylinders with one or more slits, placed in the path of the electron beam. The slits serve to catch electrons straying too much from the beam axis.

The voltage between the focusing anode A_1 and the cathode is from one-tenth to one-third of the voltage between the accelerating anode A_2 and the cathode, which is 600-5000 V.

The deflecting system in the CRO in question is of the electrostatic type. It consists of two pairs of plates, one pair deflecting

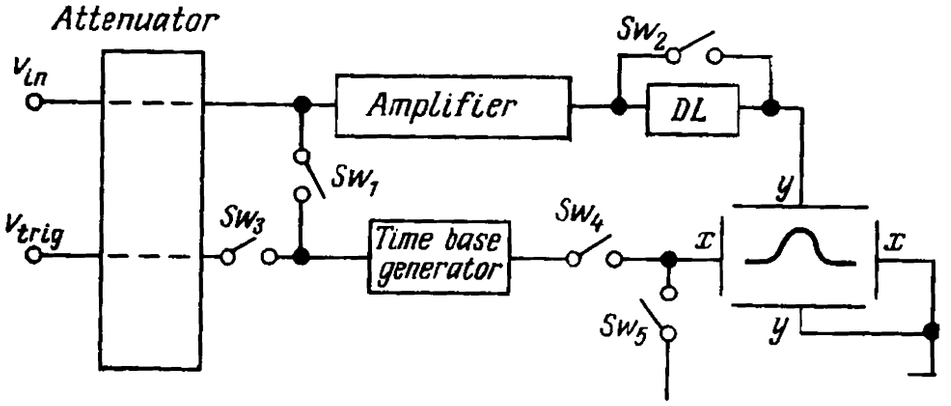


FIGURE 12.41

beam horizontally and the other pair vertically. It is usual to apply the periodic voltage under study to the vertical deflecting plates, so that the beam is deflected vertically (along the axis of ordinates) in proportion to the magnitude of the signal voltage. The horizontal deflecting plates cause the spot to move along the horizontal axis which is usually the time axis because in a large number of applications it is desired to examine the variations of a process with time. For this purpose the horizontally deflecting plates are fed with a periodic sawtooth voltage generated by a circuit called the *time base* (or *sweep*) *generator*.

In block-diagram form (Fig. 12.41), a CRO may be visualized as consisting of a number of units and switches that can be positioned so that several modes of operation can be obtained. These modes are internal triggering, external triggering, free-running, and special-function.

The front-end unit is an attenuator which is essentially a calibrated voltage divider which can reduce the signal voltage and the sync pulse voltage in any desired ratio.

In the *internal triggering mode*, switches Sw_1 and Sw_4 are closed, and the input signal v_{in} is fed via the attenuator to trigger the time base (or sweep) generator, and to the vertical deflection amplifier. The voltage generated by the time base is applied to the horizontal

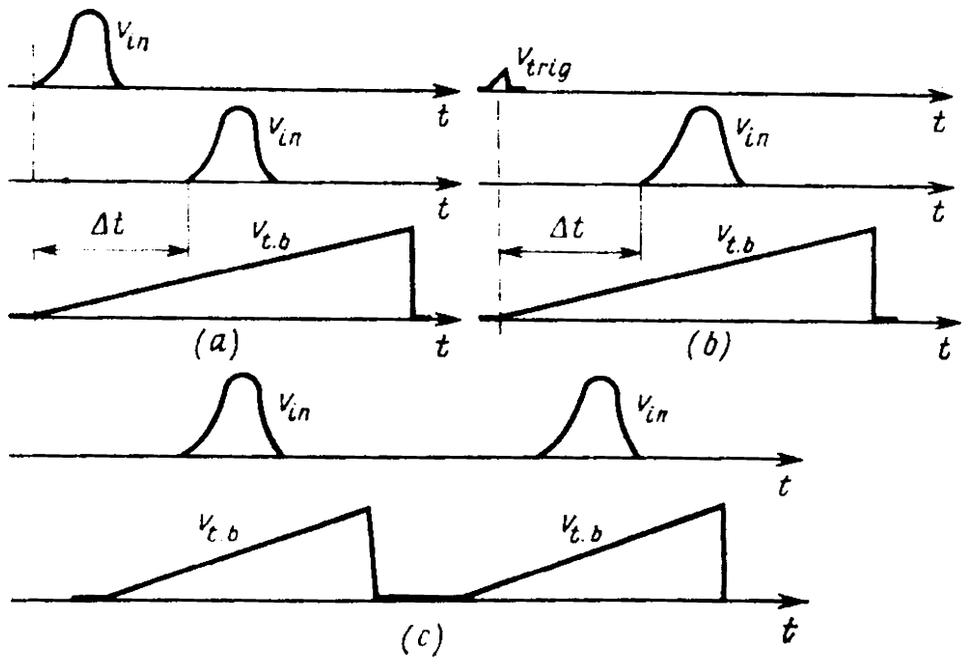


FIGURE 12.42

(x-x) deflecting plates and causes the spot to move along the time axis. From the vertical deflection amplifier, its output voltage is applied to the vertical (y-y) deflecting plates on passing through a delay line, DL, which delays it for a time Δt (Fig. 12.42 a) so

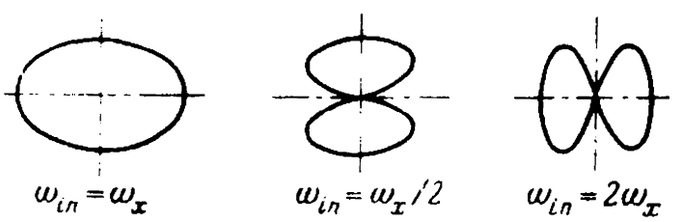


FIGURE 12.43

as to centre the electron beam. A disadvantage of the internal synchronization mode is the likely distortion of the input signal by the delay line.

In the *external triggering mode*, switches Sw_2 and Sw_4 are closed, and the time base is now triggered by a separate triggering pulse, v_{trig}, which leads the input signal, v_{in}, by a time Δt (see Fig. 12.42 b). Now no delay line is needed, and the input signal is reproduced more faithfully.

In the *free-running mode*, switches Sw_2 and Sw_4 are closed, and the time base runs continuously to generate a sawtooth voltage at frequency f_{tb}. If f_{tb} is the same as the frequency of the signal volt-

age or is its multiple, the CRO screen displays a stationary pattern (Fig. 12.42 c).

In the *special-function mode*, switches Sw_5 and Sw_4 are closed, and a suitable periodic voltage, v_x, is applied to the horizontal deflecting plates. This mode is used for some special measurements. For example, if the horizontal deflecting plates are fed a sinusoidal voltage

$$v_x = V_m \sin \omega_x t$$

we can measure the angular frequency of the input voltage, if it is likewise sinusoidal and has a frequency ω_{in} which is a multiple of the angular frequency ω_x. Depending on the ratio of angular frequencies, ω_{in}/ω_x, various shapes, called Lissajous figures (Fig. 12.43), are produced on the CRO screen.

The ratio of the vertical deflection of the light spot on the screen (in mm) to the value of the input voltage (in volts) is called the *sensitivity* of a CRO. The sensitivity of the cathode-ray tube alone (without an amplifier) is relatively low, being somewhere between 0.5 and 1 mm/V. With an amplifier, the sensitivity can be raised to 1-2 mm/mV.

Cathode-ray oscilloscopes can be fitted with suitable photographic cameras and accessories in order to make permanent records of the waveforms observed.

13 Direct-Current Electric Machines

13.1 General

Direct-current electric machines (generators and motors) have a wide field of application. The primary advantage of d.c. motors is that their speed can be readily varied as desired and that they can develop large starting torques. For this reason d.c. motors are widely used for electric traction (on tramways, trolleybuses and railways), and as drives in a variety of manufacturing and process plant.

Low-power d.c. machines are employed to actuate final control elements and to serve as speed sensors in automatic control systems.

Direct-current generators are used to supply power for radio equipment, for battery charging, for electrolytic cells, and the like.

A disadvantage common to all d.c. machines is the complexity of design, mainly due to the use of brushgear. Apart from its complex design, brushgear is subject to sparking. This reduces their reliability and limits their applicability. A substantional limitation is that a.c. supply for d.c. motors has to be first converted to d.c.

13.2 Construction of D.C. Machines

A d.c. machine consists essentially of a stationary part, called the field structure, which serves to excite the main magnetic field, and a rotating part, called the armature, in which an emf is induced by the main magnetic field. The current produced by this emf reacts with the main magnetic field and generates a braking torque when the machine is operating as a generator, and a driving torque when the machine is operating as a motor.

The field structure consists (Fig. 13.1) of a yoke which carries main poles to excite the main magnetic flux, and commutating poles to improve commutation (see Sec. 13.7) in the machine.

A *main pole* consists of a pole core (or shoe) built up of electrical-sheet steel laminations and bolted to the yoke, and a field coil wound on the core. At its free end, the core has a pole face suitably shaped to ensure a uniform distribution of the magnetic flux around the armature.

The *yoke* completes the path for the main magnetic flux Φ (Fig. 13.2). It is fabricated from cast steel rather than from laminations because the magnetic flux in it is relatively constant in magnitude.

The *commutating poles* (or *compoles*) are bolted to the yoke midway between the main poles. The compoles carry windings which are connected in series with the armature.

The *armature* is that part of a d.c. machine where an emf is induced as it rotates relative to the main field. The armature consists of a toothed core, a winding dropped in the core slots, and a commutator mounted on the armature shaft. The armature core is composed of electrical-sheet steel laminations (Fig. 13.3 a) usually 0.5 mm thick, insulated from one another with varnish. The armature winding consists of sections, or coils (Fig. 13.3 b).

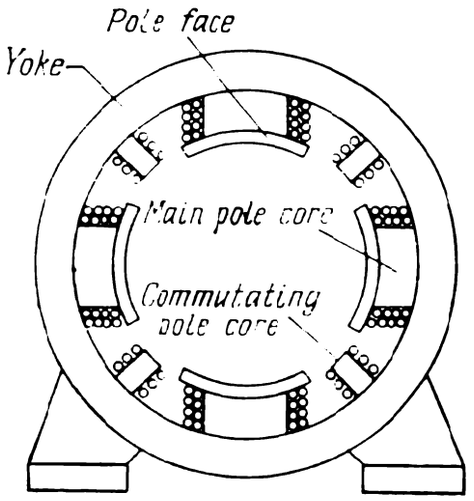


FIGURE 13.1

The *commutator*, which is a typical component of d.c. machines, is a hollow cylinder assembled from wedge-shaped copper bars, 1, insulated from one another and from the shaft with mica (Fig. 13.4). Conductors 2 connect the commutator bars to the coils of the armature winding in the slots. The armature winding is connected to the external circuit by

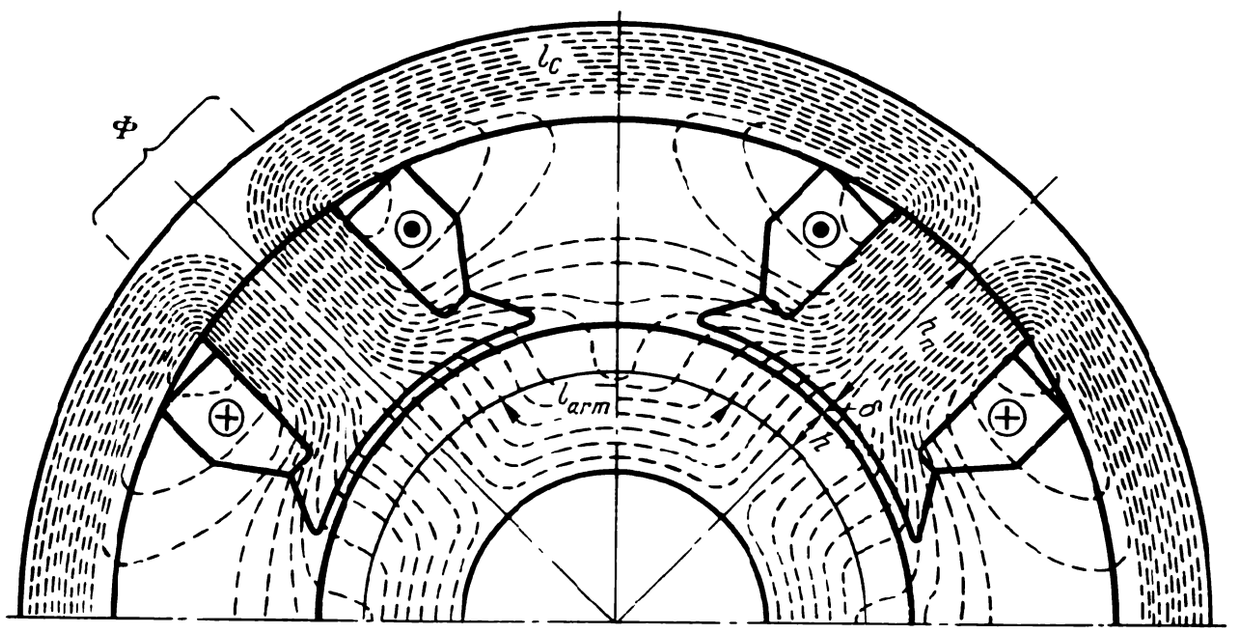
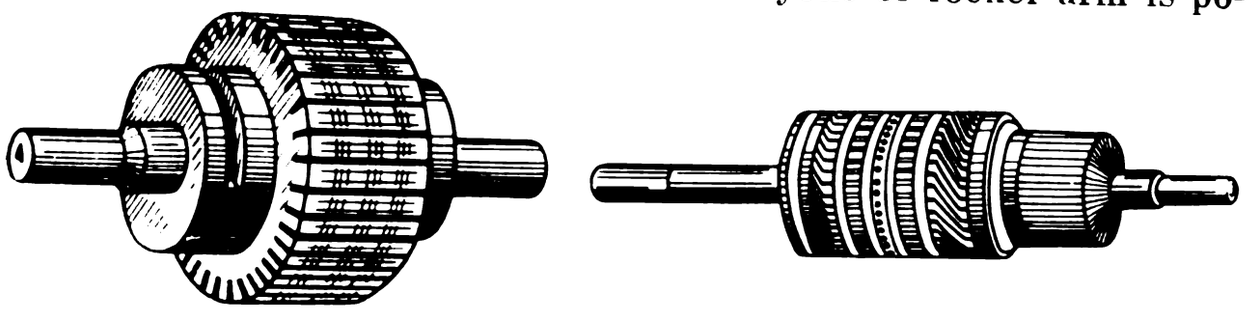


FIGURE 13.2

the sliding contact that is formed between the commutator bars and brushes.

The *brushes*, 1, are mounted in a box type of brushholder (Fig. 13.5) and are held on the commutator by a spring, 2, which gives a sufficient pressure. Current is conducted away from a brush

(as in a generator) or to a brush (as in a motor) by a pigtail. The brushholders are mounted on brush arms (also called brush studs or brackets) which pass through a hole, 3, in each brushholder. In turn, the brush arms are secured to a brush yoke or rocker arm from which they are electrically insulated. The brush yoke or rocker arm is po-



(a) (b)

FIGURE 13.3

sitioned coaxially with the armature so that it can be turned to change the location of the brushes relative to the poles of the machine.

As is the case with all electrical machines, a d.c. machine is reversible. That is, it can operate as a generator if it is driven by a

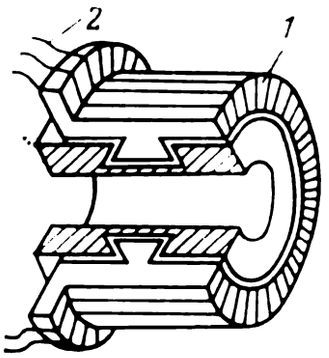


FIGURE 13.4

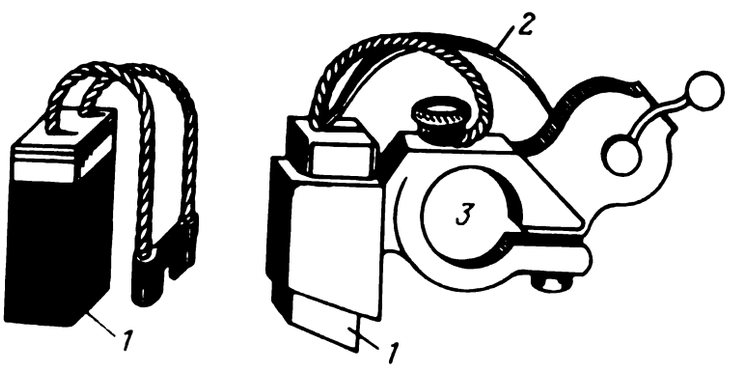


FIGURE 13.5

primary mover, the main magnetic field is excited and the armature circuit is connected via the brushes to a load. In the circumstances, the emf induced in the armature winding gives rise to a current in the armature and the load. The interaction of the armature current with the main magnetic field produces at the shaft a braking torque which is overcome by the prime mover. In this way, a d.c. generator converts mechanical energy into electric energy.

When a d.c. machine is operating as a motor, the armature and field circuits are connected to a power source. The interaction of the armature current with the main field produces a driving torque which overcomes the load torque applied to the shaft, thereby converting electric energy into mechanical energy.

In other words, one and the same d.c. machine can be used as a generator and as a motor.

Direct-current machines may be classed according to several features, the main form of classification being according to the method

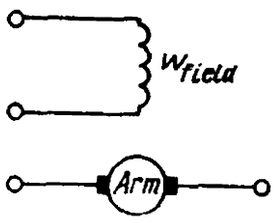


FIGURE 13.6

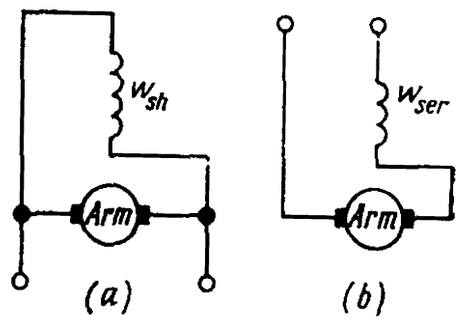


FIGURE 13.7

of excitation of the main magnetic field. In some machines this purpose is served by using permanent magnets on the poles. In many present-day machines, the main magnetic field is excited by electromagnets which are called field coils because they consist of coils carrying the field-excitation current and put on the field-structure poles. The performance of d.c. machines, used as generators or as motors, depends on the manner in which the field winding is connected relative to the armature winding. The connection may be shunt, series, and compound (that is, shunt-series), or the two windings may be arranged to be independent of each other. Accordingly, there are shunt-wound, series-wound and compound-wound machines. Since in all of them the excitation current is supplied internally, the machines are called self-excited. When the armature and field windings are independent of each other, the excitation current is supplied from some external source, and the machines are called separately excited.

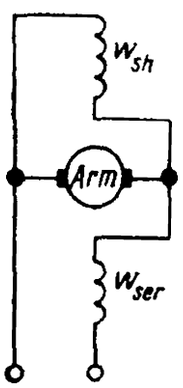


FIGURE 13.8

A fact of practical importance is that with any form of connection the power taken by the field structure is relatively small, being about 5% of the power rating for small machines, and less than 1% for large machines. This offers an economical means for controlling the operation of d.c. machines (the voltage of a generator and the angular speed of a motor).

As already noted, in a *separately excited d.c. machine* the field winding is connected to some external source (Fig. 13.6), so the current in the machine does not depend on the voltage across the armature terminals. The conductor size for the field winding of a separately excited machine is chosen to suit the voltage of the external source. What is characteristic of separately excited machines is that the main field is independent of the load sustained by the machine.

In a *shunt machine*, the field winding is connected in parallel with the armature winding (Fig. 13.7 a). Now the excitation current, I_{field}, is a small fraction (0.05-0.01) of the armature current, whereas the same voltage V exists between the terminals of the armature winding and the field winding. In consequence, the resistance of the field winding, $r_{field} = V/I_{field}$, must be relatively high. Therefore, the field winding of a shunt machine is wound with many turns, w_{sh}, of fine wire. What is characteristic of shunt machines operating within a large-power system is that the main magnetic flux is nearly constant at all loads.

In a *series machine*, the armature current, I_{arm}, is equal to the field current (Fig. 13.7 b), so it is wound with a heavy-gauge conductor. The current in a series field winding is large, so the desired mmf, $I_{arm}w_{ser}$, can be obtained with a relatively small number of turns, w_{ser}. Therefore, the resistance of a series field winding, r_{ser}, is relatively low. The feature typical of series machines is that the main magnetic flux varies noticeably with changes in load due to changes in the armature current which is at the same time the field current.

In a *compound machine*, each pole carries two windings (Fig. 13.8). That connected in parallel with the armature is the main winding, and the mmf it produces, $I_{sh}w_{sh}$, sets up the main magnetic field. The other is a series winding, w_{ser}, which only assists in controlling the magnetic field. Depending on which of the two mmfs predominates, the performance may be that of a series machine with a small shunt winding, or of a shunt machine with a small series winding. In most compound machines, the two windings are connected to aid each other, so that their mmfs are combined. Sometimes, the series winding is arranged to oppose the shunt winding, which means that their mmfs act in opposite directions. This arrangement is used in a few special cases.

13.3 Operation of Brushgear and the Commutator

An alternating emf is induced in the rotating armature winding of a d.c. machine, and it needs rectification, which is done by the commutator.

The rectification of the emf induced in a d.c. machine can conveniently be traced using as an example a d.c. generator (Fig. 13.9 a) in which the armature core is absent, the magnetic field set up by the main poles is uniform, and the armature winding is made up of two separate turns, 1 and 2, arranged to be at right angles to each other and connected to the commutator.

When the turns 1 and 2 rotate clockwise at an angular velocity ω, an emf is induced in them, its positive direction being given by the corkscrew rule (see Fig. 2.5 b), and its magnitude being given

by Eq. (2.15)

$$e_1 = -\,d\Phi_1/dt = E_m \cos \omega t$$

$$e_2 = -\,d\Phi_2/dt = E_m \sin \omega t$$

where $\Phi_1 = -\,\Phi_m \sin \omega t$ and $\Phi_2 = \Phi_m \cos \omega t$ are the components of the main flux Φ_m, normal to the planes of the turns 1 and 2.

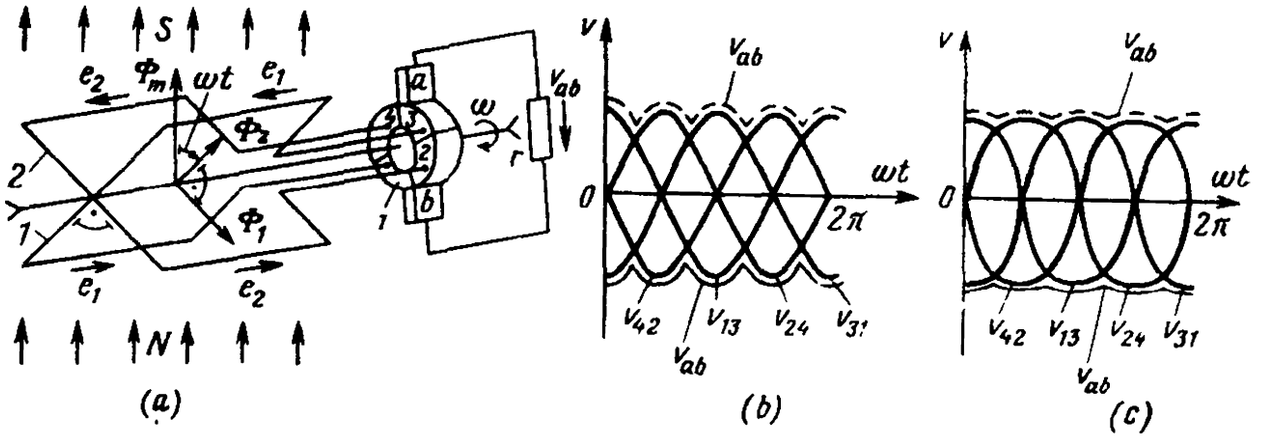


FIGURE 13.9

Since the turn 1 is connected to commutator bars 1 and 3, and the turn 2 to commutator bars 2 and 4, the voltages existing between the respective pairs of bars will be

$$v_{13} = e_1 = E_m \cos \omega t$$

$$v_{24} = e_2 = E_m \sin \omega t$$

When the armature is rotating clockwise, the external circuit is connected to the turns 1 and 2 via the stationary brushes located in a vertical plane and the commutator in such a way that during one complete revolution of the commutator its bars will come in contact with brush a in the sequence 3, 4, 1, 2, 3 and with brush b in the sequence 1, 2, 3, 4, 1. As a result the voltage between brushes a and b, which is the terminal voltage of the generator, will vary in the same sequence as the brushes come in contact with the commutator bars: $v_{ab} = v_{31}$, $v_{ab} = v_{42}$, $v_{ab} = v_{13}$, $v_{ab} = v_{24}$, and $v_{ab} = v_{31}$ (represented by the full line in Fig. 13.9 b), where

$$v_{31} = -\,v_{13} = -\,E_m \cos \omega t$$

$$v_{42} = -\,v_{24} = -\,E_m \sin \omega t$$

If we reverse the direction of rotation of the turns and the commutator, two things will happen. Firstly, the component of the main flux normal to the plane of turn 1 will change sign so that $\Phi_1 = \Phi_m \sin \omega t$, and so will the emf induced in that turn, $e_1 = -\,d\Phi_1/dt = -\,E_m \cos \omega t$. Secondly, the commutator bars will now come in contact with brush a in the sequence 3, 2, 1, 4, 3, and with brush b in the sequence 1, 4, 3, 2, 1. Accordingly, the terminal voltage of the generator will change in the same sequence: $v_{ab} =$

$= v_{31}$, $v_{ab} = v_{24}$, $v_{ab} = v_{13}$, $v_{ab} = v_{42}$, and $v_{ab} = v_{31}$ (represented by the dashed line in Fig. 13.9 b), where

$$v_{13} = e_1 = -E_m \cos \omega t$$

$$v_{31} = -v_{13} = E_m \cos \omega t$$

$$v_{24} = e_2 = E_m \sin \omega t$$

$$v_{42} = -v_{24} = -E_m \sin \omega t$$

Owing to the use of a ferromagnetic armature core and pole faces, the magnetic induction B is uniformly distributed in the air gap δ of the machine (Fig. 13.10), thereby minimizing pulsations (or ripple) in the generator voltage. If in the d.c. generator of Fig. 13.9 a we place the turns 1 and 2 in the slots of an armature rotating in the magnetic field set up by a pair of main poles ($p = 1$) fitted with pole faces, the terminal voltage of the generator (Fig. 13.9 c) will show a reduced ripple as compared with the rotation of the turns in a uniform magnetic field (see Fig. 13.9 b).

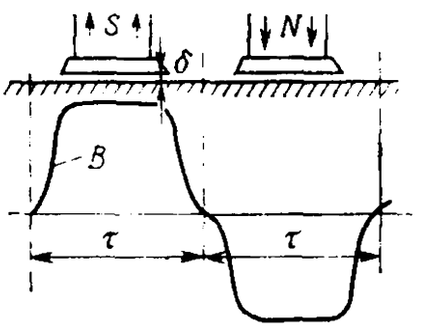


FIGURE 13.10

If a machine has more than one pair of poles, it has to have a greater number of brushes in order to interconnect the parallel circuits in the armature winding. Figure 13.11 a shows the connection

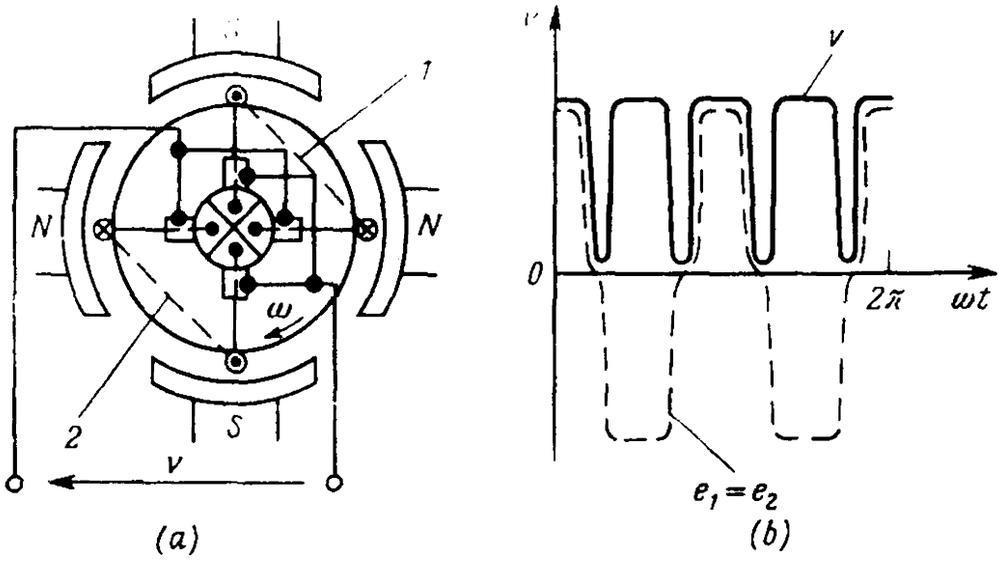


FIGURE 13.11

for a four-pole d.c. generator ($p = 2$) in which two single-turn windings 1 and 2 are connected in parallel. The identical alternating emfs ($e_1 = e_2$) induced in the turns 1 and 2 vary at an angular frequency 2ω and are rectified by the commutator (see Fig. 13.11 b).

In a generator, brushes and a commutator are required in order to rectify the alternating emf induced in the armature winding. In a motor, brushes and a commutator are needed in order to keep the armature rotating in one and the same direction. In all conductors of the parallel paths in the armature winding the current is the same and equal to $I = I_{arm}/2a$, where a is the number of pairs of parallel paths. When all of these conductors are acted upon by an electromagnetic force in the same direction, the motor develops a maximum torque. Just as a conductor passes from one pole to another the brushes and the commutator reverse the current in that conductor. As a result, the torque always acts in one and the same direction.

13.4 Drum Armature Windings

The conductors placed in the armature slots must be interconnected in a proper way so as to make up the armature winding.

Most of the present-day d.c. machines use a drum armature which is a cylinder (see Fig. 13.3) assembled from electrical-sheet steel laminations insulated from one another. The sides of each turn in

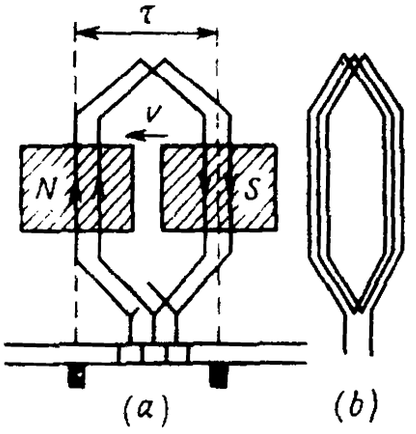


FIGURE 13.12

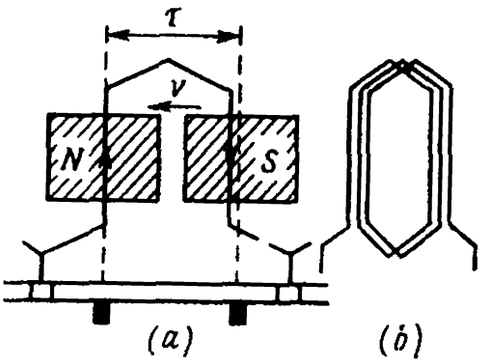


FIGURE 13.13

a drum winding are spaced approximately a full pitch, τ, apart, so that the emfs induced in both sides of a turn will combine. In this way, each turn encloses nearly all of the flux in one pole of a machine. Sometimes, short-pitch turns are used, that is, those in which the width of a turn is less than the pole pitch. This is done in order to reduce the size of the coil ends.

Instead of one turn the slots usually receive multiturn coils (Fig. 13.12 b). The coils may be connected to form a complete armature winding in any one of two basic ways. With one the result is what is called a *lap winding*: for connection to the next coil, the first must return to the pole from which it started, so the coil throw is bar 1, to bar 2, to bar 3, etc. In a diagram, the individual conductors are not shown, but only the coil sides. The brushes divide a lap

winding into as many parallel circuits as there are poles, so for a machine having a lap winding $a = p$.

The other way of connection results in what is called a *wave winding* (Fig. 13.13 a). Now the coil (see Fig. 13.13 b) lying under one pole is connected to the next one around the circumference of the armature. The number of parallel circuits in the case of a wave (or, rather a simple wave) winding is two ($2a = 2$), no matter how many poles the machine has. For a simple wave winding to close upon itself, one has to go round the circumference of the armature several times. A lap winding closes on itself first time around the armature.

The simple wave winding is used on small and medium machines (not over 500 kW) at a voltage of 110 V and higher. The simple lap winding is used on small-size two-pole machines (not over 1 kW) and on machines in sizes up to 500 kW.

Multiplex windings are formed by placing m simple windings in the armature slots; as a result, the number of parallel circuits is increased m times. Multiplex (or multiple) windings are used on large machines.

13.5 The EMF and the Electromagnetic Torque of a D.C. Machine

For any d.c. machine the basic equations are those describing the emf induced in the armature winding and the electromagnetic torque produced by the interaction of the armature current with the main magnetic field of the machine.

As a conductor of the armature winding moves in the magnetic field under a pole (Fig. 13.14), the conductor cuts the magnetic lines of force, and this induces in it an emf given by

$$e_1 = lvB$$

where l is the active length of the conductor and v is the peripheral speed of the armature.

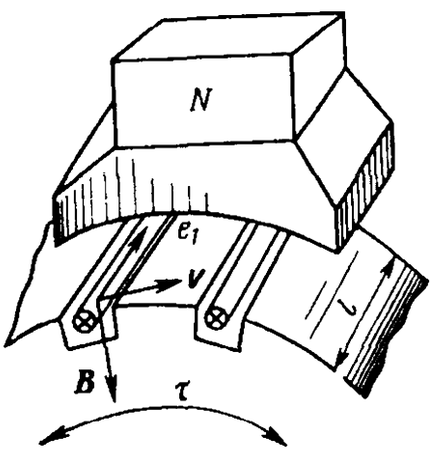


FIGURE 13.14

The above equation defines the instantaneous emf which varies with changes in the magnetic induction along the pole pitch. To find the average emf, we substitute in the above equation the average magnetic induction B_{av}, that is one under a pole but averaged over the pole pitch

$$E_{1,av} = lvB_{av}$$

The peripheral speed v may be expressed in terms of the rotational speed n of the armature (in rpm), the pole pitch τ, and the number of poles $2p$

$$v = \pi D n/60$$

$$\pi D = \tau \times 2p$$

where D is the diameter of the armature core.

Therefore,

$$v = 2pn\tau/60$$

and

$$E_{1,av} = l\tau \times 2pnB_{av}/60$$

Noting that $l\tau = A_{pole}$ is the area of a pole pitch (see Fig. 13 14) and $A_{pole}B_{av} = \Phi$ is the magnetic flux per pole, we obtain

$$E_{1,av} = 2pn\Phi/60$$

The armature winding consists of N active conductors. The brushes divide the winding into $2a$ parallel circuits. Thus, each parallel path contains $N/2a$ series-connected active conductors. The armature emf is the emf of one parallel circuit, and this is equal to the sum of the emfs induced in the conductors that make up the parallel path. Hence, the armature emf is

$$E_{arm} = E_{1,av}N/2a$$

or

$$E_{arm} = (p/a)\, N\Phi\, (n/60) = c_E\Phi n \qquad (13.1)$$

where c_E is the constant of the winding.

In a motor, this emf acts against the current, and so it is called the *counter-emf*.

The armature emf can be adjusted by varying the magnetic flux or the speed (rpm) of the armature.

When a d.c. machine is operating as a generator, the interaction of the armature current with the main magnetic field produces a braking torque which is overcome by the prime mover. When a d.c. machine is operating as a motor, the interaction of the armature current with the main magnetic field produces a driving torque. The direction of energy transfer is different in the two modes of operation, but the origin of the *electromagnetic torque* acting on the armature is the same.

Each of the N armature-winding active conductors located under the poles of a machine is acted upon by a force $f = lBI$*. The sum

* Actually, this force is applied to the core teeth.

of the forces constituting the electromagnetic torque applied to the armature

$$T_{EM} = (D/2) \sum_{k=1}^{N} f_k$$

or, using the average magnetic induction under a pole,

$$T_{EM} = (D/2)\, NlB_{av}I$$

On expressing the circumference of the armature in terms of the pole pitch, $\pi D = 2p\tau$, and on replacing $B_{av}l\tau = \Phi$, we obtain

$$T_{EM} = pN\Phi I/\pi$$

Finally, on replacing the current I in one conductor with the total armature current $I_{arm} = I \times 2a$, we get

$$T_{EM} = (1/2\pi)\ (p/a)\, N\Phi I_{arm} = c_T \Phi I_{arm} \qquad (13.2)$$

where $c_T = c_E \times 60/2\pi$ is the torque constant of a given machine.

13.6 Armature Reaction

Armature reaction refers to the effect that the armature current has on the main magnetic field of a d.c. machine. In most cases, this is an undesirable factor as it distorts the main magnetic field and impairs the performance of the machine. That is why measures are provided for during the design of a machine to minimize the armature reaction.

So long as the magnetic field in a machine is produced solely by the current in the field winding ($I_{arm} = 0$), it is symmetrical about the axis of the pole cores and is nearly uniform under the poles. In sketch form, this magnetic field for a two-pole machine ($p = 1$) is shown in Fig. 13.15 a. The line n-n' running at right angles to (or in quadrature with) the main pole axis (the direct, or d-, axis) and dividing the armature circumference into the North and South pole regions is called the *geometric neutral axis* or, more generally, the quadrature (q-) axis. Under the conditions shown, it is also the *physical neutral axis*, that is, the line passing through the points on the armature circumference where the magnetic induction is zero. The brushes are shown riding the armature, although actually they are in contact with the commutator. In the diagram they are set on the q-axis.

When the armature winding is energized, the armature becomes an electromagnet whose axis runs along the brush axis (Fig. 13.15 b). The armature field axis is in quadrature with the d-axis—so long as the brushes are set on the q-axis, the armature field

is the q-axis field. In present-day d.c. machines the brushes are set on the q-axis. If, however, the brushes are shifted away from the q-axis, there appears, in addition to the q-axis component, also the direct (or d-) axis component of the armature field, which has a demagnetizing effect.

When the machine is loaded, the armature-reaction field interacts with the main field and produces a resultant field (approximately shown in Fig. 13.15 c). The flux in the machine is shifted in the direction of rotation for a generator or against the direction

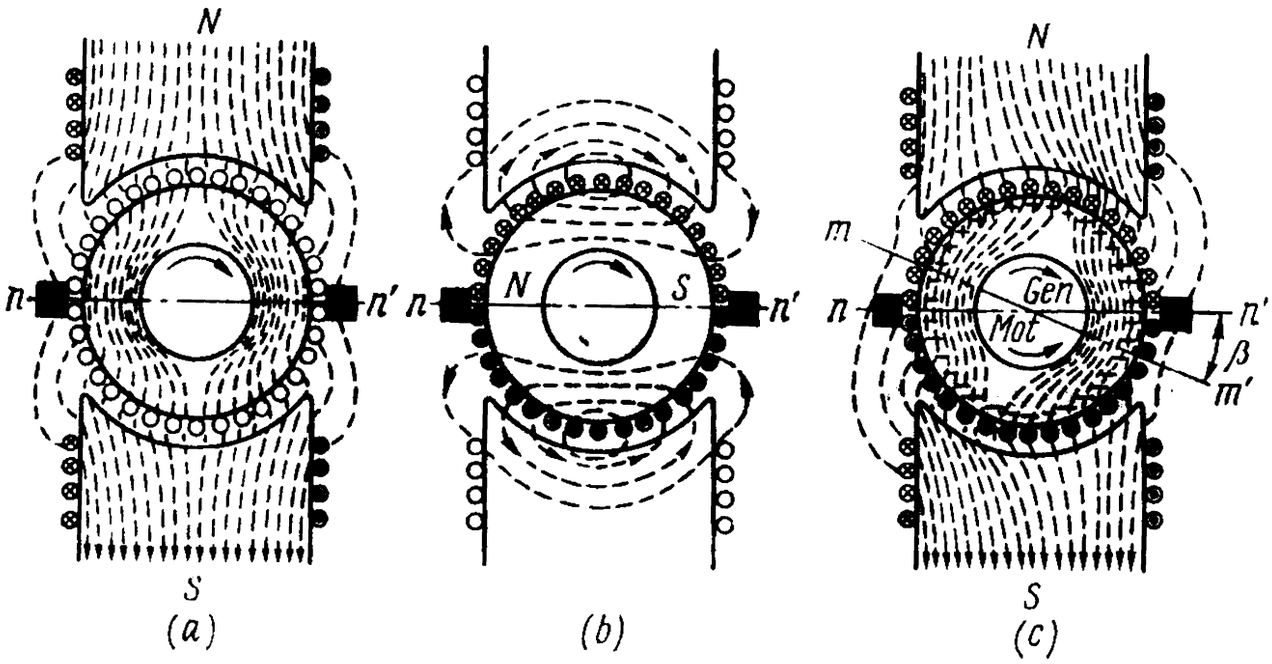


FIGURE 13.15

of rotation for a motor. As a result, the flux is distributed unsymmetrically about the pole axis: It is weakened at the one pole tip and it is strengthened at the opposite pole tip. At the same time the physical neutral line m-m' is shifted away from the geometric neutral line by an angle β, and the brushes are now off the physical neutral.

Let us see how the magnetic induction is distributed under the poles as a result of armature reaction. So long as the field is produced only by the main poles, it is symmetrical about the pole axis and is nearly uniform under the poles (curve 1 in Fig. 13.16). The armature winding is distributed among the slots made around the armature circumference, and the current in the armature winding produces an mmf which changes stepwise around the circumference. Since the number of slots is great, the stepped curve may be replaced with a straight line. The armature mmf is a maximum on the brush axis (curve 2). If we consider the armature field independently of the main field, the distribution of its magnetic induction will to a great extent be decided by the reluctance in the path of the armature flux. This reluctance is relatively small and constant around the

circumference under the poles, but grows abruptly between the poles. Now the distribution of the magnetic induction due to the armature field is shown by a saddle-shaped curve 3.

If the pole cores are not saturated when the machine is running, the machine field under load can be found by superimposing the armature field on the main field. This superimposition leaves unchanged the field that induces the armature emf, but its distribution around the armature circumference is changed (curve 4). Now

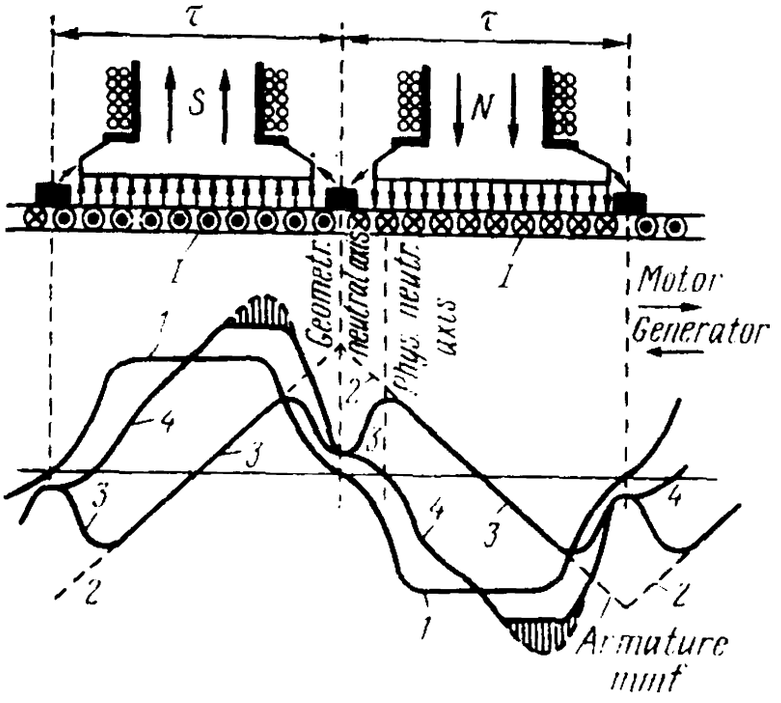


FIGURE 13.16

the physical neutral axis does not coincide with the geometric neutral axis. Since, however, the brushes are set on the geometric neutral axis, they are shifted away from the physical neutral axis due to armature reaction. Because of this shift, each parallel circuit between the brushes will contain several coils whose emfs will oppose those of the remaining coils, and this will weaken the armature emf, E_{arm}. For a generator, this causes a fall in the generator voltage; for a motor this causes a variation in the torque and rpm.

The distortion of the magnetic field under the poles is accompanied by a marked local increase in the magnetic induction. Since the instantaneous emf induced in each coil is proportional to magnetic induction, this local increase may raise the voltage between adjacent commutator bars so much (to over 30-50 V) that a stable arc is established. This phenomenon is called flashover.

So far we have not considered the effect of core saturation caused by armature reaction. The rise in magnetic induction at one pole tip may be so high that the armature and pole-shoe teeth within that region are saturated (the shaded area on curve 4 in Fig. 13.16).

As a result, the armature field weakens the main field at one side of the pole ($-\Delta B$, Fig. 13.17) and builds up the main field on the other side of the pole ($+\Delta B$, Fig. 13.17). In this way, armature reaction reduces the useful magnetic flux to which the armature emf is proportional.

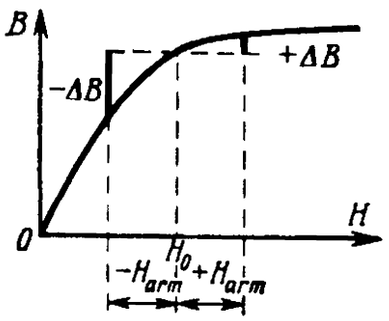


FIGURE 13.17

Armature reaction can be minimized by designing a machine with an increased reluctance in the path of the armature flux. This is done by increasing the air gap between the armature and the pole faces, and by choosing the cross-sectional area for the armature teeth so that a high value of induction is obtained in them. A further increase in magnetic induction leads to a higher reluctance which is equivalent to an increase in the air gap on the path of the armature flux. However, in order to maintain the desired useful flux in the machine as the reluctance is increased, one has to increase the main-pole mmf and, as a consequence, the size and weight of the machine.

Most present-day machines use *commutating* (or *intermediate*) poles (abbreviated to compoles or interpoles, respectively). Apart from improving commutation (see Sec. 13.7), they serve to prevent the shift of the physical neutral axis due to armature reaction. They are set on the geometric neutral axis, and their windings are connected via the brushes in series with the armature winding so that the direction of the field intensity due to the compoles is opposite to the direction of the field intensity due to armature reaction. In this way the compoles balance out the armature-reaction field inside the relatively narrow commutation zone and prevent the shift of the physical neutral axis away from the geometric neutral axis.

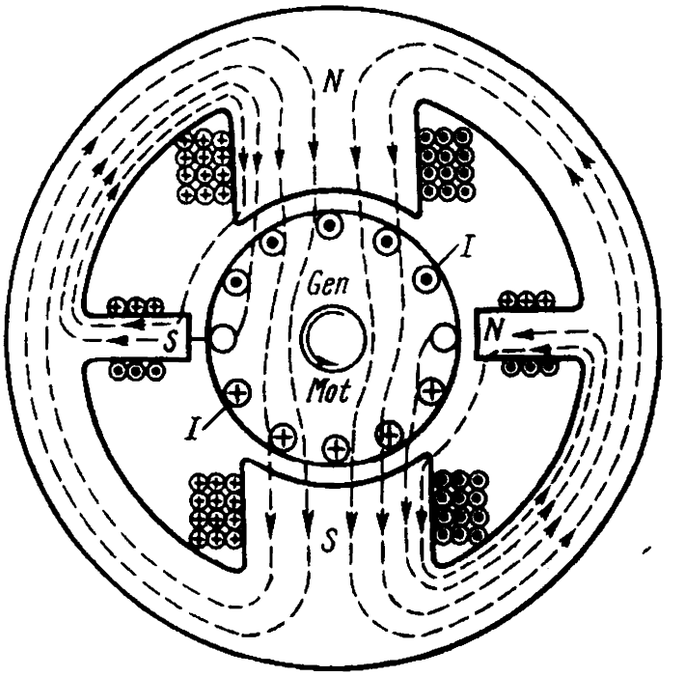


FIGURE 13.18

Since the field intensity due to the compoles is produced by the armature current, the extent of armature reaction compensation is adjusted automatically at any loading, the only requirements being that the compole cores should not be saturated.

For a generator, the compoles must be of the same polarity as

the main poles towards which the armature moves. For a motor their polarity must be that of the main poles from which the armature recedes (Fig. 13.18).

Compoles fail to prevent completely the nonuniform distribution of magnetic induction under the poles and the decrease in the useful flux caused by armature reaction. Because of this, especially in the case of large or heavy-duty machines (such as frequently reversed motors), an excessive local build-up in magnetic induction under the

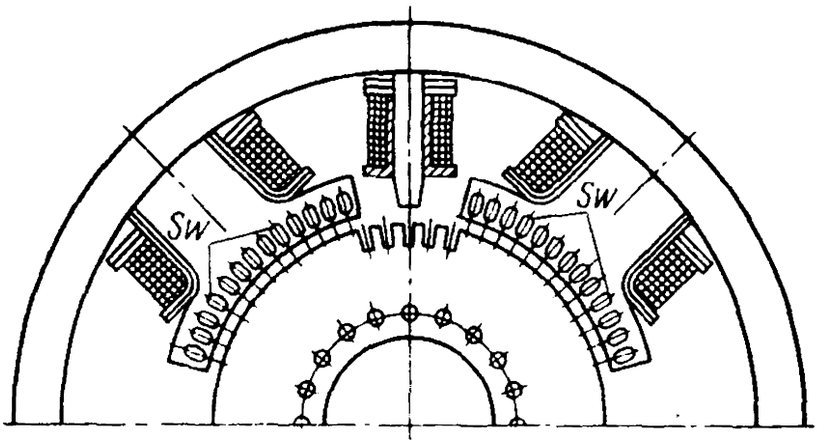


FIGURE 13.19

poles might cause a break-down of the air gap between the commutator bars and, as a consequence, a flashover. The only way to avoid this is to cancel armature reaction completely.

Although the armature rotates, its magnetic flux remains stationary relative to the yoke. Therefore, the effect of armature reaction can be completely cancelled by an opposing effect. This opposing effect is produced by a stationary compensating winding. This winding, *SW*, is a set of coils normally embedded in pole-face slots (Fig. 13.19) and connected in series with the armature winding so that each of its bars and the armature-winding bar below it make up a bifilar system setting up practically no magnetic field.

In fact, the compensating winding supplements the compoles, and they together cancel out armature reaction almost completely. Unfortunately, the use of a compensating winding makes the machine more expensive and its design more elaborate. For this reason, compensating windings are used only in extreme cases.

13.7 Commutation in D.C. Machines

If allowed to occur, sparking under the brushes leads to a rapid destruction of the commutator, because an electric spark has a very high temperature and is capable of severely damaging the hardest metals and alloys. Therefore, the prevention of sparking under the

brushes is an essential condition for a machine to have a long service life.

Sparking under the brushes may be caused by mechanical and electrical factors. The mechanical factors include a poor contact between the commutator and brushes because of a rough commutator surface or brush chatter in their boxes, etc. The electrical factors include poor commutation.

In the narrow sense, the term 'commutation' refers to the reversal of the current in an armature coil as it passes under a brush. In a wider sense, it refers to the processes accompanying this current reversal. Both senses can be better understood from reference to

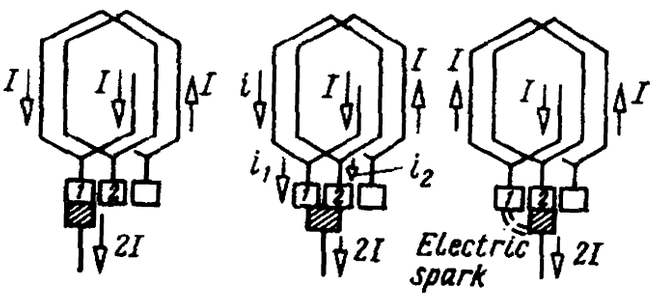


FIGURE 13.20

a simple lap armature winding (Fig. 13.20). The current in an armature coil must reverse during the short interval when the coil is short-circuited by a brush. This time interval is called the *commutation time T*. During this time the current in the short-circuited coil must change from $+I$ to $-I$, that is, by $2I$. As the speed of the armature is increased and the brush width b_b is decreased, the commutation time becomes progressively shorter

$$T = b_b/v_c \tag{13.3}$$

where v_c is the peripheral speed of the commutator.

If no emf were induced in the coil undergoing commutation, the commutation process would solely be decided by the relative magnitude of the contact resistance between a brush and the two commutator bars it overlaps, with one bar gradually moving from under the brush and the other moving, as gradually, under the brush. For simplicity, we assume that the brush width is the same as the commutator bar width and that the resistance of the conductors connecting the coil to the commutator and of the coil itself is negligible in comparison with r_b, which is the brush contact resistance. During commutation, a brush touches two commutator bars at a time, so there appear two contact resistances, each being greater than r_b, because the area of contact between the brush and a commutator bar is now smaller than it is when the brush resides on one bar, and the contact resistance is inversely proportional to the area of contact.

In consequence, the contact resistance, r_1, between the brush and the bar leaving it increases during commutation, tending to infinity at the end of the commutation time. On denoting this time as t, we get

$$r_1 = r_b T/(T - t)$$

At the same time, the contact resistance between the brush and the other commutator bar decreases

$$r_2 = r_b T/t$$

The brush current I is shared between the two commutator bars in inverse proportion to their contact resistances

$$i_1/i_2 = r_2/r_1 = (T - t)/t$$

By Kirchhoff's current law, the current in the coil being commutated is

$$i = i_1 - I$$

and the current in the external circuit during commutation is the sum of the two currents across the two brush contact resistances

$$2I = i_1 + i_2$$

Hence,

$$2I = i_1 T/(T - t)$$

$$i_1 = 2I\ (T - t)/T$$

and the current in the coil being commutated is

$$i = 2I\ (T - t)/T - I = I\ (T - 2t)/T$$

At the start of commutation ($t = 0$), this current is $i = I$. At the end of commutation ($t = T$), it is $i = -I$.

The current I is the current in only one parallel circuit of the armature winding. In the general case, with $2a$ parallel circuits the armature current will be $I_{arm} = 2aI$. Therefore, the current in the coil undergoing commutation is

$$i = (I_{arm}/2a)\ (T - 2t)/T \qquad (13.4)$$

Equation (13.4) shows that under the idealized conditions assumed the current in the coil being commutated is a linear function of time (see Fig. 13.21 a). Accordingly, this form of commutation is called *linear*. In this ideal case the current density is the same over the entire surface area of a brush and constant for the duration of commutation. In consequence, commutation cannot give rise to sparking.

We have considered the case where the brush width b_b is the same as that of a commutator bar. Actually, the brush is always

several bars wide. However, this does not affect the linear form of commutation so long as no emf is induced in the coil being commutated.

In reality, however, the situation is different. The coil turns are embedded in the armature core, so the coil has a substantial inductance L. For this reason, any change in current during commutation must induce an emf of self-induction, e_L, in the coil. To this emf we should add the emf of mutual induction, e_M, induced by changes in the current of adjacent coils commutated at the same time. By

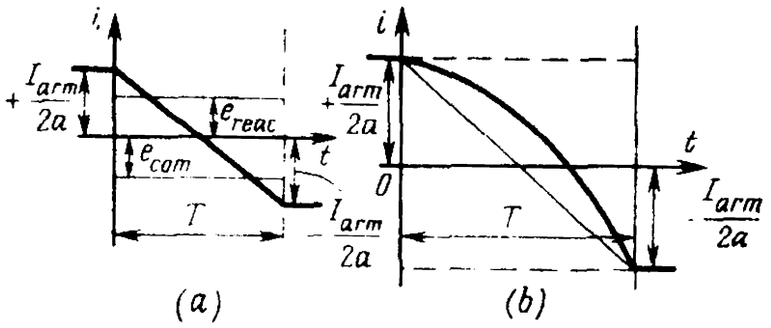


FIGURE 13.21

Lenz's law, the two emfs oppose the change in the coil current, and retard this change. The resultant emf induced by changes of current has come to be known as the *reactance emf* (or *voltage*)

$$e_{reac} = e_L + e_M$$

It can be expressed as the emf of self-induction, Eq. (2.2), in terms of the coil inductance, L_c, and the time rate of change of current

$$e_{reac} = -L_c di/dt$$

The reactance emf retards the change of current (see Fig. 13.21 b) and delays the commutation for the greater part of a period. Towards the end of a period, however, when the trailing edge of the brush comes off the commutator bar, the coil current is forced to take up the value $(-I_{arm}/2a)$. Thus, at the end of a period the change of current is inevitably speeded up, and this leads to an increase in the reactance emf. At the same time the current density under the brush becomes nonuniform, being substantially higher at the trailing tip of the brush, and this might cause severe heating of the brush and bar tips. Each factor may cause sparking—miniature electric arcs at the trailing edge of the brush.

The detrimental effect of the reactance emf can be minimized or even eliminated by counteracting it with a second emf, called the *emf of commutation*, e_{com}. Thus, the condition for ideal commutation is

$$e_{reac} = -e_{com} = 0$$

The commutation emf is generated as the turns of the coil undergoing commutation cut an external magnetic field. In most cases, e_{com} is induced by the magnetic field set up by the *commutating poles*, Φ_{com}. Therefore,

$$e_{com} = - w_c d\Phi_{com}/dt$$

As the loading on the machine changes, so does the reactance emf which is proportional to the armature current. But the commutation emf also changes to the same extent and at the same time, because the compole winding is connected in series with the armature winding. Hence, a change in load does not affect the balance of e_{reac} and e_{com}.

By adding more turns to the compole winding, it may be so arranged that $e_{com} > e_{reac}$, thus leading to *early commutation*. Early commutation is used in machines intended for exceptionally heavy duty involving frequent reversals.

In small-size machines the necessary commutating magnetic field is set up by shifting the brushes off the q-axis (the geometric neutral axis) rather than by use of compoles. As a result of the *brush shift*, e_{com} is induced in the coil being commutated by the main magnetic field. In a generator, the brushes must be shifted in the direction of armature rotation; in a motor, they must be shifted against the direction of armature rotation (see Fig. 13.15 c).

An extremely detrimental case of unfavourable commutation is the *flashover* when a strong arc strikes all the way round the commutator and may even jump over to the frame, causing a severe damage to the machine. The flashover occurs as follows. A sudden change in the current of the armature coil undergoing commutation leads to a rise in the reactance emf, and the latter initiates a strong arc between the commutator bar and the trailing edge of the brush. The arc is not extinguished, but is extended due to the rotation of the commutator, so that the commutator is shorted, and the short-circuit destroys both the commutator and brushes.

13.8 The Separately Excited Generator

In the case of separate excitation, the field circuit is electrically independent of the armature circuit (Fig. 13.22). Therefore the field (or excitation) current is independent of the terminal voltage of the generator and, in consequence, of loading. This makes it possible to regulate the magnetic flux and, hence, the generator voltage over a broad range. To this end the field circuit is extended to include a field rheostat, r_f. The field rheostat must be designed and connected so as to prevent any sudden interruption in the excitation current, such as when the supply source, E_f, is disconnected. The point is that the field winding has an appreciable inductance, so

an interruption of its circuit may be accompanied by a large emf of self-induction detrimental to the insulation of the field winding and by heavy sparking between the separating contacts.

The power taken by the field excitation circuit accounts for as little as 1-3% of the rated power, so the generator voltage can be controlled in a very economical way.

A change in the excitation current brings about a change in the main magnetic field and in the armature emf proportional to it, $E_{arm} = c_E \Phi n$. The dependence of the emf on the excitation current with the armature circuit open ($I_{arm} = 0$) and with the speed (rpm)

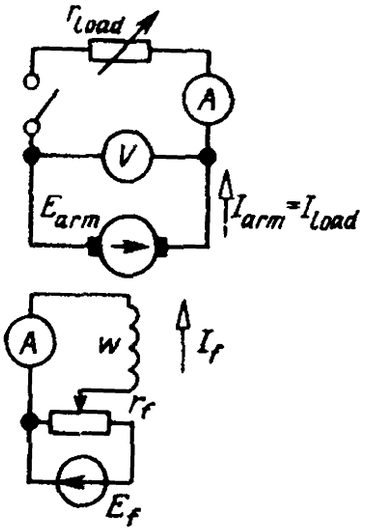


FIGURE 13.22

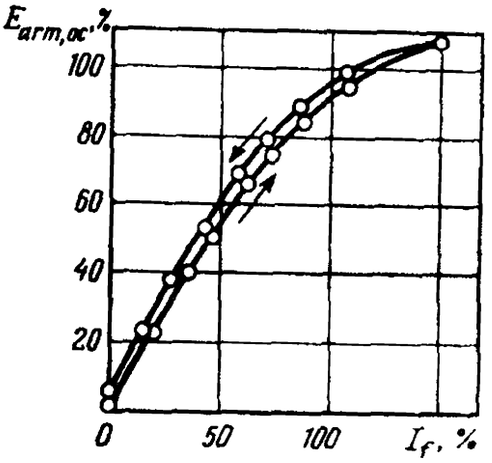


FIGURE 13.23

held constant, $n = \text{const}$, is called the open-circuit characteristic of a machine, $E_{arm,oc} = F(I_f)$. When drawn to a different scale, it can serve as the magnetization curve.

When the field circuit is open ($I_f = 0$), the frame of the machine retains some residual magnetic induction. When the armature rotates in the field due to the residual magnetic induction in the absence of armature current, a small open-circuit emf, $E_{arm,oc}$, is induced in the armature winding.

The open-circuit characteristic of a machine is plotted by applying a voltage to the field winding and gradually increasing the excitation current to its maximum permissible value while noting the values of emf. This produces the up-sloping portion of the characteristic (see Fig. 13.23). Then the excitation current is gradually reduced, and the corresponding values of emf are again noted and plotted to give the down-sloping part of the characteristic. It usually lies somewhat above the up-sloping part due to hysteresis. If, now, the excitation current is turned off, the residual magnetic induction and the corresponding emf will be greater than they were initially. However this residual magnetic induction is not stable and a short time after the excitation current is turned off, the flux and the emf

it induces as the armature rotates go down to their initial values.

Thus, the open-circuit characteristic is the shape of a loop which is a natural consequence of hysteresis (see Fig. 6.5 b) in the magnetic circuit of the machine, mainly its substantial frame. At the top the open-circuit curve has a smaller slope due to saturation in the magnetic circuit of the machine. In most cases d.c. generators operate within the saturation region of the open-circuit characteristic. Owing to saturation the generator voltage is less subject to variations with changes in loading, because the effect of armature reaction on the emf of the machine is reduced.

Another important characteristic of a d.c. generator is the dependence of the terminal voltage V on the armature current I_{arm}, with the field current I_f and the speed n held constant. This dependence is called the *load* or *external characteristic* (Fig. 13.24). Experimentally it is plotted as follows. The generator is first loaded to the rated current I_{rtd}, with the terminal voltage held at its rated value. The terminal voltage is set at its rated value by adjusting the field current I_f. Then the armature current is gradually decreased to zero. In the circumstances, the field current of a separately excited d.c. generator will remain unchanged.

The terminal voltage of a separately excited d.c. generator is somewhat smaller than its emf due to the voltage drop across the resistance, r_{arm}, of the armature winding and the brush contact resistance

$$V = E_{arm} - r_{arm} I_{arm}$$

If the armature emf, E_{arm}, remained constant, the V (I_{arm}) plot would yield a straight line (shown dashed in Fig. 13.24). However, as load is increased, the useful magnetic flux of the machine decreases mainly due to armature reaction, and so does the proportional armature emf, E_{arm}. As a result, the external characteristic bends towards the axis of abscissae. On the basis of the external characteristic, it is convenient to find the rated voltage regulation of the machine. It is defined as

$$\Delta V_{rtd} = \frac{V_{oc} - V_{rtd}}{V_{rtd}} \times 100\% \qquad (13.5)$$

that is, as the final change in voltage when the specified load is reduced gradually to zero, expressed as a percent of rated-load voltage, the speed and the excitation current being held constant.

For generators without a compensating winding, the voltage regulation ΔV_{rtd} is about 5-15%.

If we increase the armature current over and above its rated value, the generated voltage will keep falling, but the machine will be overloaded. Its armature winding, brushes and commutator will be heavily heated and a severe sparking will occur under the brushes.

For the normal operation of loads it is essential to maintain the voltage of each of them at a constant value despite variations in the overall loading. This is effected by controlling the field current. The dependence of the field current I_f on the armature current I_{arm}, with the terminal voltage V and the speed n being held constant, is

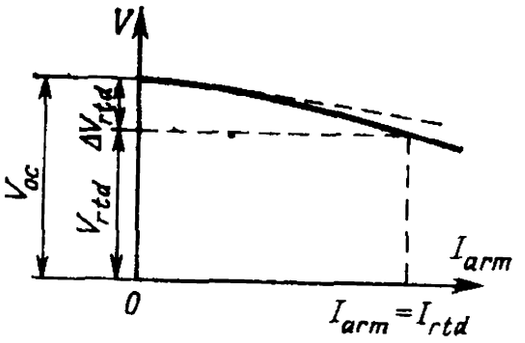


FIGURE 13.24

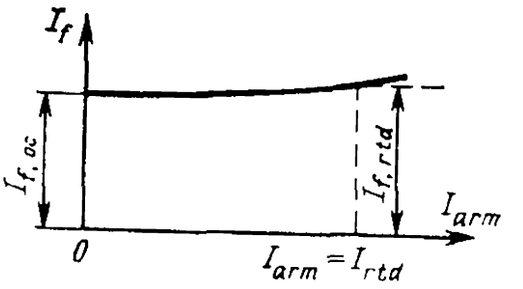


FIGURE 13.25

called the *control characteristic* of a separately excited d.c. generator. This characteristic shows how the field current should be varied in order to keep the terminal voltage constant despite variations in load. At first this curve is nearly straight (Fig. 13.25), but finally bends upwards due to the saturation of the magnetic circuit of the machine.

13.9 The Process of Self-Excitation

In most d.c. generators the field current is part of the armature current. At starting there is no current in the armature and, as a consequence, in the field winding, but the substantial frame always retains some residual magnetic flux Φ_r amounting to 1-3% of the normal operating magnetic flux of the machine. When a prime mover begins to rotate the armature, the residual flux induces a small emf in the armature winding. In a shunt generator this emf, $E_{arm\,oc}$, gives rise to a current, i_f, in the field winding, and there appears an excitation mmf. It may act aiding or in opposition with respect to the residual magnetic flux Φ_r, that is, it may magnetize or demagnetize the magnetic circuit of the machine. For self-excitation to take place, it must act aiding, and this requirement is fulfilled by properly connecting the field winding to the armature winding. With such a connection, the field set up by the excitation current will augment the magnetic field of the machine, and the resultant field will induce a large emf in the armature winding. The increase in armature emf leads to a further increase, or build-up, of excitation current. This spontaneous build-up of flux and current is limited by the saturation of the magnetic circuit of the machine.

After the transient build-up process is completed, the armature emf, E_{arm}, and the field current, I_f, take on their steady-state

values. They can be found by reference to the open-circuit characteristic of the machine (Fig. 13.26). If we neglect the resistance of the armature circuit, r_{arm}, in comparison with that of the field circuit, r_f, the steady-state field current I_f will be such that $E_{arm} = r_f I_f$. This equality corresponds to the intersection of the open-circuit characteristic $E_{arm}(I_f)$ and the straight line $E_{arm} = r_f I_f$, that is, the balancing point A. The slope of the $E_{arm} = r_f I_f$ line, defined as the tangent of the angle α, depends on r_f. If we reduce I_f by, say, bringing the field rheostat in the field circuit, the balancing point will shift to the left (A'). When the resistance of the field circuit is at its critical value, the machine will not build up at all.

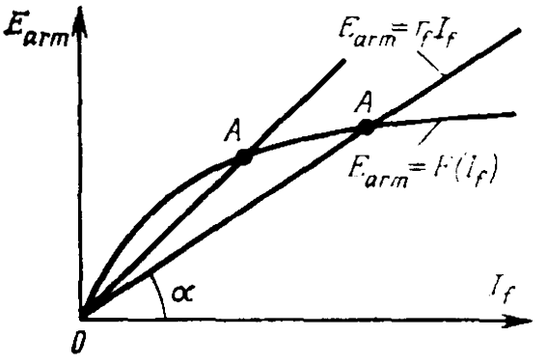


FIGURE 13.26

If no residual magnetization has remained in a machine (because its armature winding has been shorted or the machine itself was jarred), it can be restored by connecting an external d.c. source to the field winding of the demagnetized machine for a short time (so as to "flash the field" of the machine). Once restored, the residual magnetic flux can be used for normal self-excitation.

Self-excitation is used in shunt and compound generators.

13.10 Shunt, Series and Compound Generators

In a *shunt generator*, a fraction of the armature current serves to excite the main magnetic field of the machine (Fig. 13.27). Shunt generators are widely used as sources of direct current as they need no external source for excitation, which fact markedly simplifies their use. On the other hand, their terminal voltage varies but little with changes in loading.

The *open-circuit characteristic* of a shunt generator does not practically differ from that of a separately excited machine, because variations in $r_f I_f$ and armature reaction due to the field current have but a negligible effect on the behaviour of a shunt generator. This statement is also true of the control characteristic.

In contrast, the *external characteristic* of a shunt generator (Fig. 13.28 a) runs far below than it does in the case of separate excitation (Fig. 13.28 b). This is due to the decrease in the field current with a decrease in the terminal voltage because $I_f = V/r_f$. In the case of separate excitation, the decrease in the terminal voltage with a rise in the armature current is due to two factors: the increase in the voltage drop across the armature resistance and armature reaction. In the case of shunt excitation, a third factor has to be

added to the first two, namely the decrease in the field current. So long as the field current corresponds to the condition of saturation of the magnetic circuit (the gradually sloping part of the magnetization curve), the decrease in the armature emf is smaller than the decrease in the field current (Fig. 13.29). In these conditions, a decrease in the load resistance is accompanied by a rise in the armature current. The conditions drastically change, however, when the increase in the armature current and the associated decrease in the terminal voltage result in such a reduction in the field current that the magnetic circuit is no longer saturated. Now the linear portion of the magnetization curve applies, and the decrease in the field current brings about a proportionate decrease in the flux and armature emf. This brings about a further decrease in the field current which leads to a new decrease in the emf, and so on. In a way, the generator experiences self-demagnetization which ends in that, with the armature short-circuited, the machine only retains a residual magnetization sustaining a limited short-circuit current (which is smaller than the rated current).

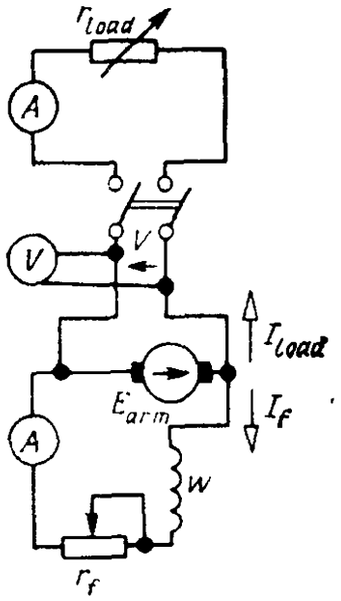


FIGURE 13.27

The armature current at which a machine experiences self-demagnetization is called *critical*, I_{cr}. It is 2 to 2.5 times the rated current. The portion of the external characteristic below I_{cr} (shown dashed in Fig. 3.28) corresponds to an unstable mode of operation.

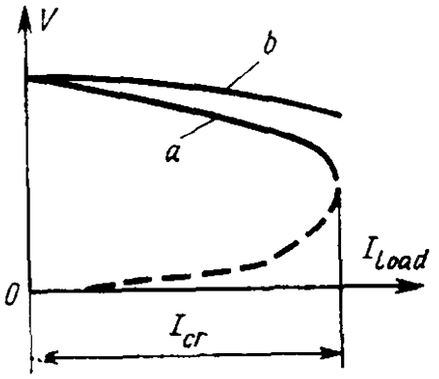


FIGURE 13.28

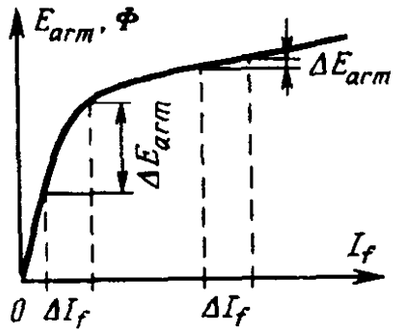


FIGURE 13.29

The voltage regulation of shunt generators is markedly greater than that of separately excited machines, being equal to 8-15%.

In a *series generator* the armature winding is connected in series with the field winding, so the load current is at the same time the field current (Fig. 13.30). The field winding of a series generator

is wound with a heavy-gauge wire to suit a heavy armature current, and with a small number of turns.

In the open-circuit condition the emf induced in the armature winding of a series generator is solely due to the residual flux. Therefore, we cannot plot an open-circuit characteristic for such a machine. Nor has it a control characteristic.

The voltage of a series generator (Fig. 13.31) initially rises with increasing armature current. Then the shape of the characteristic

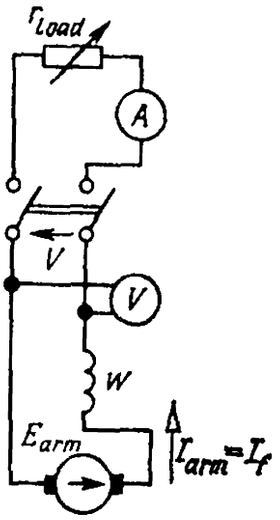


FIGURE 13.30

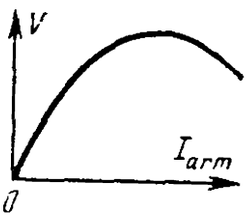


FIGURE 13.31

changes due to magnetic saturation (the armature emf ceases to build up, while the voltage drop across the armature resistance keeps rising) and to the demagnetizing effect of armature reaction. Therefore a further increase in load brings about a decrease in the terminal voltage of the generator. Because of an unstable terminal voltage, series generators are used only in a few special cases.

A *compound generator* has two field windings, one shunt w_{sh} and the other series w_{ser} (Fig. 13.32). The terminal voltage of a compound generator remains practically constant as the load varies within certain limits. This is achieved through the use of series excitation in order to make up for the increase in the voltage drop across the armature resistance, the decrease in the current of the shunt field winding, and the demagnetizing effect of armature reaction when the load current goes up. Owing to the series field winding, the main magnetic flux and also the armature emf go up as the load is increased. By choosing the right number of turns in the series field winding, it is possible to secure the same voltage in the open-circuit condition (at no load) and under the rated load (curve *a* in Fig. 13.33).

Compound generators are convenient to use in relatively small units as a means of preventing marked changes in voltage when some of the loads are disconnected. However they are not well adapted

for parallel operation because even a chance drop in the speed of the prime mover may cause the generator emf to fall to a level below

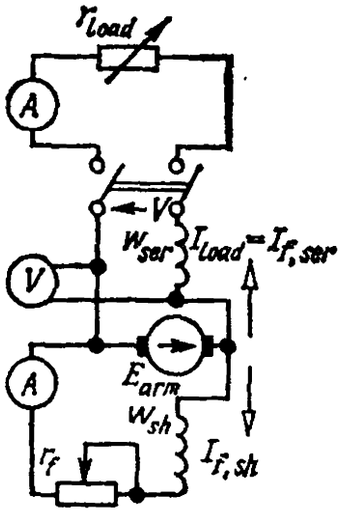


FIGURE 13.32

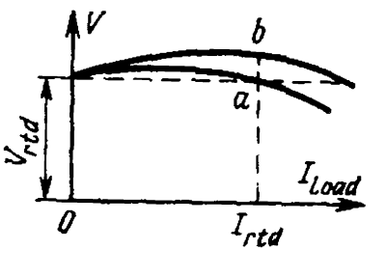


FIGURE 13.33

that of the line voltage so that the current in the armature and series field windings is reversed entailing a polarity reversal of the generator and a heavy damage to the installation as a whole.

13.11 Parallel Operation of Shunt Generators

In practice cases often occur where it is advantageous to feed loads from two or more generators operating in parallel. Let us see how this is done, taking the simplest example, that of shunt generators.

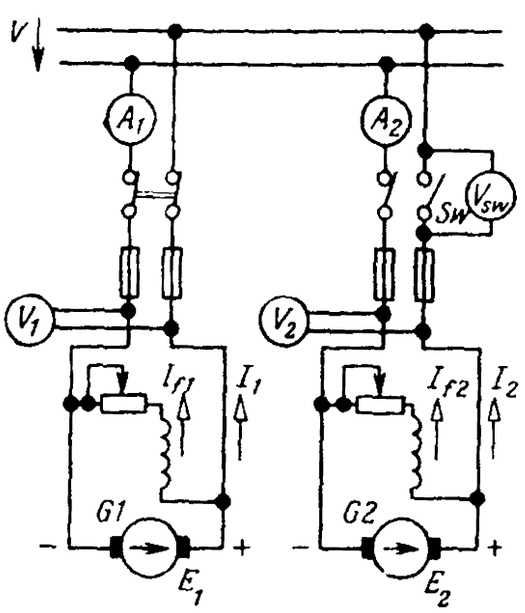


FIGURE 13.34

Suppose we are to bring a second generator (Fig. 13.34) into a line where a first generator, G_1, maintains a voltage V. The procedure is as follows. First the armature of the incoming generator is brought up to the speed of the first generator with the aid of a prime mover (a turbine, a Diesel engine, etc.). Now the field current, I_{f_2}, of the second generator, G_2, is adjusted so that its emf, E_2, is equal to the line voltage. Then a check is made to see that the generator has the same polarity as the line by reference to a voltmeter V_{sw}: if the voltmeter reads zero, the single-pole switch *Sw* may be closed, thus bringing the second generator in.

Since the generator emf is balanced by the line voltage, its current after switch-on is

$$I_2 = (E_2 - V)/r_{\text{arm},2} = 0$$

The load on the second generator is raised by increasing its field current. This leads to an increase in the emf E_2 of the second generator, with the result that this, firstly, raises the load current of the second generator to I_2 and, secondly, raises the line voltage V. The emf of the first generator which carried all the load remains unchanged, so the increase in the line voltage leads to a reduction in the load on the first generator.

On interacting with the magnetic field set up by the poles, the armature current I_2 of the second generator produces a braking torque which causes the generator to slow down. In response, the speed governor of the second generator increases the amount of working fluid (steam, water, oil, etc.) fed to the prime mover, and the second generator is restored to its specified speed. In this way, the second generator and its prime mover have taken over their share of the total load on the line. The situation is reversed for the first generator, $G1$, in which the reduction in I_1 unloads its prime mover.

If it is required to shift all of the load onto the second generator, it will suffice to reduce the excitation of the first gradually and to increase the excitation of the second, while maintaining the line voltage V at a constant value. When the emf of the first generator becomes equal to the line voltage, its current will drop to zero (which is another way of saying that it is carrying no load any longer), and it may be disconnected from the line. The speed governors of the prime movers will assist in load shifting.

If we leave the first generator connected to the line and keep reducing its field current, its armature current will be reversed

$$I_1 = (V - E_1)/r_{\text{arm},1}$$

Now, a driving rather than braking torque will be developed, and the machine will begin running as a motor. Since this might cause a heavy damage to the prime mover, all generators operating in parallel are fitted with a reverse current relay which automatically disconnects the associated generator just as its current undergoes a reversal.

Thus, by varying the excitation of paralleled generators, we can divide the load between them in any ratio.

Let us see how the load is divided between two generators whose external (load) characteristics differ in slope (Fig. 13.35). Suppose that initially their excitation was adjusted in such a way that the two generators were loaded identically, and their load curves intersected at the balancing point a for a given line voltage V. If, now, we increase the load current, the current in each generator will al-

so go up, and so will the voltage drop across the armature winding resistance and armature reaction. In consequence, the line voltage will go down by ΔV. However, as can be seen from the load characteristics, this reduced voltage

$$V' = V - \Delta V$$

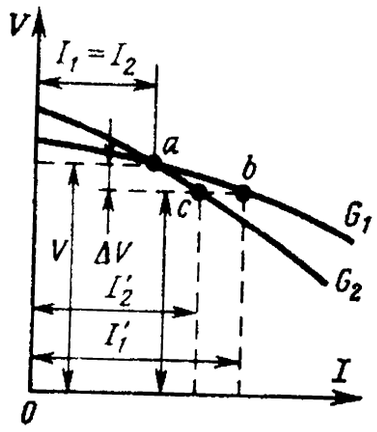


FIGURE 13.35

corresponds to different values of current, I_1' and I_2'. This means that the load is no longer shared equally between the two generators: the generator having the flatter load characteristic takes a greater share of the load, I_1'. This inequality is easy to correct (by, say, increasing the excitation of the second generator), however this complicates the duty of attending personnel. In fact, the correction is difficult to accomplish if the variations are rapid. For this reason it is required that the generators to be used for parallel operation should have identical external characteristic or that suitable automatic control should be provided.

13.12 Motor Action

Owing to the identity of generator and motor structure, a generator may be made to operate as a motor. This is especially simple to do in the case of a shunt generator operating into a d.c. line. In this case, it will suffice to bring down the field current so that the armature emf becomes smaller than the line voltage. As a result, the greater line voltage will cause a reversal of the armature current I_{arm} which will now be produced by the difference between the line voltage and the armature emf

$$I_{arm} = (V - E_{arm})/r_{arm} \qquad (13.6)$$

By interacting with the magnetic field of the machine, this current will produce a driving rather than a braking torque. Owing to the driving torque, the armature will no longer need a prime mover for its rotation, and it can be disengaged. In this way, an electric machine can be switched from generator action into motor action.

Of course, proceeding in reverse, we can switch a shunt-wound machine from motor action into generator action, provided there is a prime mover available to drive it. It should be stressed, however, that such a transition from generator action to motor action without making any changes in internal connections applies only to shunt-wound machines. In the case of a series-wound machine, a transition from motor to generator action necessitates the reversal of current in the armature or field winding.

Figure 13.36 shows how power is distributed in a motor. The power P_1 taken from a supply line is divided between the armature circuit, P_{arm} (which is the greater portion of the total), and the field circuit, $P_f = VI_f$ (which amounts to a few percent of the total). A small fraction of the power supplied to the armature circuit is lost as heat generated in the armature winding, and the remainder is converted to mechanical power P_{mech}. By subtracting from it the core (or iron) losses P_c (due to hysteresis and eddy current), and the friction and windage losses $P_{f,w}$ (in the bearings, at the commutator, etc.), we obtain the useful power available at the motor shaft, P_2.

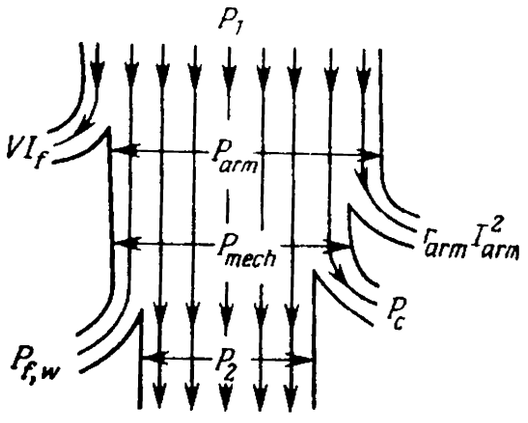


FIGURE 13.36

Special mention should be made of the part played in a motor by the armature emf, E_{arm}. Since its direction is opposite to that of the current, it is called the *counter-emf*. On the basis of Eq. (13.6), the voltage across the armature terminals is

$$V = r_{arm}I_{arm} + E_{arm}$$

This equation can be converted into a power equation if we multiply its left- and right-hand sides by I_{arm}

$$VI_{arm} = P_{arm} = r_{arm}I^2_{arm} + E_{arm}I_{arm} = r_{arm}I^2_{arm} + P_{mech}$$

because

$$VI_{arm} - r_{arm}I^2_{arm} = P_{mech} = E_{arm}I_{arm}$$

is the mechanical power developed by the machine.

As follows from the power equation, the power in the armature circuit is the sum of the copper loss, or the power lost as heat, $r_{arm}I^2_{arm}$, and mechanical power. The latter is directly proportional to the counter-emf E_{arm}. The presence of counter-emf is typical of the conversion of electric into mechanical energy by an electromagnetic device. Hence we may conclude that the higher the counter-emf E_{arm}, the higher the efficiency of a motor. For this reason, the armature voltage $r_{arm}I_{arm}$ during operation of medium and large motors is a few percent (2-5%) of the rated voltage.

We have examined how a machine can be brought out of generator into motor action. In most cases, however, an electric machine intended to operate as a motor is caused to do so by suitably starting it, for which purpose it is connected to a d.c. line. In the circumstances, no emf is induced and $V = r_{arm}I_{arm}$ so long as the armature remains stationary ($n = 0$). The armature winding resistance r_{arm} is relatively constant and small, so, unless measures to the

contrary are taken, the armature current I_{arm} at starting would be 25-40 times the operating current in the steady state. Such a starting current would be disastrous for the commutator, armature winding, and line. It is avoided by connecting a field rheostat, r_f, in series with the armature winding in all d.c. motors except those rated at not over 0.25 kW.

In the steady state, the speed of a d.c. motor, Eq. (13.1), is inversely proportional to the main magnetic field Φ and directly proportional to the terminal voltage $V \approx E_{arm}$. Therefore it can be adjusted by varying either the magnetic flux or the voltage across the armature.

A d.c. motor can be reversed by reversing the current in the field circuit or the armature circuit of the machine. If the current is reversed in the two circuits at the same time, the machine will keep rotating as before.

The performance of a d.c. motor depends on how its main magnetic field is excited.

13.13 The Shunt Motor

In a shunt motor, the armature circuit is connected in parallel with the field circuit. Therefore, the field current in a shunt motor is independent of the armature current and the load on the motor.

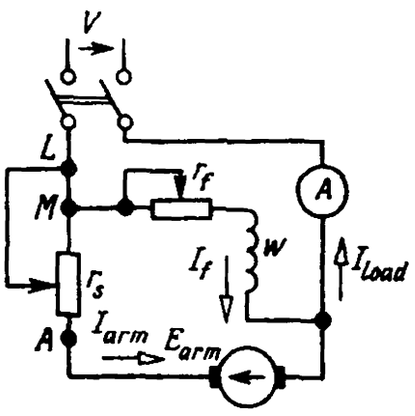


FIGURE 13.37

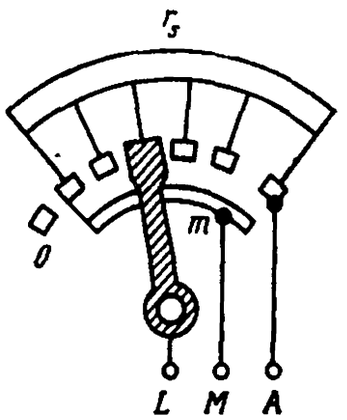


FIGURE 13.38

The basic form of connection for a d.c. motor to a starting rheostat, r_s, and a field rheostat, r_f, is shown in Fig. 13.37. The function of the starting rheostat is to limit the armature current at starting so long as the armature emf is zero or very small because, in accord with Eq. (13.6),

$$I_{arm} = (V - E_{arm})/(r_{arm} + r_s)$$

The armature emf E_{arm} is directly proportional to the magnetic flux Φ. It is desired that at starting the armature emf E_{arm} builds up as fast as practicable. For this reason, the shunt field winding

should receive full line voltage at starting, with the field rheostat r_f brought out of circuit. The driving torque developed by a motor, Eq. (13.2), is likewise directly proportional to the magnetic flux Φ. Therefore when a motor is started at a maximum flux, its torque will also be a maximum, and this will simplify the starting procedure, because the starting current may also go down at a proportionately faster rate.

The required connection of the shunt field winding is shown in Fig. 13.38. The starting rheostat has a metal arc m held in contact with a sliding arm attached to the rheostat knob. So long as the knob is positioned at the dummy contact 0, both the armature circuit and the field circuit of the motor are deenergized. At starting, when the knob is moved from contact to contact, the field winding is receiving full line voltage all the time.

The first contact of the starting rheostat is connected to the metal arc m, so when the motor is turned off, the field winding is shorted through the armature and the starting rheostat. This prevents the occurrence of a large emf which could otherwise be generated as the field circuit is opened, because the field winding has a high inductance. The closed loop containing the starting rheostat converts the energy stored in the magnetic field of the field winding into heat when the motor is stopped. The steps of the starting rheostat are designed so that at normal starting the armature current is not more than twice the rated armature current. When the motor is running, its armature current is $I_{arm} = (V - E_{arm})/r_{arm}$.

Since $E_{arm} = c_E \Phi n$, it follows that

$$n = (V - r_{arm} I_{arm})/c_E \Phi \tag{13.7}$$

which is the speed equation for a shunt motor.

The driving torque developed by a shunt motor can be deduced from the general equation of electromagnetic torque, Eq. (13.2), for a d.c. machine

$$T_d = c_T \Phi I_{arm} \tag{13.8}$$

The driving torque balances the braking torque applied to the shaft (the static moment).

As the load on the motor shaft is increased, the torque balance is upset at first, and the motor slows down a little. However, this brings about a reduction in E_{arm} and, in consequence, an increase in I_{arm}. The driving torque builds up in proportion to I_{arm}, and the torque balance is restored at a somewhat reduced speed, n. On the basis of Eqs. (13.7) and (13.8), the dependence of n on torque may be written as

$$n = (V/c_E \Phi) - T_d r_{arm}/c_E c_T \Phi^2 \tag{13.9}$$

Armature reaction may affect the performance of a motor to some extent. As the armature current I_{arm} increases, the main magnetic flux Φ decreases (especially so if the motor has no compoles). According to Eq. (13.9), a reduction in the magnetic flux causes the speed to rise, and so it opposes the reduction in speed caused by the increase in $r_{arm}I_{arm}$. On the other hand, the reduction in the magnetic flux reduces the driving torque, so that a greater armature current is needed in order to maintain the torque balance.

Most shunt motors have a compensating winding wound with a small number of turns and intended to cancel out the effect of armature reaction on the main magnetic flux. The compensating winding is connected aiding with the shunt field winding, and has practically no effect on the characteristics of the motor. For this reason, motors fitted with a series compensating winding are classed as shunt-wound, although actually they use compound excitation. In some special applications, however, this compensating winding must be considered.

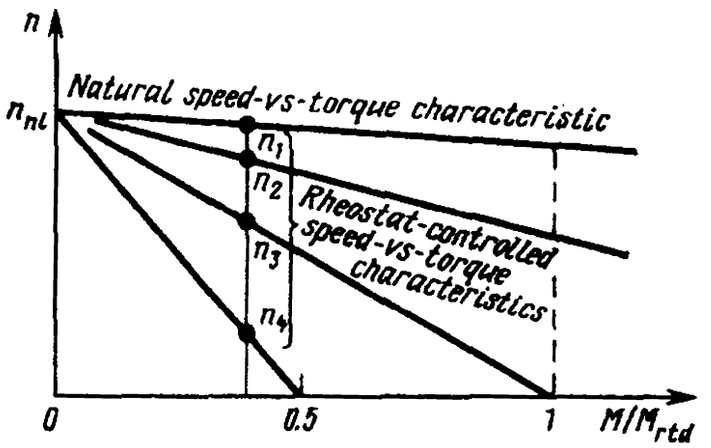


FIGURE 13.39

Thus, if we deem the magnetic flux Φ constant, then Eq. (13.9), when plotted, will yield a straight line slightly bent towards the axis of abscissae (Fig. 13.39). This is the *natural speed-vs-torque characteristic* of a shunt motor. As the load at the shaft varies from zero to full load, the speed of most shunt motors drops by as little as 3-8% (the drop decreases with increasing power rating of the motor). Therefore the natural speed-vs-torque characteristic of a shunt motor may be taken as a flat one.

The slope of the speed-vs-torque characteristic can be varied by placing a rheostat, r, in series with the armature winding. Then the speed equation will take the form

$$n = \frac{V-(r_{arm}+r)\,I_{arm}}{c_E\Phi} \qquad (13.10)$$

Equation (13.10) defines the speed-vs-armature current characteristic of a shunt motor. By varying the setting of the rheostat r, we can obtain a family of *rheostat-controlled speed-vs-torque characteristics* which are more drooping than the natural speed-vs-torque characteristics. All of these characteristics cut the axis of ordinates at the same point located so that $I_{arm} = 0$ or $E_{arm} = c_E n_{nl}\Phi = V$, where n_{nl} is the armature speed with the motor running under ideal

no-load conditions, that is, when the braking torque applied to the shaft is zero. Since the friction in the moving parts of a motor always produces a braking torque, an ideal no-load condition can only be obtained by applying an external driving torque to the shaft from an auxiliary motor.

If, with the torque at the shaft held constant, we gradually increase the resistance of the armature rheostat, the point n indicating the speed of the motor on the diagram of Fig. 13.39 will move from one curve to another (points n_1 through n_4). Thus, by varying the setting of the armature rheostat, we can control the speed of the motor. However this form of speed control is uneconomical, because the current flowing through the armature rheostat is equal to the armature current, I_{arm}, so the rI^2_{arm} loss is considerable. For this reason, armature rheostat speed control is only used on small motors. In fact, the armature rheostat controls the voltage across the armature winding. An advantage of this form of speed control is that the armature current and the torque developed by the motor remain constant. The situation typical of most drives is a constant load torque and a varying speed. Therefore, when a motor is used in such a drive, speed control by change of voltage will not cause any change in the armature current, and the armature winding will not be overloaded. For this reason, this form of control is called *constant-maximum-torque speed control.*

As follows from Eq. (13.10), the speed n is inversely proportional to the main magnetic flux Φ, and this flux may be deemed directly proportional to the field current I_f so long as the magnetic circuit is not saturated. Hence, the speed of a motor can be controlled by varying its field current for which purpose a field rheostat, r_f, is inserted in the field circuit. The speed-vs-armature current characteristic is hyperbolic in shape. In Fig. 13.40, the characteristic is shown for a motor running at no load.

Speed control at a constant braking torque proceeds as follows. An increase in the resistance of the field rheostat brings down the field current, with the result that the magnetic flux Φ and the armature emf E_{arm} are decreased. The reduction in E_{arm} leads to an increase in the armature current

$$I_{arm} = (V - E_{arm})/r_{arm}$$

and also in the driving torque and the speed of the motor. This serves to restore both the torque balance and the electric balance

$$V = E_{arm} + r_{arm}I_{arm}$$

at a higher speed and at a larger armature current. Thus, when the torque is held constant, an increase in the speed of the motor by weakening its magnetic flux brings about an increase in the armature

current and leads to an overload at the commutator and in the armature winding.

The situation is different when the driven machinery calls for a constant output power, P_{load}. Since

$$P_{load} = T\omega = c_T \Phi I_{arm} n\pi/30 = k\Phi I_{arm} n$$

a decrease in the magnetic flux will bring about an increase in the speed and a decrease in the torque while the armature current remains constant. Thus, speed control by varying the field current is advantageous when the output power remains constant. This form of control is called *constant-maximum-power speed control.*

The slope of the speed-torque characteristics of a motor is different for different levels of excitation (Fig. 13.41). Given a constant

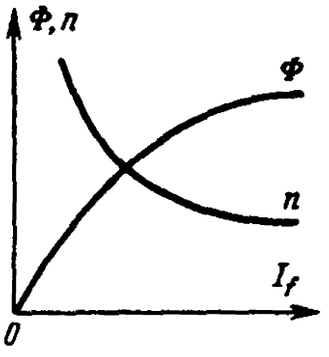


FIGURE 13.40

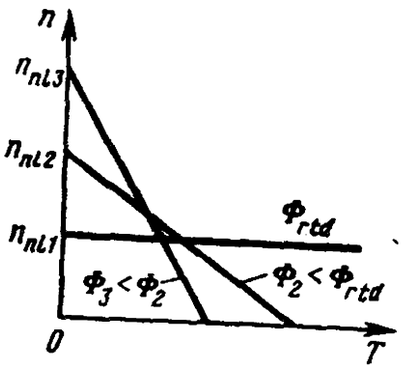


FIGURE 13.41

torque, $T = c_T \Phi I_{arm}$, a decrease in the magnetic flux necessitates a proportionate increase in the armature current. In consequence, a change in load brings about a greater change in the speed, $n = (V - r_{arm} I_{arm})/c_E \Phi$. In other words, a reduction in the magnetic flux makes the speed-torque characteristic progressively more drooping.

Since the field current is a mere 2 or 3% of the armature current, the additional $r_f I_f^2$ loss in speed control by weakening the magnetic flux is relatively small, so this form of speed control is fairly economical. However, an excessive increase in speed may produce an overload current in the commutator and armature, impair the conditions of commutation, give rise to dangerous centrifugal forces in the armature, and so on. For this reason, general-purpose shunt motors are built with speed control in the ratio 2 : 1 as a maximum. Greater ratios of speed control are possible only if suitable changes are made in motor design.

A wider range of speed control and non-rheostat starting for shunt motors is provided by use of a controlled-voltage source which may be a d.c. machine or a solid-state controlled rectifier. This form of control is especially advantageous for heavy duty, such as rolling mills.

If a d.c. generator is used, the system is called a Ward-Leonard system. As is seen from Fig. 13.42, it consists of four machines: an induction (or synchronous) motor, IM, which drives at constant speed a large d.c. generator, G, and a small shunt generator which serves as an exciter, Ex, for the package. The armature circuit of the generator is connected across the armature circuit of the controlled motor, M. The field circuits of the generator and motor are energized by the exciter. The speed of the main motor is controlled by controlling solely the field circuits of the machines. The generator voltage is controlled with a rheostat, $r_{f,g}$, placed in the field circuit of the generator. The rheostat has two contact arms and is connected as a potentiometer, so it can be used to reverse the field current of the generator. Reversal of the field current reverses the polarity of the voltage at the generator terminals and causes the motor to reverse as well.

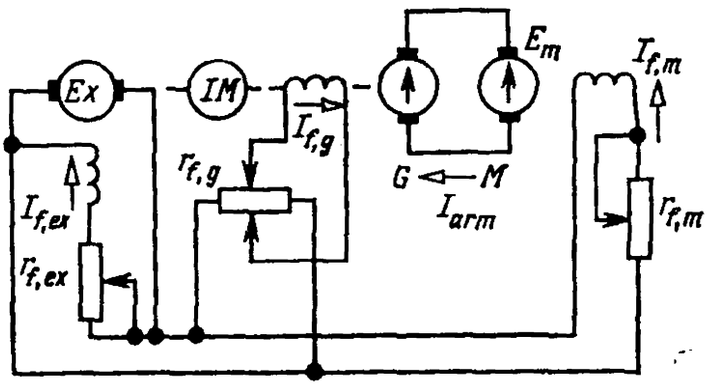


FIGURE 13.42

When starting the motor, the generator voltage is brought down by adjusting the generator excitation.

For a medium-sized installation, the Ward-Leonard system can control speed in the range from 5 : 1 to 10 : 1. The speed-vs-torque characteristics within that range are parallel straight lines (Fig. 13.43), but they make a greater angle with the axis of abscissae than the natural speed-torque characteristic of a motor. The reduced flatness of the speed-torque characteristic stems from the reduction in the generator voltage with increasing load due to the effect of the internal armature resistance, $r_{\text{arm, g}}$, of the generator. Hence, the speed equation for the motor in a Ward-Leonard system takes the form

$$n = \frac{E_g - (r_{\text{arm, g}} + r_{\text{arm, m}}) I_{\text{arm}}}{c_E \Phi} = \frac{E_g}{c_E \Phi} - T \frac{r_{\text{arm, g}} + r_{\text{arm, m}}}{c_E c_T \Phi^2} \tag{13.11}$$

The reduction of the braking torque to zero corresponds to an ideal no-load condition. If we keep applying an external torque ($T < 0$) to the motor shaft, the motor will change from the motor to the generator mode of operation.

The range of speed control offered by a Ward-Leonard system can be extended still more by varying the field (excitation) current of the motor. By weakening the magnetic flux, speed can be controlled approximately in the ratio 4 : 1. Thus, taking a Ward-Leonard system as a whole, the overall range of speed control is from 12 : 1

to 16 : 1. Speed control by weakening the magnetic flux is advantageous when the output power remains constant.

Among the disadvantages of the Ward-Leonard system are large size, low efficiency due to the losses in the three auxiliary machines, and sluggishness in response.

For a shunt machine we can construct a *universal characteristic* (Fig. 13.44). If we use an independent motor to drive the armature at a speed exceeding that for an ideal no-load condition, n_{nl}, the armature current will undergo a reversal, and the machine will operate as a generator into a d.c. line. If we apply a sufficiently large

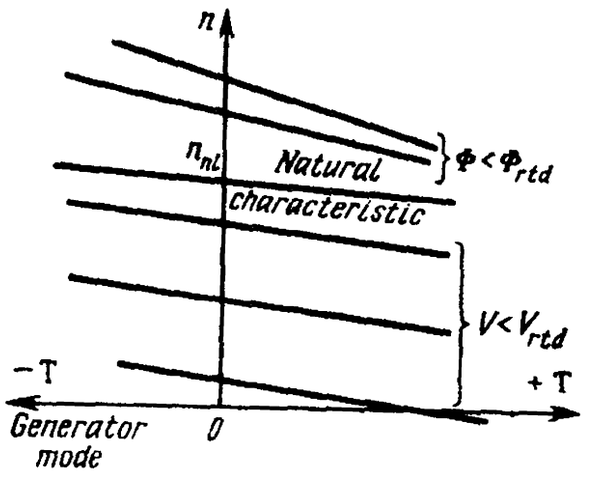


FIGURE 13.43

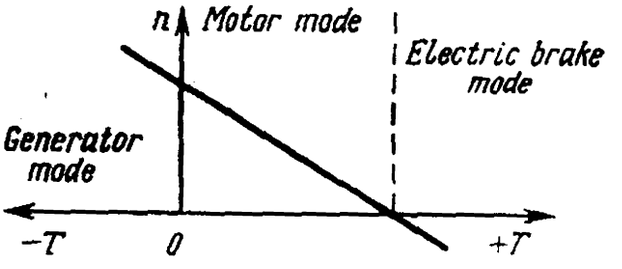


FIGURE 13.44

braking torque to the motor shaft, the motor will come to a stop. If the braking torque is supplied by, say, a rapidly falling load, the machine will change from the motor mode to the *electric brake mode.* This form of retarding the motor speed or of completely stopping the motor or reversing it by the application of power in the reverse direction so that the motor develops a torque which is opposite to the direction of rotation is called *plugging.* In plugging the armature current is

$$I_{arm} = (V + E_{arm})/(r_{arm} + r) \qquad (13.12)$$

where r is the rheostat resistance that must be cut into the armature circuit in order to limit the current. If no additional resistance were inserted in the circuit at stopping by plugging at the rated voltage, the armature current would be heavy enough to cause an irrepairable damage to the commutator and the armature winding.

If a particular application requires that a motor stop quickly, *dynamic braking* is used. In dynamic braking, the rotating armature is disconnected from the supply line, and a resistor is connected across the armature terminals. With the shunt field energized, the machine acts as a generator, and a counter-torque is developed which rapidly slows the machine. The resistor is called a dynamic-braking resistor.

13.14 The Series Motor

In a series motor the armature current is at the same time its field current, because the field winding is connected in series with the armature winding (Fig. 13.45). Owing to this connection the main magnetic flux of the machine varies in direct proportion to the armature current so long as the magnetic circuit is not saturated. As is the case with all d.c. motors, series motors are fitted with a starting rheostat, r_s.

The emf and torque equations developed for d.c. machines, Eqs. (13.1) and (13.2), can be re-written for the series motor as follows. So long as the magnetic circuit of a series motor is not saturated,

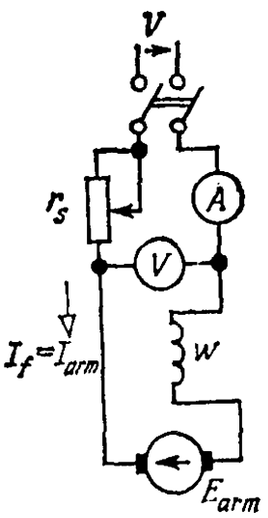


FIGURE 13.45

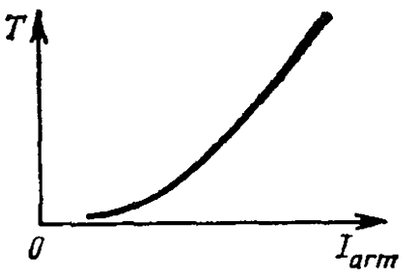


FIGURE 13.46

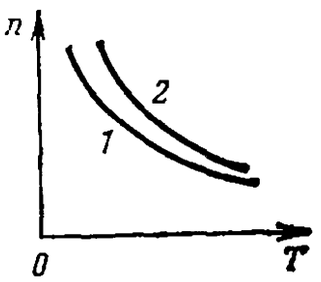


FIGURE 13.47

its magnetic flux is directly proportional to the field mmf, $I_{arm}w_f$, and inversely proportional to the reluctance of the machine, R_m

$$\Phi = I_{arm}w_f/R_m$$

On substituting the above expression into the torque and speed equations of d.c. motors, we obtain the speed equation (the speed-vs-armature current characteristic)

$$n = \frac{V-(r_{arm}+r_f)\,I_{arm}}{c_E w_f I_{arm}/R_m} \tag{13.13}$$

and the torque equation

$$T = c_T w_f I_{arm}^2/R_m \tag{13.14}$$

of a series motor.

Equation (13.14) shows that the torque of a series motor with its magnetic circuit unsaturated increases in direct proportion to the square of armature current. Because of this the initial portion of the torque characteristic is a parabolic curve (Fig. 13.46). When, however, the magnetic circuit is strongly saturated, the flux all but

ceases to rise with increasing excitation current, and the torque rises approximately in proportion to the first power of armature current. The speed of a series motor falls off nearly in inverse proportion to current until magnetic saturation becomes noticeable. The *natural speed-vs-torque-characteristic* of a series motor is shown in Fig. 13.47 (curve 1). As the load on the motor shaft is increased, the current rises at a relatively low rate, and the speed decreases in inverse proportion to the current. If we neglect the saturation of the magnetic circuit and armature reaction, it may be taken that doubling the torque will cause the current to rise to 140% of the initial value and the speed will drop to 70%. In the case of a shunt motor, doubling the torque would only slightly reduce the speed, but the current would rise to twice the original value.

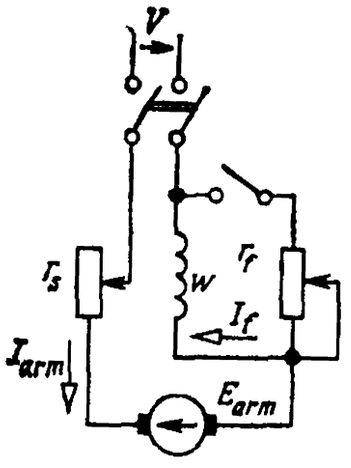


FIGURE 13.48

Thus, a series motor is capable of withstanding heavy overloads with a moderate increase in current which is a valuable property. As the load applied to its shaft is reduced, its current decreases slowly, but the speed goes up rapidly so that at loads smaller than 25% of the rated value a series motor tends to run away with most detrimental consequences to its mechanical integrity. This is the reason why a series motor ought not to be started at no or small load.

Thus, a series motor has a falling (or drooping) speed-torque characteristic because a change in torque has a strong effect on its speed.

The speed of a series motor can be controlled by connecting a field rheostat, r_f, across the field winding (Fig. 13.48). The resistance placed across the field winding brings down the magnetic flux Φ with the the result that, in accord with Eq. (13.11), the speed of the motor, n, increases in inverse proportion to the flux. At the same time, the armature current goes up because the torque is assumed to be constant

$$T \approx \Phi I_{\text{arm}}$$

Thus, if the original magnetic flux and armature current are Φ_1 and I_{arm1}, and as a result of speed control they are Φ_2 and I_{arm2}, then

$$\Phi_1 I_{\text{arm1}} = \Phi_2 I_{\text{arm2}}$$

Curve 2 in Fig. 13.47 is the *resistance-controlled speed-torque characteristic* of a series motor, that is, one plotted when the field winding is shunted by a fixed resistor, r_f.

Another form of speed control is by varying the armature terminal voltage. However it is uneconomical because the armature has

to be connected in series with a rheostat. This form of speed control is used only when the speed has to be reduced by a large amount but for a short time.

The high overload capacity and the falling speed-torque characteristic of series motors make series motors suitable for driving hoists and cranes, for traction and rope haulage.

13.15 The Compound Motor

The shunt motor has a flat speed-torque characteristic and the series motor has a falling speed-torque characteristic. In some applications it is desirable to have some intermediate form. The simplest way to achieve this objective is to use a compound motor. In a com-

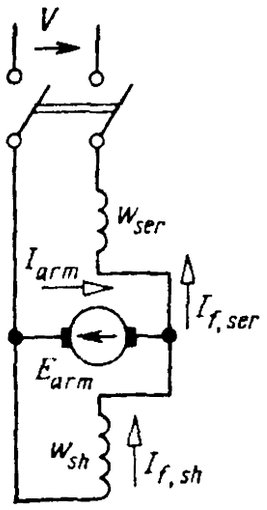


FIGURE 13.49

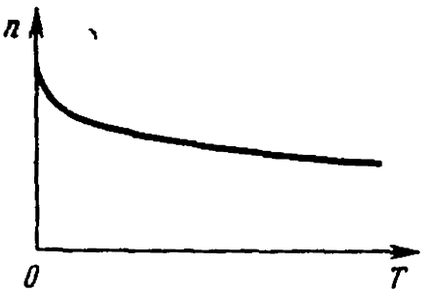


FIGURE 13.50

pound motor, one winding (series or shunt) supplies the bulk (at least 70%) of the mmf, and the other winding (shunt or series) supplies the balance (Fig. 13.49). In most cases the series and shunt-field windings are so connected that their mmfs aid each other, and the machine is said to be "*cumulative compound*".

If we consider the two windings of a compound motor separately, that is, neglecting the effect of magnetic saturation, the speed equation for such a motor may be written in the following general form

$$n = \frac{V-(r_{arm}+r_{f,\,ser})\,I_{arm}}{c_E\,(\Phi_{ser}+\Phi_{sh})} \tag{13.15}$$

where $r_{f,ser}$ is the resistance of the series-field winding.

The series flux Φ_{ser} may be expressed in terms of the armature current and the reluctance of the machine (see Sec. 13.14), so that Eq. (13.15) can be re-arranged as follows:

$$n = \frac{V-(r_{arm}+r_{f,\,ser})\,I_{arm}}{c_E\,(I_{arm}w_{ser}/R_m+I_{sh})}$$

The above equation defines the *speed-vs-armature current characteristic* of a compound motor.

In a *series-shunt motor*, series excitation is predominant, but the shunt-field winding limits the rise in speed as the load on the motor is decreased (Fig. 13.50).

In a *shunt-series motor*, shunt excitation is predominant, and the weak series-field winding stabilizes the main magnetic flux and makes the speed-torque characteristic more drooping.

A compound motor is reversed by interchanging the connections to the armature only, so as to reverse the direction of armature current.

13.16 A.C. Commutator Machines

A most valuable quality of d.c. motors is that their speed and torque can be controlled gradually and economically. However, on a commercial scale electricity is generated in the a.c. form which has first to be converted to d.c. in order to make it available to d.c. motors. This makes the use of d.c. motors more complicated and costly. It is therefore natural that attempts have been made to use commutator machines directly in a.c. circuits.

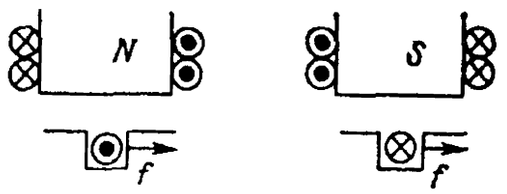


FIGURE 13.51

If we change the direction of current flow both in the armature and field windings of a d.c. motor, the direction of its torque will remain unchanged (Fig. 13.51). Therefore, if the direction of the current in the two windings of a d.c. motor were reversed periodically and simultaneously, as would happen if it were connected to an a.c. supply line, the motor will develop some average torque of constant direction. Before, however, a d.c. motor can be connected to an a.c. line, its design must be adapted to operation in an alternating magnetic field. Among other things, it is necessary to reduce eddy currents in the frame as they greatly weaken the main magnetic flux and to reduce the inductance of the armature winding so as to improve the power factor and commutation.

In order to reduce *eddy currents*, the frame of an a.c. commutator motor is built up of electrical-sheet steel laminations (see Fig. 7.19 b). The *inductive reactance* of the armature winding w_{arm} is reduced by balancing out its magnetic field with a compensating winding which is spread uniformly all the way round the periphery of the stator and connected in series opposition with the armature (rotor) winding.

A further reduction in the magnetic field around the armature is achieved by using nonsalient-pole stators in a.c. commutator machine. A nonsalient-pole stator is a hollow cylinder with slots which

receive (Fig. 13.52) the field winding w_f, the compensating winding w_{com}, and the compole winding (not shown in the figure).

It is to be noted that the field winding, too, has an inductance, but the magnetic flux set up by this winding is essential for the generation of torque, so it may not be balanced out. In a series motor the inductance of the field winding somewhat impairs the power factor. In a shunt motor the high inductance of the field winding brings about a phase shift of nearly 90° between the excitation flux and the voltage, while the armature current is nearly in phase with the voltage. As a result, the average torque produced by the interaction of the excitation flux and the armature current which are nearly in quadrature with each other would be small. This is the reason why special measures have to be taken in present-day a.c. commutator shunt motors, and their field of application is very limited.

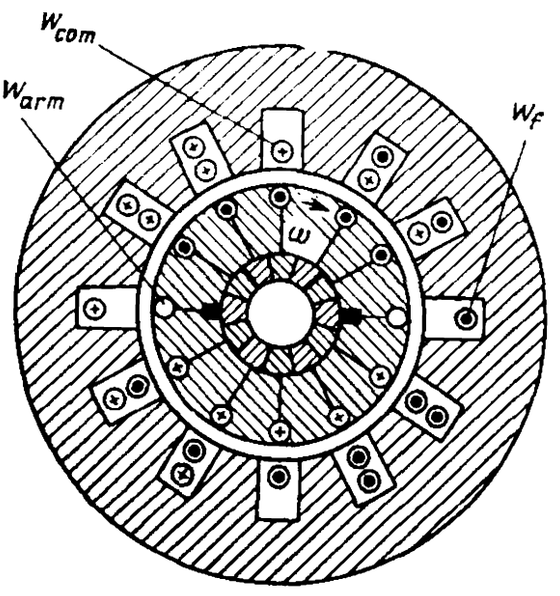


FIGURE 13.52

The task of achieving satisfactory commutation in an a.c. commutator motor poses a number of specific problems. In the case of a d.c. motor, two emfs are induced in a coil being commutated (see Sec. 13.7), one being the reactance emf, e_{reac}, which is the emf of the self-induction and mutual induction, and the other being the commutation emf, e_{com}, induced as the coil sides cut the magnetic field set up by the compoles. This commutation emf, which is proportional to the armature current, can completely cancel the reactance emf and ensure ideal commutation in which case $e_{reac} + e_{com} = 0$, or even lead to early commutation when $e_{com} > e_{reac}$. All this is utilized in a.c. commutator motors: they are fitted with compoles in the same way as d.c. machines are.

In contrast to a d.c. machine, a third (transformer) emf, e_t, is induced in a coil undergoing commutation in an a.c. commutator motor. It owes its origin to the periodic changes in the main magnetic flux threading the coil sides. With respect to the alternating magnetic flux the coil being commutated and short-circuited by a brush acts as the closed secondary winding of a transformer (Fig. 13.53). This emf may be defined in the same way as the emf on a transformer

$$e_t = -w_c d\Phi/dt$$

where w_c is the number of turns in a rotor coil.

The transformer emf lags in phase behind the magnetic flux that induces it, and is in quadrature lagging with the field current. It is independent of the rotor speed and retains its value even when the rotor is at rest.

The conditions for ideal commutation in the case of an a.c. commutator motor are the same as they are for a d.c. motor, namely the algebraic sum of the emfs in the coil undergoing commutation must be zero. To satisfy this condition, the designer has at his disposal only the commutation emf, e_{com}, induced by the compoles. Unfortunately, e_{com} is proportional to the rotor speed and is zero at starting.

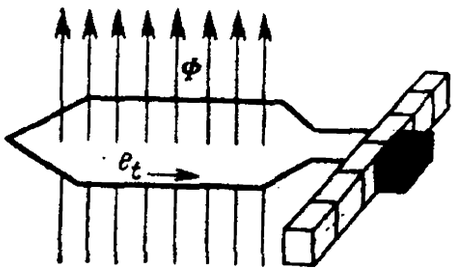


FIGURE 13.53

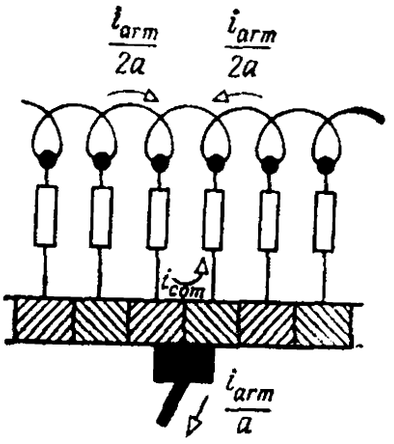


FIGURE 13.54

Therefore the compole flux can be used to create conditions for ideal commutation only at some definite speed. This is probably the most important limitation of a.c. commutator motors. So, since the presence of e_t in the coil undergoing commutation is inevitable, the designer can only, firstly, take measures so as to reduce e_t as much as practicable and, secondly, limit the short-circuit current produced by the emfs in the coil being commutated.

The rms value of transformer emf, Eq. (7.2c), is

$$E_t = 4.44 f w_c \Phi_m$$

Therefore, in order to reduce it, the designer uses fewer turns in each coil of the rotor winding. At the same time, in order to preserve the total number of turns in the rotor winding, he uses a greater number of coils and, in consequence, a greater number of commutator bars. In fact, an oversize commutator is a distinguishing feature of an a.c. commutator motor. Additionally, the designer can reduce the flux Φ_m set up by each pole, but in order to preserve the overall flux he has to use a greater number of poles.

A radical way to reduce e_t is to reduce the supply frequency. This also entails a reduction in the reactance of the motor. Of course, this measure may only be applied when an a.c. commutator mo-

tor operates off an isolated a.c. line. In practice. a reduced frequency ($50/3 = 16^2/_3$ Hz) is sometimes used in electric traction on a.c.

The short-circuit current appearing in the coil undergoing commutation can be limited by inserting series resistors in the coil circuit (Fig. 13.54). Since the rotor current also flows through these resistors, their resistance ought not to be very high, or else heat losses would be prohibitive. This purpose can alternatively be served by carbon brushes which have a high resistance.

14 Induction Machines

14.1 General

Of the many types and forms of present-day electric machines those most commonly used are induction machines, ordinarily employed as motors.

An *induction machine* is an a.c. two-winding unit in which only one (usually the stator) winding is supplied with an alternating current at a constant frequency from an external source, while in the other (usually the rotor) winding currents arise from induction. The fact that the rotor currents are produced by induction is the basis for the name of this class of machines.

Induction machines are also called "asynchronous" because their operating speed is slightly different from the angular velocity of the revolving magnetic field set up by the stator winding.

An important advantage of induction machines is that they have no readily damaged or wearing parts (such as a commutator).

Large induction machines are made three-phase. Small induction machines may be single- and two-phase, so they can operate off a two-wire supply line. Such units are widely used in household appliances.

A disadvantage common to induction machines is that speed and torque control is complicated and uneconomical.

14.2 Construction of a Three-Phase Induction Machine

As any induction machine, it consists of a stationary part, or the stator, and a rotating part, or the rotor.

The stator. The stator of an induction machine is a hollow cylinder built up of electrical-sheet steel laminations insulated from one another by varnish (Fig. 14.1 a). Slots on its inner periphery carry three phase windings. Each phase winding consists of one or several coil groups connected in series and spaced equidistantly from one another. Figure 14.1 b shows the arrangement of one phase winding in stator slots. It consists of two coil groups, with A marking the start and X the finish of the phase winding.

The phase windings may be interconnected in a star or a delta and to a three-phase power supply. The currents flowing in the phase windings set up in the machine a revolving stator field which has p pole pairs, equal to the number of coil groups per phase winding.

This is achieved by disposing the phase winding in such a way that the coil groups in one are displaced around the stator circumference through an angle $120°/p$ with respect to the coil groups in the adja-

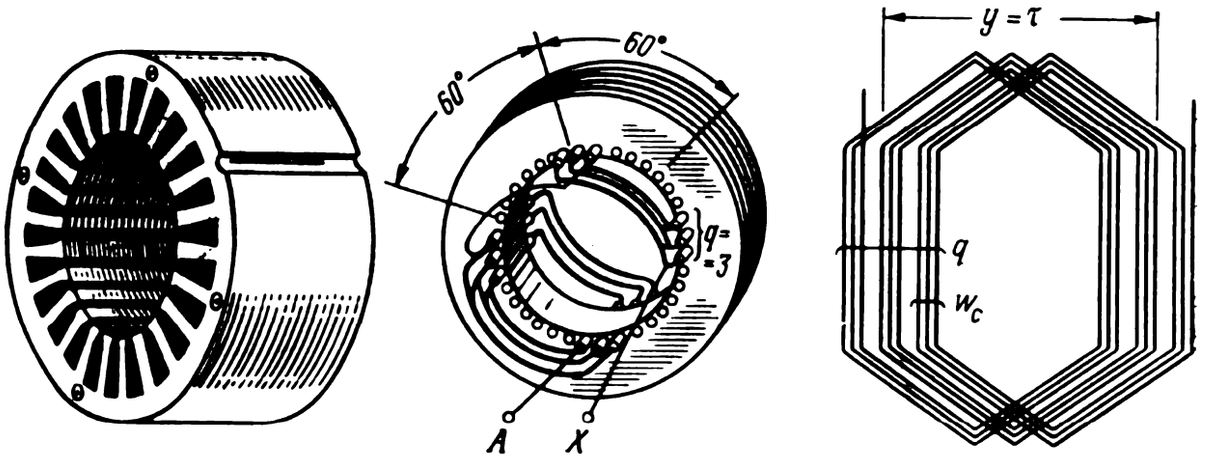


FIGURE 14.1

cent phase winding. As an example, for a four-pole machine ($p = 2$) this angle is 60° (Fig. 14.1 b).

Each multiturn coil group of the stator winding is made up of q series-connected coils each having w_c turns. The coils may be wound with the same or a different coil pitch, y. In the former case the sides of a coil are spaced $180°/p$ apart around the stator periphery, which corresponds to one pole pitch, $y = \tau$ (Fig. 14.1 c). In the latter case, $\tau < y < \tau$, the coils in a coil group are nested within one another (Fig. 14.1 b).

The distribution of the phase windings among many slots not only improves the utilization of a cylindrical stator, but also optimizes the distribution of the magnetic field in the air gap between the stator and the rotor.

The stator core of an induction machine may have open slots (Fig. 14.2 a) or semi-closed slots (Fig. 14.2 b). An advantage of semiclosed slots is that the machine has a lower reluctance and so it needs a smaller magnetizing current. On the other hand, it is simpler

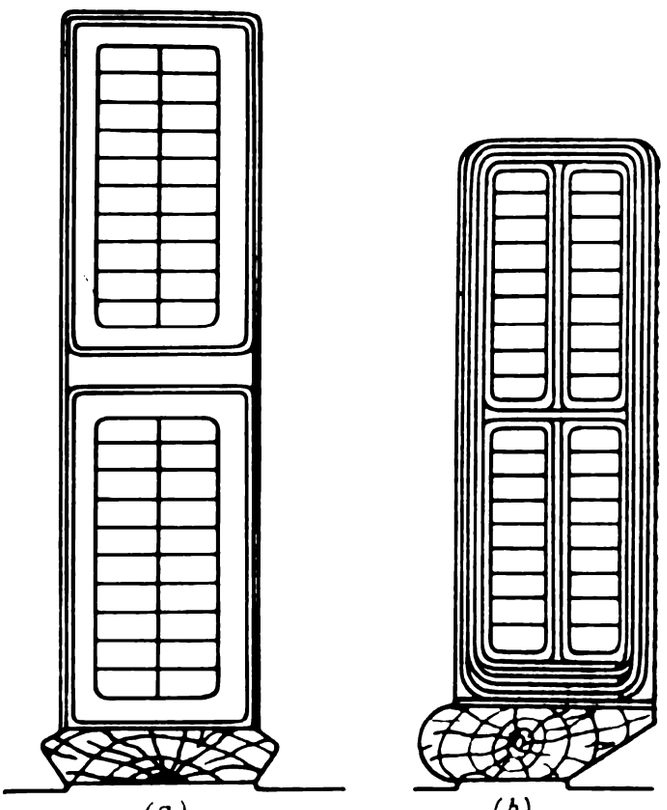
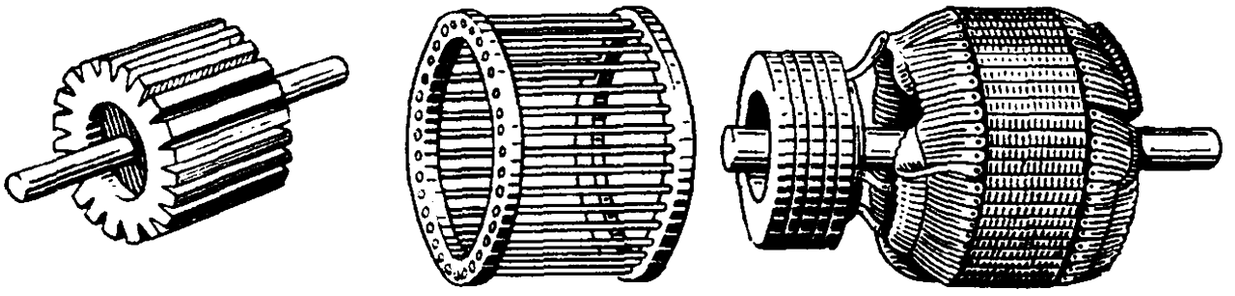


FIGURE 14.2

to place coils in open slots and the coils can be better insulated which is an important consideration for high-voltage machines.

The rotor. Induction machines mainly differ in the rotor which may be the wound type and the squirrel-cage type. In either case



(a) (b) (c)

FIGURE 14.3

the rotor is a cylindrical core (Fig. 14.3 a) assembled from electrical-sheet steel laminations insulated from one another with varnish. The rotor core is mounted on a shaft which is supported in bearings. Slots on its outer periphery carry the turns of the rotor winding.

Most induction motors use a *squirrel-cage rotor*. It is less expensive to make and, what is very important, a squirrel-cage induction

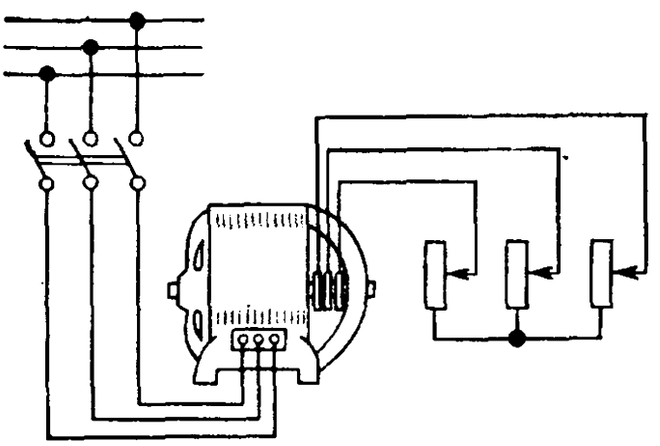
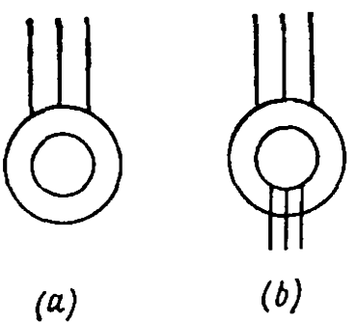


FIGURE 14.4



(a) (b)

FIGURE 14.5

motor is simpler to service and operate. The winding of a squirrel-cage rotor consists of solid uninsulated copper or aluminium bars placed in rotor slots. These rotor bars are shorted together at the two ends of the rotor by end rings to form a structure resembling a squirrel cage which is why such induction motors are called "squirrel-cage". Frequently, the core laminations are stacked in a mould which is then filled with molten aluminium, so that the rotor bars, end rings and cooling-fan blades are cast at the same time in one economical process.

The winding of a *wound rotor*, also called a slip-ring rotor (Fig. 14.3 c) consists of coils of insulated wire, placed in slots in the rotor core. It is almost always three-phase and has the same number of coils as the stator. They are connected in a star on the rotor, and the three terminal leads are brought to three slip rings mounted on, but insulated from, the shaft. The slip rings rotate in contact with brushes set up in fixed brush holders. The slip rings and the brushes connect the rotor winding in series with a three-phase rheostat (Fig. 41.4). This rheostat improves the starting characteristics of the machine by reducing the starting current and by raising the starting torque. Also the rheostat can be used for speed control.

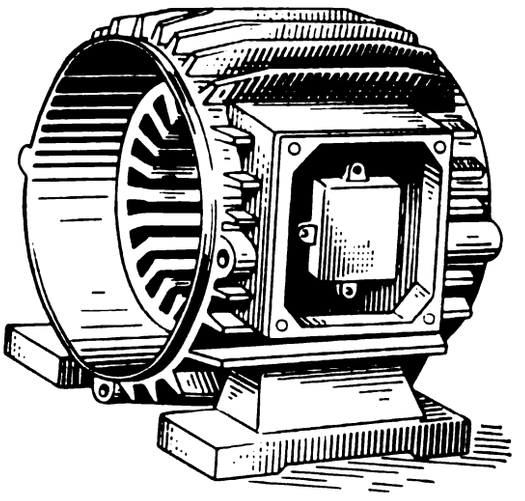


FIGURE 14.6

The graphic symbols used (in the Soviet Union) to represent squirrel-cage (a) and wound-rotor (b) induction machines in diagrams are given in Fig. 14.5, and Fig. 14.6 shows the frame of an induction machine with its stator unwound.

14.3 The Revolving Magnetic Field of the Stator

When three-phase voltages are applied to the stator-winding terminals, balanced three-phase currents flow in the phase windings and a rotating magnetic field is produced in the air gap of the machine. This field induces an emf in the rotor winding shorted together or connected in series with a starting rheostat. The currents thus induced produce a revolving magnetic field of their own. The speed and direction of rotation of the two fields are the same, so they combine to yield the operating revolving field of the machine.

Consider the revolving stator field, assuming that the rotor circuit is open. Above all this field depends on the geometric arrangement of the phase windings on the stator.

A two-pole revolving field. A two-pole revolving field will be produced when the three identical phase windings are positioned on the stator so that they are spaced 120° apart. In Fig. 14.7 c, each phase winding is represented by a single turn and has its terminals labelled as A, B, and C for the starts, and X, Y and Z for the finishes. When the phase windings are connected in a star (see Fig. 14.7 a) or in a delta and energized from a three-phase power supply, the co-

il turns will carry the following currents:

$$i_A = I_m \sin \omega t$$
$$i_B = I_m \sin (\omega t - 120°)$$
$$i_C = I_m \sin (\omega t - 240°)$$

(see Fig. 14.7 b). The currents in the phase windings produce magnetic fields with the respective magnetic inductions given by

$$B_A = B_m \sin \omega t$$
$$B_B = B_m \sin (\omega t - 120°) \qquad (14.1)$$
$$B_C = B_m \sin (\omega t - 240°)$$

The pattern of the magnetic induction vectors is shown in Fig. 14.7 c. On combining vectorially, the magnetic inductions yield the magnetic induction due to the resultant stator field.

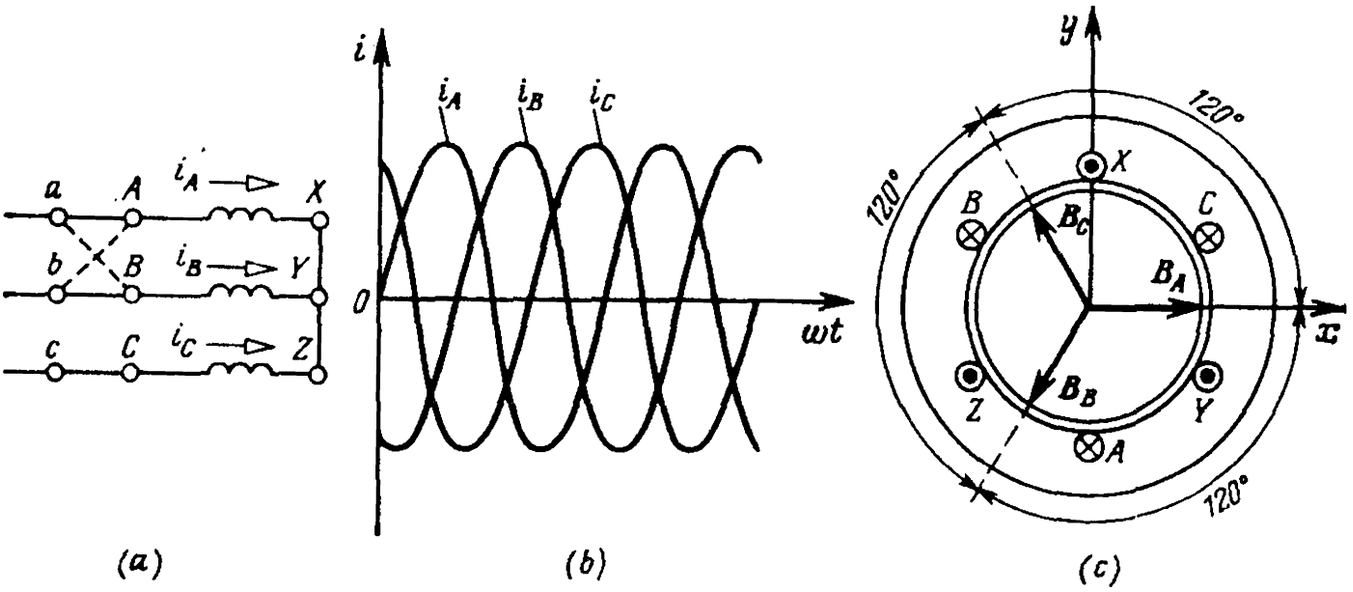


FIGURE 14.7

The properties of the resultant stator field can conveniently be defined in terms of its components directed along two mutually perpendicular axes, x and y, with the x-axis assumed to be along the axis of the phase A coil.

Now we set out to determine the magnetic field component along the x-axis. It is equal to the algebraic sum of the projections of the three instantaneous magnetic inductions on that axis

$$B_x = B_A \cos 0° + B_B \cos (-120°) + B_C \cos (-240°)$$
$$= B_A + B_B (-1/2) + B_C (-1/2)$$

On substituting for the magnetic inductions from Eq. (14.1), we get

$$B_x = B_m \left[\sin \omega t - \frac{1}{2} \sin (\omega t - 2\pi/3) - \frac{1}{2} \sin (\omega t - 4\pi/3) \right]$$

$$= B_m \left[\sin \omega t + \frac{1}{4} \sin \omega t + \frac{\sqrt{3}}{4} \cos \omega t + \frac{1}{4} \sin \omega t \right.$$

$$\left. - \frac{\sqrt{3}}{4} \cos \omega t \right] = 1.5 B_m \sin \omega t \qquad (14.2)$$

The magnetic field component along the y-axis is

$$B_y = B_A \sin 0° + B_B \sin (-120°) + B_C \sin (-240°)$$

$$= B_B (-\sqrt{3}/2) + B_C \sqrt{3}/2$$

Or, on substituting for the magnetic inductions from Eq. (14.1),

$$B_y = B_m \left[-\frac{\sqrt{3}}{2} \sin (\omega t - 2\pi/3) + \frac{\sqrt{3}}{2} \sin (\omega t - 4\pi/3) \right]$$

$$= 1.5 B_m \cos \omega t \qquad (14.3)$$

Thus the resultant magnetic induction of the stator field

$$B_{st} = \sqrt{B_x^2 + B_y^2} = 1.5 B_m \sqrt{\sin^2 \omega t + \cos^2 \omega t} = 1.5 B_m \qquad (14.4)$$

is constant in magnitude and the angle α between the magnetic lines of force and the y-axis is such that

$$\tan \alpha = B_x / B_y = \sin \omega t / \cos \omega t = \tan \omega t$$

Hence,

$$\alpha = \omega t$$

In other words, the magnetic field of the stator revolves clockwise at a constant speed ω in the plane containing the coil axes. It is consecutively aligned with the axis of that phase winding in which the current is a maximum. This means that it revolves in the direction of the phase sequence of the three-phase system of currents in the phase windings.

The direction of rotation of the stator field can be reversed by simply interchanging the connection of any two phase windings to a three-phase power supply, such as shown dashed in Fig. 14.7 a.

The overall distribution of the magnetic lines of force in the revolving stator field of a two-pole induction machine at time t_1 is shown in Fig. 14.8 a. The distribution of the magnetic induction

B in the air gap between the stator and rotor as a function of the distance z from the line O-O taken as reference at times t_1 and $t_2 > t_1$ is shown in Fig. 14.8 b. The linear speed at which the magnetic field travels along the air gap depends on the diameter D and is equal to $v = D\omega/2$. At the commercial frequency ($f = 50$ Hz), the stator magnetic field of a two-pole induction machine revolves at $n = 50 \times 60 = 3000$ rpm. In most practical cases, motors are made to have a lower speed. This is achieved through the use of multipole stator windings.

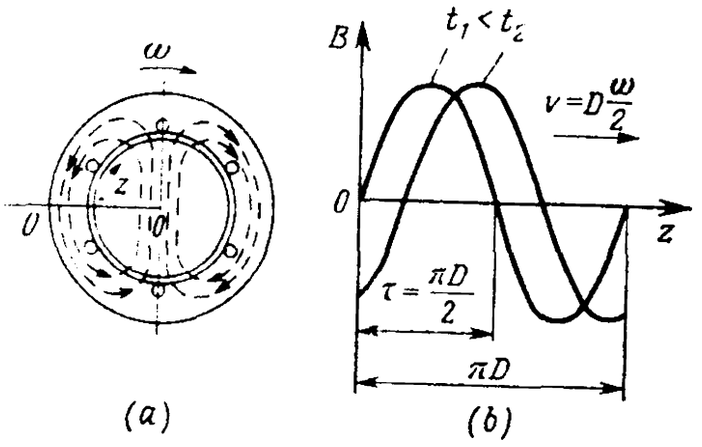


FIGURE 14.8

A multipole revolving field. In the case of a multipole stator winding, each pole pair of the revolving field corresponds to one coil group in each phase winding, which works out to just three coil groups for three phase windings. Therefore, if the field is to have p pole pairs, the three phase windings must be divided into

$$k = 3p \tag{14.5}$$

equal parts, or p parts per phase. As an example, Fig. 14.9 shows a simplified diagram of the stator winding of a six-pole ($p = 3$) machine. Here the phase windings are divided into $3p = 9$ parts, and each phase winding into three parts. In the simplified diagram each coil group of a phase winding is shown as a single-turn coil and the connections are labelled only for phase A, with the connections made on the rear side of the stator core being shown dashed. In a multipole machine, the sides of such a coil group are spaced $180°/p$, or one pole pitch τ, apart along the stator periphery. For a six-pole machine this angle is $180°/3 = 60°$.

Figure 14.9 also shows the waveforms of instantaneous stator currents making up a balanced three-phase system. The magnetic field patterns labelled 1 through 4 are given for instants t_1 through t_4. The directions of currents in the phase conductors shown in the leftmost figure correspond to time t_1 when the current in the first phase is a maximum. In accord with these current directions, the magnetic lines of force produced by the stator enter the rotor at three points and leave it likewise at three points, thus forming three pole pairs ($p = 3$). At time t_2, the directions of currents and, as a consequence, the position of the stator field are changed. During one cycle of alternating current the magnetic field moves through one-third of a

circle, that is, through the arc occupied by the phase windings on the stator. This arc corresponds to two pole pairs ($2p$) of the revolving

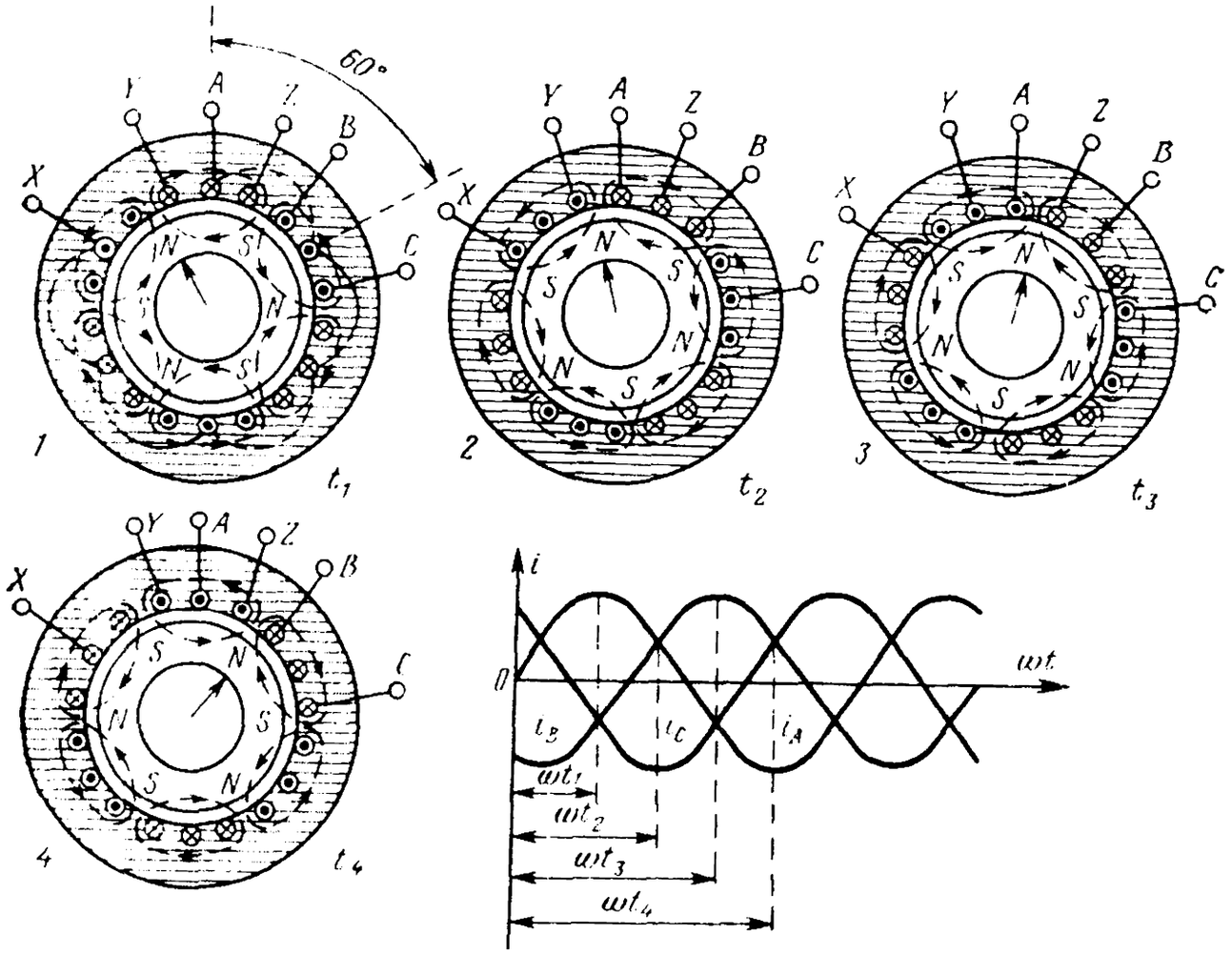


FIGURE 14.9

field and is called the double pole pitch, 2τ. Thus, the *pole pitch* τ is the arc of the stator circumference corresponding to one pole of the magnetic field

$$\tau = \pi D/2p \tag{14.6}$$

where D is the inside diameter of the stator core.

During one cycle, T, of alternating current the revolving field of the stator moves through one double pole pitch, 2τ, and completes one revolution during p cycles. Thus, every second the field completes $1/pT = f/p$ revolutions, and the speed (in rpm) is

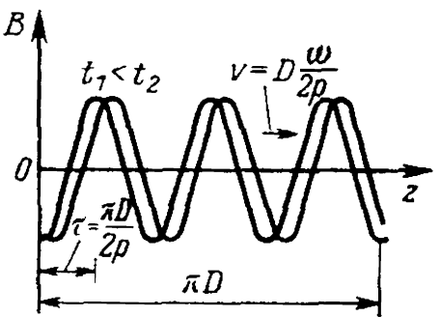


FIGURE 14.10

$$n_1 = f \times 60/p \tag{14.7}$$

The distribution of magnetic induction B due to the stator magnetic field in the air gap of a six-pole machine at times t_1 and t_2 is shown in Fig. 14.10, with the distance z along the air gap counted from the same datum as in Fig. 14.8 a.

14.4 The Modes of Operation of a Three-Phase Induction Machine

The mode of operation for a three-phase induction machine depends on the interaction between the currents in the stator and rotor windings.

The interaction of the revolving magnetic field produced by the currents in the stator winding with the currents in the rotor winding causes the rotor to rotate with the revolving field. However, as the speed of the rotor increases, the emfs induced in its winding and, in consequence, the associated currents decrease. If the speed of the field is n_1 and the speed of the rotor is n, the mode in which an induction machine operates can be defined in terms of the slip which is given the symbol s and is defined as follows:

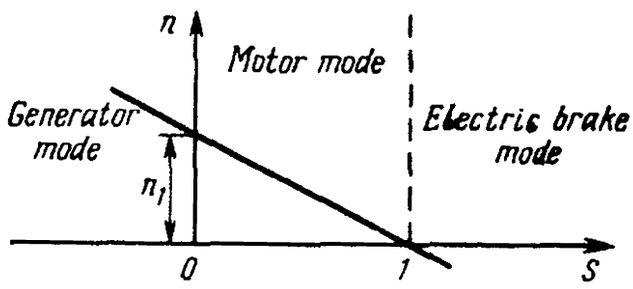


FIGURE 14.11

$$s = (n_1 - n)/n_1 \qquad (14.8)$$

A plot of the rotor speed as a function of the slip is shown in Fig. 14.11.

Depending on the magnitude of the slip, an induction machine will operate as a motor, a generator, or an electromagnetic brake.

In the *motor mode* ($0 < s < 1$), a three-phase induction machine converts electric energy into mechanical energy. The rotor should rotate more slowly than the field at an asynchronous speed such that the currents in the rotor winding interact with the revolving magnetic field, produced by the currents in the stator winding, and develop a driving torque. The latter balances the braking torque which exists at the shaft due to friction and the load.

In the *generator mode* ($s < 0$), a three-phase induction machine converts mechanical energy into electric energy. The rotor rotates with the revolving magnetic field set up by the currents in the stator winding at a speed exceeding that of the revolving field.

In the *brake mode* ($s > 1$), the rotor of a three-phase induction machine rotates against the revolving field set up by the currents in the stator windings. In this mode, a large proportion of input energy is dissipated in the windings through hysteresis and eddy currents

As already noted, induction machines are mostly used as motors. Therefore our subsequent discussion will be mainly limited to the operation of an induction motor.

14.5 The Revolving Rotor Field and the Working Revolving Field in an Induction Motor

To begin with, let us assume that the rotor circuit is open so that there is no current flowing in it, no electromagnetic forces are acting on the rotor, and it is at rest. Then the magnetic field in the machine will be solely due to the revolving magnetic field of the stator.

The winding of a squirrel-cage rotor consists of N bars. The phase difference between the emfs induced in two adjacent bars by the revolving magnetic field of the stator is $360° \, p/N$. It may be taken that the number of phases in a squirrel-cage rotor is equal to the number of bars, $m_2 = N$, and that each phase has $w_2 = 1/2$ turn.

Similarly, the circuit of a wound rotor is a three-phase system, $m_2 = 3$, with w_2 turns per phase. (Here and elsewhere the quantities related to a rotor phase will be labelled "2", and those related to a stator phase will be labelled "1".)

So long as the rotor is at rest, the frequency of the emf induced in the rotor winding is equal to the frequency f of the current in the stator circuit. When the rotor is rotating with the revolving stator field at the speed n, the frequency of the emf induced in the rotor winding goes down. This frequency can be deduced from Eq. (14.7) in which n_1 should be replaced with the difference $n_1 - n$ because the revolving stator field can cut the rotor turns only when the rotor is lagging in speed behind the field

$$f_2 = p \, (n_1 - n)/60$$

If we do not complete the rotor circuit, the currents in it will form a multiphase system ($m_2 = N$) in the case of a squirrel-cage rotor, and a three-phase system ($m_2 = 3$) in the case of a wound rotor. Acting similarly to the currents in the stator winding, they will produce a revolving magnetic field. The speed of this field relative to the rotor, n_{rel}, can be found from the general expression, Eq. (14.7), for the speed of a multipole field

$$n_{rel} = f_2 \times 60/p$$

Since the rotor itself rotates in the same direction at the speed n rpm, the rotor field revolves at

$$n_{rel} + n = (n_1 - n) + n = n_1$$

that is, the rotor field revolves at synchronous speed (at the same speed as the field), and faster than the rotor does.

Thus, the revolving fields of the stator and rotor are stationary with respect to each other, which is the decisive condition for a complete transfer of energy from the stator to the rotor. On being combined, the two revolving fields yield the working revolving magnetic field of an induction motor. This field also acts as a link between the stator and rotor windings, similarly to the alternating magnetic field transferring energy from the primary to the secondary in a transformer.

In our subsequent discussion we will call the working revolving field simply the revolving field of an induction machine. It is this field that is used when analysing the events taking place in the stator and rotor circuits.

An induction motor can operate in any one of three characteristic modes: (1) the *rated condition* corresponding to the rated slip $s = s_{rtd}$, the rated supply voltage $V_1 = V_{1,rtd}$, and the rated supply current $I_1 = I_{1,rtd}$; (2) the *operating mode* when the line voltage is close or equal to the rated value, $V_1 \approx V_{1,rtd}$, and the load on the motor is determined by the braking torque applied to the shaft at the slip $s \leqslant s_{rtd}$ and at the supply current $I_1 \leqslant I_{1,rtd}$; and (3) the *starting condition* which arises when the supply voltage is just turned on and the rotor is at rest, $s = 1$.

The conditions existing in any one phase of the stator winding is the same as in the remaining two. This is also true of the rotor phases. That is why it is convenient to analyse the operation of only one phase of an induction motor.

14.6 The Equation of State for a Stator Phase of an Induction Motor

The emf induced in each turn of the stator winding by the revolving magnetic field, is, in accord with Eq. (2.2), given by

$$e_{turn} = - \, d\Psi/dt$$

Since it is legitimate to assume that the normal component of the magnetic induction is distributed sinusoidally around the circumference of the rotor, the flux linkage of the turn as the field revolves will also vary sinusoidally with time

$$\Psi = \Phi_{turn} \sin \omega t$$

Therefore the rms value of the emf induced in a stator turn can be found from the equation for transformer emf, Eq. (7.2c)

$$E_{turn} = 4.44 f \, \Phi_{turn}$$

In contrast to a transformer, however, Φ_{turn} symbolizes the average (constant) rather than peak value of the revolving magnetic flux threading the turn

$$\Phi_{turn} = l \int_0^{\tau} B_m \sin(\pi/\tau)\, z\, dz = (2\tau l/\pi)\, B_m$$

where l is the rotor length and τ is the pole pitch.

When defining the emf, e_1, induced in each of the stator phase windings, it should be remembered that this emf is induced by a revolving rather than an alternating magnetic field. Usually the requisite number of turns in the stator winding consists of several coils located in q adjacent slots rather than of one coil each side of which is located in one slot. These coils are connected in series to form a *coil group* (see Fig. 14.1 c). The angular width of a tooth in the stator core determines the central angle belween two adjacent slots or the slot angle in mechanical degrees

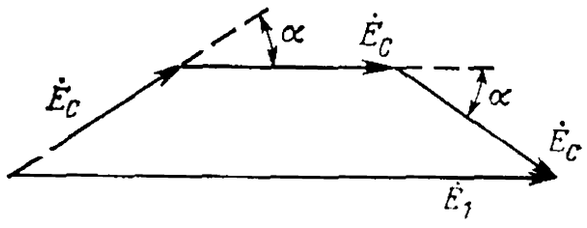


FIGURE 14.12

$$\beta = 360°/z$$

where z is the number of stator-core teeth or, which is the same, the number of stator-core slots.

The revolving magnetic field cuts the coil sides at different times, so there is a phase difference between the emfs induced in the adjacent coils of the same coil group. This phase angle α is p times the slot angle β in mechanical degrees

$$\alpha = p\beta = p \times 360°/z \qquad (14.9)$$

Hence, each time a $2p$-pole field rotates through 360° mechanical the phase of the induced emfs changes by $p \times 360°$ electrical. Because of this phase angle between the emfs e_c induced in the coils of a coil group, the rms emf of a coil group, E_{1c}, is smaller than the product of the rms emf in one turn, E_{turn}, by the number of turns, w_{1c}, in a coil group (Fig. 14.12)

$$E_{1c} < E_{turn} w_{1c}$$

This reduction in emf is thus caused by the distribution of coil sides among many slots. A second cause of the reduction in the winding emf is chording or the use of short-pitched coils in which the angular distance between the two sides of a stator-winding turn is less than the pole pitch τ. Chording is ordinarily used in order to reduce the length of the coil-overhang connections. As a result of chording, a

stator turn links less than all of the pole flux, so a smaller emf is induced in it.

The reduction in the induced emf due to the distribution of the stator winding in several slots and chording is accounted for in design by applying the winding factor, k_w. As a rule, this factor is less than unity. For induction machines the figure is 0.92 to 0.96.

Therefore, the rms emf induced by the revolving magnetic field in each phase winding with $w_1 = w_{1c}p$ turns is equal to

$$E_1 = pE_{1c} = w_1 k_{w1} E_f = 4.44 f w_1 k_{w1} \Phi_f \tag{14.10}$$

It should be noted that the winding factor and the phase angle between induced emf apply not only to induction machines, but al-

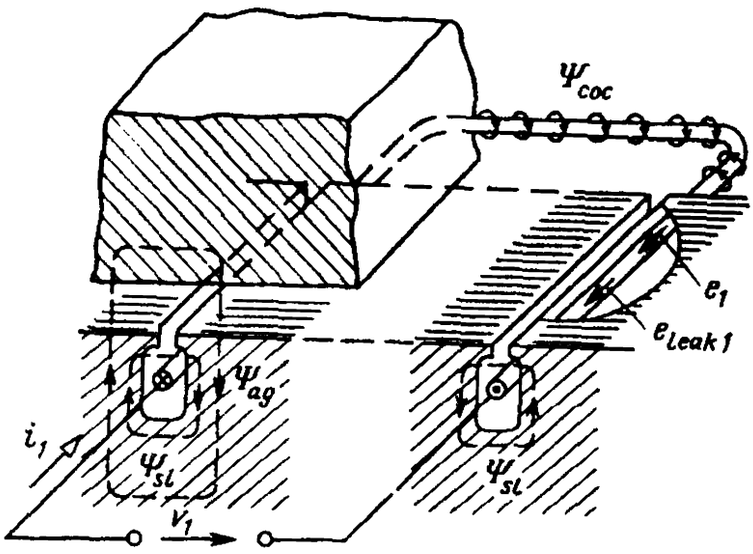


FIGURE 14.13

so to all other machines which utilize a revolving magnetic field in one form or another.

In addition to the air-gap flux crossing the clearance between the stator and the rotor and linking both windings, there is also a flux which avoids the rotor. It is called the leakage flux, and the flux linkage associated with it is called the *leakage flux linkage*, $\Psi_{leak,1}$. As is seen from Fig. 14.13 where a stator phase winding is shown as a single-turn coil, the stator leakage flux linkage is the sum of the flux linkages across the core slots, Ψ_{sl}, around the coil-overhang connections, Ψ_{coc}, and across the air-gap between the stator and rotor, Ψ_{ag}. The stator leakage flux linkage increases with increasing air-gap, increasing length of coil-overhang connections, and increasing slot depth. Since most of the path for the magnetic flux is through air, we may take it, as we did for a transformer, that the stator leakage flux linkage is directly proportional to the stator current i_1 and is in phase with that current.

In each phase winding of the stator the leakage flux linkage induces a leakage emf $e_{\text{leak},1}$ in accord with Eq. (2.2). This emf is aligned with the stator current i_1 (see Fig. 14.13). Referring to Fig. 14.13, e_1 is the emf induced by the revolving magnetic field in each phase winding, and v_1 is the phase voltage of the supply line.

Thus, the current in each phase winding may be looked upon as produced by the joint action of the phase supply voltage and two emfs, one induced by the revolving field and the other induced by the leakage flux.

On representing all sinusoidal quantities in complex form and using Kirchhoff's voltage law, we may write for a stator phase circuit

$$\dot{V}_1 + \dot{E}_1 + \dot{E}_{\text{leak},1} = r_{\text{turn},1}\dot{I}_1$$

where $r_{\text{turn},1}$ is the resistance of the turns in a stator phase winding.

On the basis of Eqs. (2.32) and (2.33), the emf, $-\dot{E}_{\text{leak},1}$, needed to balance the stator leakage emf may be written as

$$-\dot{E}_{\text{leak},1} = jx_{\text{leak},1}\dot{I}_1 = j\omega L_{\text{leak},1}\dot{I}_1$$

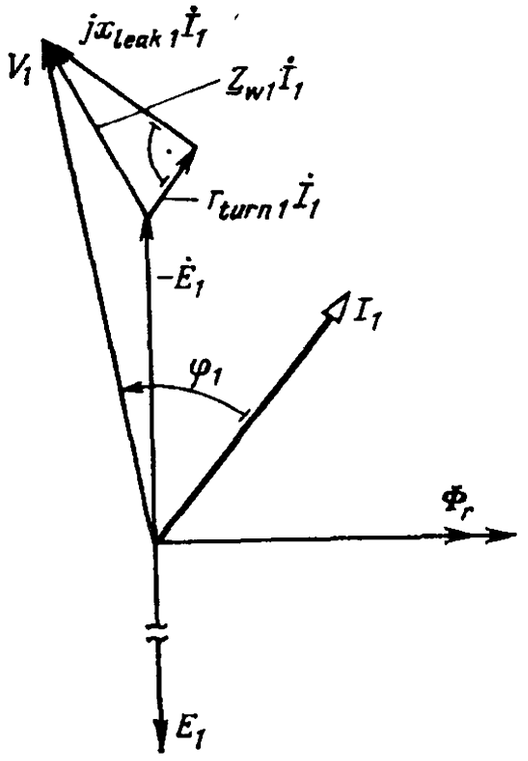


FIGURE 14.14

where $L_{\text{leak},1} = \Psi_{\text{leak},1}/i_1$ is the leakage inductance of the stator phase, and $x_{\text{leak},1}$ is the leakage inductive reactance in the stator phase winding.

Hence, an equation of electric state for a stator phase may be written as

$$\dot{V}_1 = (-\dot{E}_1) + (r_{\text{turn},1} + jx_{\text{leak},1})\dot{I}_1 = (-\dot{E}_1) + \underline{Z}_{w1}\dot{I}_1 \qquad (14.11a)$$

where

$$\underline{Z}_{w1} = r_{\text{turn},1} + jx_{\text{leak},1}$$

is the complex impedance of a stator phase winding.

Equation (14.11a) does not differ from that of electric state for the transformer primary, Eq. (8.11a) which is quite natural because in an induction machine energy is transferred from the primary (stator) circuit into the secondary (rotor) circuit by induction as this is done in a transformer.

Graphically, the equation of electric state for a stator phase of an induction motor is illustrated in the phasor diagram of Fig. 14.14

where $\underline{Z}_{w1}\dot{I}_1$ is substantially greater than its counterpart is in the equation of the transformer primary. This results from the existence of an air gap in the magnetic circuit of an induction machine (Fig. 14.14).

In medium and large induction machines, the voltage drop $\underline{Z}_{w1}I_1$ at $I_1 \leqslant I_{1.rtd}$ is relatively small, and V_1 may, in accord with Eq. (7.2c), be approximately written as

$$V_1 \approx E_1 = 4.44\, f w_1 k_{w1} \Phi_r \qquad (14.11b)$$

Since the voltage between the terminals of a phase winding, V_1, remains unchanged, the revolving magnetic flux Φ_r in operation may likewise be taken approximately as constant and independent of load.

14.7 The Equation of Electric State for a Rotor Phase of an Induction Motor

The revolving magnetic field induces in a rotor phase an emf, e_2, at frequency $f_2 = p\,(n_1 - n)/60$. We can express this frequency in terms of the supply-line frequency on dividing and multiplying the right-hand side of this expression by n_1. Then, in view of Eqs. (14.7) and (14.8), we get

$$f_2 = fs \qquad (14.12)$$

which tells us that the slip frequency, f_2, is equal to the product of the supply-line frequency by the slip.

The emf in a stator phase winding, e_1, and the emf in a rotor phase winding, e_2, are induced by a revolving magnetic field which is common to the two windings and is produced by the joint action of the stator current i_1 and the rotor current i_2. However, e_1 opposes a change in i_1, while e_2 induces i_2. Therefore, while the positive directions of e_1 and i_1 are the same (see Fig. 14.13), it is convenient to assume that the positive direction of i_2 is opposite to that of e_2. This corresponds to the connection of the stator and rotor phase windings in opposition so that the resultant mmf of the motor is the difference between the stator mmf and the rotor mmf. This is also true of the primary and secondary emfs, currents and mmfs of a transformer (see Fig. 8.5 b).

The rotor phase current i_2 sets up a magnetic field a part of which avoids the stator winding. This part is responsible for the leakage flux linkage of a rotor phase winding, $\Psi_{leak.2}$.

Thus, the equation of electric state in complex form for a rotor phase is

$$-\dot{E}_2 = r_{turn.2}\dot{I}_2 + js\omega L_{leak.2}\dot{I}_2 \qquad (14.13)$$

where $r_{\text{turn.2}}$ is the resistance of the rotor turns, $L_{\text{leak.2}} = \Psi_{\text{leak.2}}/i_2$ is the leakage inductance of a rotor phase and $s\omega L_{\text{leak.2}}$ is the inductive reactance of a rotor phase.

By analogy with Eq. (14.11b), the rms value of the rotor phase emf is

$$E_2 = 4.44 f s w_2 k_{w2} \Phi_r \tag{14.14}$$

where for a squirrel-cage rotor $w_2 = 1/2$ and $k_{w2} = 1$.

The equivalent circuit diagram of a rotor phase answering Eq. (14.13) appears in Fig. 14.15, and the corresponding phasor diagram is shown in Fig. 14.16.

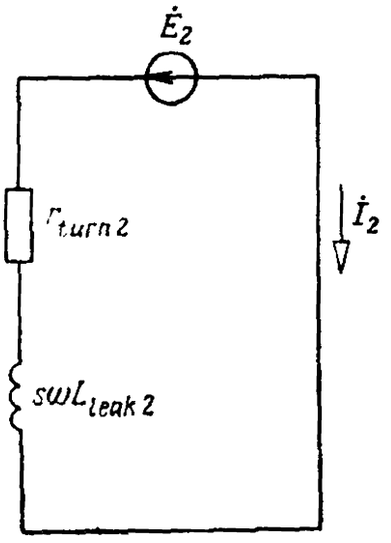


FIGURE 14.15

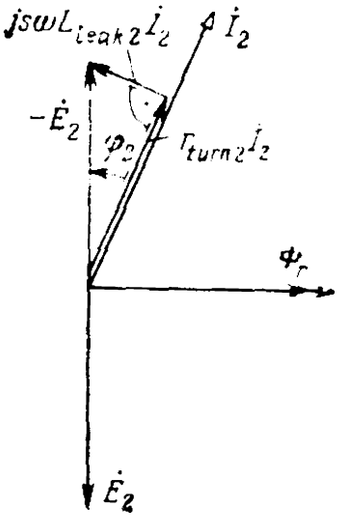


FIGURE 14.16

It is to be noted that Eq. (14.11a) and Eq. (14.13) are similar to Eq. (8.11a) describing the electric state of a transformer primary and Eq. (8.11b) describing the electric state of a transformer secondary in a short-circuit condition, that is, at $\dot{V}_2 = 0$.

14.8 The Balance of MMFs in an Induction Motor

The revolving magnetic field set up by the stator of an induction machine remains stationary relative to that produced by its rotor. Therefore, with the rotor spinning, the mmfs produced by the stator and rotor currents may be treated as phasors which, when combined vectorially, give the mmf exciting the revolving magnetic field of the machine. In calculating these mmfs, it is important to remember that they are produced by currents in the coil sides that are distributed among several slots and are out of phase in space. This is accounted for by applying the winding factor which usually is less than unity.

Thus, by analogy with the mmf balance for a transformer, Eq. (8.4), the mmf balance for an induction motor, with its rotor stationary or spinning, is

$$3w_1k_{w1}\dot{I}_1 - m_2w_2k_{w2}\dot{I}_2 = 3w_1k_{w1}\dot{I}_{1,oc} \tag{14.15}$$

Here we have accounted for the fact that the resultant mmf for a three-phase stator and rotor winding is 3/2 times that given by Eq. (14.4), and that the resultant mmf of an m_2-phase winding is $m_2/2$ times that of a single phase mmf.

On the basis of Eq. (14.15), the stator current may be expressed as

$$\dot{I}_1 = \frac{m_2w_2k_{w2}}{3w_1k_{w1}}\dot{I}_2 + \dot{I}_{1,oc} = \dot{I}_2' + \dot{I}_{1,oc} \tag{14.16}$$

where

$$\dot{I}_2' = \frac{m_2w_2k_{w2}}{3w_1k_{w1}}I_2$$

is the *rotor current referred to the stator side.* It is that part of the stator current which counteracts the demagnetizing effect of the rotor current.

When the motor is running ($s \leqslant s_{rtd}$), the magnetizing current $\dot{I}_{1,oc}$ is practically equal to the stator current in the ideal no-load (open-circuit) condition. In order to determine the latter from a test, the rotor must be spinned by an auxiliary motor at the synchronous speed which is the speed of the rotating magnetic field ($s = 0$). In the circumstances, the rotor current becomes equal to zero. Thus, with no load torque applied to the shaft, the open-circuit (no-load) current of an induction motor exceeds the ideal no-load (open-circuit) current due to the loss of power through friction in the rubbing parts, as heat in the winding rotor, etc.

14.9 The Equivalent Circuit for a Phase of an Induction Motor

In calculating the process taking place in an induction motor, resort is usually made to the equivalent circuit of one phase. This circuit is assumed to include fixed resistive and inductive elements and also a variable resistive element to represent the mechanical load applied to the motor shaft.

The difficulties in developing such an equivalent circuit arise because, firstly, the frequency of the stator phase currents is f and that of the rotor phase currents is $f_2 = fs$, secondly, the stator phase has w_1 turns and the rotor phase has w_2 turns, thirdly, the two wind-

ings differ in the winding factors k_{w1} and k_{w2}, and fourthly, the stator has $m_1 = 3$ phases and a squirrel-cage rotor has $m_2 = N$ phases. Therefore, all rotor phase quantities must be referred (or transferred) to the respective stator quantities.

To begin with, we will refer the rotor phase quantities to the stator phase frequency. To this end we consider an equivalent stationary rotor such that the rotating magnetic field of the motor remains unchanged and corresponding to the operating conditions. Since the rotating magnetic field remains unchanged, this implies that the energy transferred from the stator to the revolving rotor in operation is the same as that transferred to the equivalent stationary rotor.

Since the rotating magnetic field remains constant, the rms value of the emf in the phase winding of the equivalent rotor is given by the same equation as the rms value of the emf in the stator phase winding in operation, Eq. (14.11b), so, subject to Eq. (14.14), it is

$$E_{2,st} = 4.44 f w_2 k_{w2} \Phi_r = E_2/s \tag{14.17}$$

where, for a squirrel-cage rotor, $w_2 = 1/2$ and $k_{w2} = 1$.

The term E_{2st} is the rotor phase emf referred to the stator frequency. It is one of the key quantities used in the design of induction motors, and its value is about twice that of the rms emf induced actually in a phase winding of the stalled rotor. This is because in the latter case the stator phase current is about 6.5 times the rated value, so that the voltage drop $Z_{w1}I_1$ across it increases, and there is a decrease in the magnetizing current, Eq. (14.11a), which excites the rotating magnetic field, Eq. (14.11b).

For the rotating magnetic field to remain unchanged when the spinning rotor is replaced with a stationary one, it is necessary that the currents in their phases, which have frequency $f_2 = fs$ in the former case and frequency f in the latter, be the same in amplitude and phase shift relative to the emfs that induce these currents. This is achieved by referring the phase current of the spinning rotor to the frequency f of the stationary rotor. In view of Eq. (14.17), the phase current of the spinning rotor in Fig. 14.15 may be written as

$$I_2 = \frac{E_2}{\sqrt{r_{w2}^2 + (s\omega L_{leak,\,2})^2}} = \frac{E_{2,\,st}}{\sqrt{(r_{w2}/s)^2 + (\omega L_{leak,\,2})^2}}$$

The right-hand side of the above equality corresponds to the equivalent circuit (Fig. 14.17, in which $\dot{E}_{2br} = \dot{E}_{2,st}$) of a phase of the equivalent stationary rotor, where the current frequency is f, and the term r_{w2}/s is represented by the sum of the rotor phase winding resistance r_{w2} and some additional resistance r_2 (s) which is many times r_{w2}.

As follows from comparison of the equivalent phase circuits for the spinning and the stationary equivalent rotor, the current in

them lags behind the emf by the same angle

$$\varphi_2 = \arccos \frac{r_{w2}}{\sqrt{r^2_{w2} + (s\omega L_{leak.\,2})^2}} = \arccos \frac{r_{w2}/s}{\sqrt{(r_{w2}/s)^2 + (\omega L_{leak.\,2})^2}}$$

Thus, for design purposes an operating induction motor may be replaced with an equivalent stationary motor in which each rotor phase is shorted by a resistor

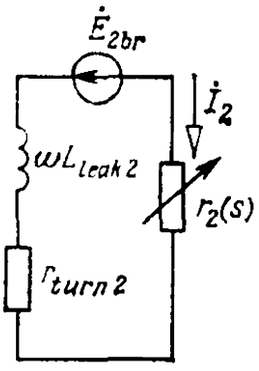


FIGURE 14.17

$$r_2(s) = r_{w2}(1 - s)/s$$

The power dissipated in this resistor, $r_2 I_2^2$, is equal to the mechanical power developed by the rotor in one phase.

Now we can refer the phase quantities of the equivalent rotor to the number of turns, the winding factor and the number of stator phases in much the same way as the secondary quantities of a transformer are referred to the primary turns (see Sec. 8.5).

The stator phase emf E_1 is connected to the phase emf of an equivalent stationary rotor, $E_{2.st}$, by a relation of the form

$$E_1 = (w_1 k_{w1}/w_2 k_{w2})\, E_{2.st} = k_e E_{2.st}$$

where k_e is the *voltage transformation ratio* of an induction motor.

As follows from the equivalent circuit (see Fig. 14.17) for a phase of a stationary rotor

$$-\dot{E}_{2.\,st} = (r_{w2}/s + j\omega L_{leak.\,2})\,\dot{I}_2$$
$$= (r_{w2} + r_2 + j\omega L_{leak.\,2})\,\dot{I}_2 = \underline{Z}_{w2}\dot{I}_2 + r_2\dot{I}_2$$

Therefore,

$$-\dot{E}_1 = (\underline{Z}_{w2} + r_2)\, k_e \dot{I}_2$$

The rotor phase current, $\dot{I}_2$, may be replaced with the current referred to the stator side, Eq. (14.16)

$$\dot{I}_2 = (3w_1 k_{w1}/m_2 k_{w2})\,\dot{I}'_2 = k_i \dot{I}'_2$$

where k_i is the *current transformation ratio* of an induction motor.

On substituting, we get

$$\dot{E}_1 = -(\underline{Z}_{w2} + r_2)\, k_e k_i \dot{I}'_2$$

The product $k_e k_i = k$ is the *transformation ratio of an induction motor*.

Now we will introduce the rotor circuit impedance and resistance referred to the stator side into the equation of electric state for a stator phase of an induction motor

$$k_e k_i \underline{Z}_{w2} = \underline{Z}'_{w2} \quad \text{and} \quad k_e k_i r_2 = r'_2$$

In the stator phase circuit these elements would draw the same power and with the same phase angle between current and voltage as they do in the rotor phase circuit shown in Fig. 14.17.

Thus, the stator phase emf is

$$\dot{E}_1 = -(\underline{Z}'_{w2} + r'_2)\,\dot{I}'_2$$

and the stator phase voltage, in accord with Eq. (14.11a), is

$$\dot{V}_1 = -\dot{E}_1 + \underline{Z}_{w1}\dot{I}_1 = \underline{Z}_{w1}\dot{I}_1 + (\underline{Z}'_{w2} + r'_2)\,\dot{I}'_2 \qquad (14.18)$$

On the other hand, $\dot{E}_1$ is proportional to the magnetizing current $\dot{I}_{1.oc}$. So, for the same reason as for a transformer,

$$-\dot{E}_1 = \underline{Z}_{12}\dot{I}_{1.oc}$$

where the term $\underline{Z}_{12}$ whose magnitude has the dimensions of ohms, represents in the equivalent circuit the magnetic circuit of the motor. Hence, the stator phase voltage may alternatively be defined as

$$\dot{V}_1 = \underline{Z}_{w1}\dot{I}_1 + \underline{Z}_{12}\dot{I}_{1.oc} \qquad (14.19)$$

Along with the stator current equation

$$\dot{I}_1 = \dot{I}_{1.oc} + \dot{I}'_2 \qquad (14.20)$$

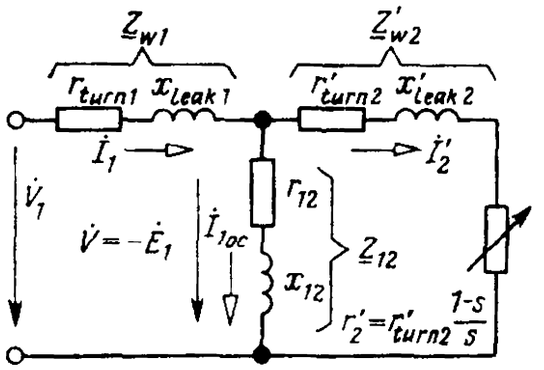


FIGURE 14.18

we may regard Eqs. (14.18) and (14.19) as corresponding to the Kirchhoff laws for the circuit in Fig. 14.18. Here, $\underline{Z}_{w1}$ represents a stator phase, $\underline{Z}'_{w2}$ represents a rotor phase, $\underline{Z}_{12}$ represents the magnetic circuit of the machine, and r'_2 represents mechanical load.

In accord with Eq. (14.16), at synchronous speed the resistive element representing the mechanical load in the equivalent circuit is

$$r'_2 = r'_{w2}/s - r'_{w2} = r'_{w2}(1 - s)/s = \infty$$

If we bring the motor to a complete stop, then $s = 1$ and $r'_2 = 0$.

For this reason the test in which a motor is brought to a complete stop (the blocked-rotor test) may be called a *short-circuit test* since it is usually carried out at a very low stator voltage.

14.10 The Phasor Diagram of an Induction-Motor Phase

Under running conditions, the frequency of the current in the stator circuit is f and that of the current in the rotor circuit is $f_2 = fs$, so no common phasor diagram may be constructed for the two circuits. However such a phasor diagram may be plotted if we assume an equivalent blocked rotor in the motor for which a model circuit representing one stator and rotor phase is shown in Fig. 14.19. This figure also shows the phasor diagram thus constructed and the phasor diagram for a stator phase carried over from Fig. 14.14 unchanged. The reference vector is $\dot{\Phi}_r$ with respect to which the stator phase emf $\dot{E}_1$ and the phase emf of the equivalent blocked rotor, $\dot{E}_{2,br}$, are in quadrature lagging.

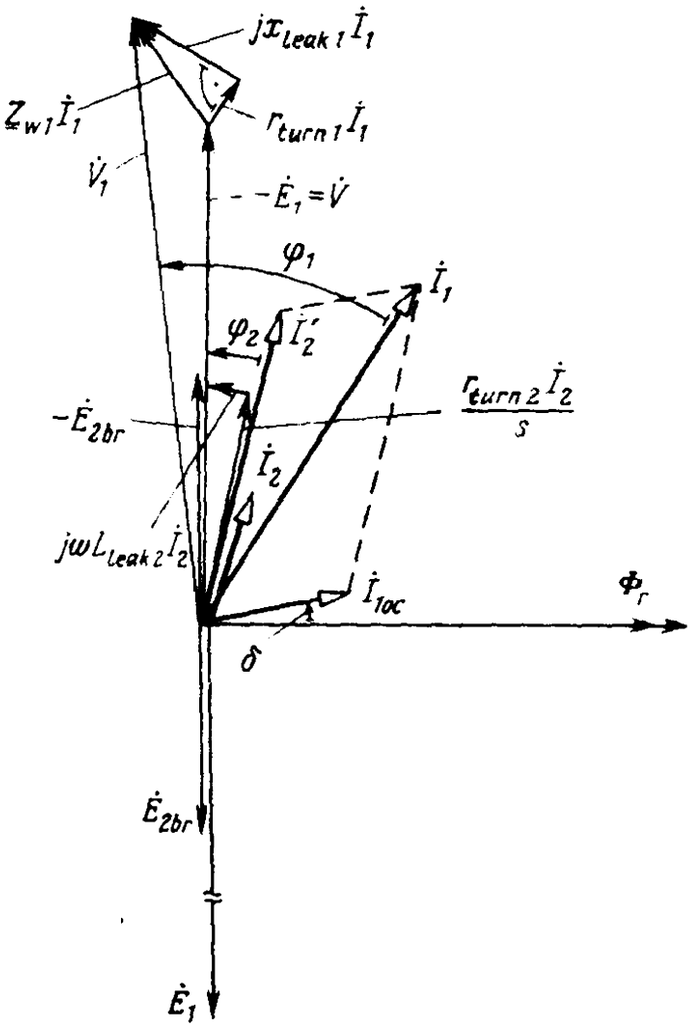


FIGURE 14.19

With respect to $-\dot{E}_{2,br}$, the rotor current $\dot{I}_2$ lags behind by an angle

$$\varphi_2 = \arctan\,[\omega L_{leak,\,2}/(r_{w2}+r_2)] = \arctan\,(s\omega L_{leak,\,2}/r_{w2})$$

which is the same phase difference as is observed between $-\dot{E}_2$ and $\dot{I}_2$ when the rotor is revolving (see Fig. 14.16).

The secondary circuit includes a resistance

$$r_{w2} + r_2 = r_{w2}/s$$

and an inductive reactance $\omega L_{leak.2}$. Accordingly, $\dot{E}_{2,br}$ consists of an active component, $r_{w2}\dot{I}_2/s$, and a reactive component $j\omega L_{leak,2}\dot{I}_2$.

The vector of the rotor phase current referred to the stator side is

$$\dot{I}_2' = (m_2 w_2 k_{w2}/3 w_1 k_{w1})\, \dot{I}_2$$

and the stator phase current is

$$\dot{I}_1 = \dot{I}_2' + \dot{I}_{1,n1}$$

In the stator phase winding the referred current $\dot{I}_2'$ corresponds to the rotor phase current $\dot{I}_2$ which it compensates. The magnetizing current $\dot{I}_{1,n1}$, which accounts for a part of the stator phase current, excites the revolving magnetic field of the motor. The magnetizing current phasor leads the magnetic flux phasor by an angle δ owing to hysteresis and eddy-current losses in the core. The phasor diagram of one phase of an induction motor with its rotor blocked is in effect the same as the phasor diagram of a transformer (see Fig. 8.11).

The stator phase voltage phasor is plotted on the basis of the following equation:

$$\dot{V}_1 = -\dot{E}_1 + r_{w1}\dot{I}_1 + jx_{leak,1}\dot{I}_1$$

14.11 The Power Balance of an Induction Motor

The total (apparent) power of an induction motor is

$$\underline{S} = P_1 + jQ_1 = 3V_1 I_1 \cos\varphi_1 + j3V_1 I_1 \sin\varphi_1$$

where P_1 is its active power and Q_1 is its reactive power.

The active power P_1 defines the average power associated with the irreversible conversion of the electric energy drawn by the motor from the supply line into mechanical, thermal and other forms of energy.

The reactive power Q_1 defines the maximum time rate of energy exchange between the supply source and the magnetic field of the motor, Eq. (2.52).

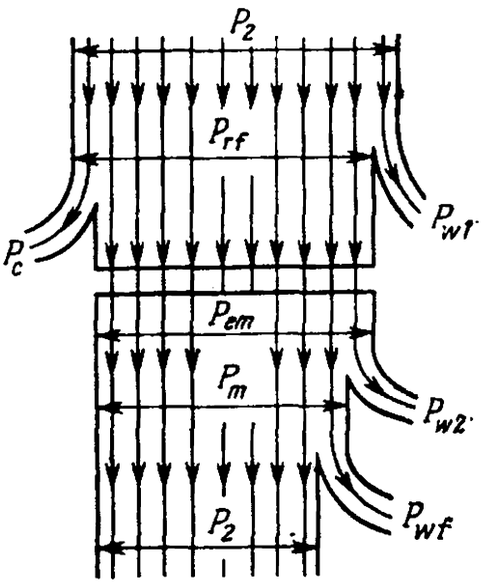


FIGURE 14.20

The active power and efficiency of an induction motor. The way in which electric power is converted in an induction motor is illustrated in Fig. 14.20. In this diagram, $P_1 = 3V_1 I_1 \cos\varphi_1$ is the power drawn by the motor from a three-phase supply line. P_{w1} is the power lost as heat in the stator conductors. The remaining power, P_{rf}, is converted to the power of a rotating magnetic field. P_c is the core

loss which includes hysteresis and eddy-current losses in the stator core. The rotor core loss is of no practical importance because at the rotor frequency f_2 (1-3 Hz) hysteresis and eddy-current losses in the rotor core are negligible.

The remainder of the rotating magnetic field power is the electromagnetic power of the rotor

$$P_{em} = P_{rf} - P_c$$

On substracting the power lost as heat in the rotor conductors, P_{w2}, from P_{em} we obtain the mechanical power developed by the rotor, P_m. Hence

$$P_m = P_1 - P_{w1} - P_c - P_{w2}$$

However, the useful mechanical power available at the motor shaft, P_2, is less than the developed mechanical power P_m by the power lost through windage and friction, P_{wf}, so

$$P_2 = P_m - P_{wf}$$

The ratio of the available mechanical power P_2 to the active power drawn from the supply line, P_1, gives the efficiency of an induction motor

$$\eta = P_2/P_1$$

Under rated operating conditions the efficiency of present-day induction motors is 80%-95%.

The reactive power and power factor of an induction motor. The reactive power Q_1 is associated with the reversible exchange of energy between the magnetic field of the motor and the supply line. The existence of a magnetic field is essential to the operation of an induction motor, so reactive power is inevitable.

In the design and operation of induction motors, it is important to know the relation between active and reactive power. This relation is defined in terms of the *power factor*

$$\cos \varphi_1 = P_1/\sqrt{P_1^2 + Q_1^2} \qquad (14.21)$$

As follows from an analysis of Eq. (14.11b), when the stator-phase terminal voltage V_1 is constant and $I_1 \leqslant I_{1.rtd}$ the magnetic flux Φ_r associated with the rotating magnetic field of the machine is likewise constant and independent of load. This implies that the energy stored in the magnetic field of an induction motor and the reactive power are constant, too, and independent of load. However, as the load is increased, the active power of an induction motor goes up. Therefore, it follows from Eq. (14.21) that an increase in load causes the power factor of the machine to increase too. If, at no load the power factor of an induction motor is 0.1-0.15, it usually goes up to 0.8-0.9 at rated load.

14.12 The Torque of an Induction Motor

An expression for the torque of an induction motor may be derived from the equation of the mechanical power, P_m, available at its shaft

$$T = P_m/\omega_{rot} \tag{14.22}$$

where ω_{rot} is the angular velocity of the rotor.

Since the angular velocity of the rotor is

$$\omega_{rot} = 2\pi n/60$$

it follows that the angular velocity of the magnetic field of a motor, ω_f, and that of its rotor, ω_{rot}, are connected by the slip

$$s = (n_1 - n)/n_1 = (\omega_f - \omega_{rot})/\omega_f$$

or

$$\omega_{rot} = \omega_f (1 - s)$$

The angular velocity of the rotating magnetic field, ω_f, is equal to the angular frequency of the sinewave current in the stator phase windings, ω, for a two-pole motor ($p = 1$). In the general case of a multipole motor the angular velocity of the rotating magnetic field is

$$\omega_f = \omega/p$$

where p is the number of pole pairs.

On expressing ω_{rot} in terms of ω in Eq. (14.22), we get

$$T = \frac{p}{(1-s)\,\omega} P_m \tag{14.23}$$

From the equivalent circuit of a motor phase, Fig. 14.18, it follows that the mechanical power at the shaft is

$$P_m = 3r_2' (I_2')^2 = 3r_{w2}' \frac{1-s}{s} (I_2')^2 = r_{w2} m_2 \frac{1-s}{s} I_2^2 \tag{14.24}$$

Substituting Eq. (14.24) into Eq. (14.23) yields an expression for the torque of an induction motor

$$T = m_2 p r_{w2} I_2^2/\omega s \tag{14.25}$$

Since, on the basis of the phasor diagram in Fig. 14.15,

$$r_{w2} I_2 = E_2 \cos \varphi_2$$

and

$$E_2/s = E_{2.st}$$

it follows that

$$T = E_{2.\,st} I_2 \frac{m_2 p}{\omega} \cos \varphi_2 \tag{14.26}$$

On recalling Eq. (14.12),

$$E_{2.\,st} = 4.44 f w_2 k_{w2} \Phi_r = \omega w_2 k_{w2} \Phi_r / \sqrt{2}$$

Eq. (14.26) may be re-cast as follows

$$T = (1/\sqrt{2})\, m_2 p w_2 k_{w2} \Phi_r I_2 \cos \varphi_2 = \text{const} \times \Phi_r I_2 \cos \varphi_2 \tag{14.27}$$

That is, the torque of an induction motor is proportional to the product of the rotating magnetic flux and the current in the rotor winding.

14.13 The Torque-Slip Curve and Speed-Torque Characteristic of an Induction Motor

For an induction motor to run steadily, it is essential that its torque should balance the applied load automatically—as the load applied to the shaft of a motor is increased, the torque developed by the motor must go up in proportion. In a running induction motor, this condition of balance is achieved in the following way. As the load applied to the shaft is increased, the opposing torque exceeds the driving torque, the rotor slows down, and the slip goes up. This increase in the slip brings about an increase in the driving torque, and the two torques, driving and opposing, strike a balance at an increased slip.

The dependence of the driving torque on the slip is not so straightforward as it might appear from the foregoing, however. The point is that in Eq. (14.27) defining the driving torque of an induction motor, all the three quantities involved (I_2, Φ_r, and $\cos \varphi$) are functions of the slip. The rotor current I_2 (see Sec. 14.7) increases rapidly with increasing slip due to the increase in E_2 which is proportional to the slip. At first, so long as the leakage reactance of the rotor winding, $s\omega L_{leak.2}$, remains small in comparison with its resistance, r_{w2}, the increase in the slip is accompanied by a rapid growth in the rotor current. Then, at $s\omega L_{leak.2} \geqslant r_{w2}$, this growth in rotor current slows down.

The power factor of the rotor circuit

$$\cos \varphi_2 = r_{w2} / \sqrt{r_{w2}^2 + (s\omega L_{leak.2})^2}$$

first decreases very slowly as the slip increases, then its decrease becomes more rapid. It seems useful to explain the physical con-

ditions that govern the effect of $\cos \varphi_2$ on the driving torque. To this end, we assume the extreme ideal conditions. First we assume that $\cos \varphi_2 = 0$ or, which is the same, that the rotor winding only has an inductive reactance. In the circumstances, the rotor current would be a maximum in those of the rotor conductors where the induction was a maximum a quarter of a cycle previously (Fig. 14.21). The current would be maximum where there is no induction associated with the rotating magnetic field, so the forces acting on the remaining rotor conductors cancel out one another and the driving torque at the shaft is zero.

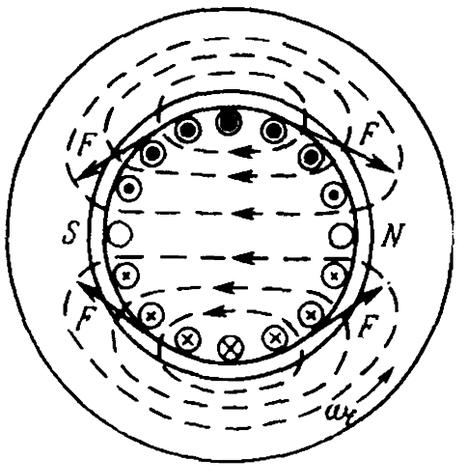


FIGURE 14.21

In reality, the rotor circuit of an induction motor has both a resistance and an inductive reactance, with the latter varying in proportion to the slip so that the driving torque of the machine cannot but be affected.

The slip also affects the main flux, Φ_r, of the machine, although to a lesser degree. As the slip increases, the stator current I_1 goes up, but the stator emf

$$\dot{E}_1 = \dot{V}_1 - Z_{w1}\dot{I}_1$$

goes down, and so does Φ_r proportional to it, because, in accord with Eq. (14.10)

$$\Phi_r = E_1/4.44 f w_1 k_{w1}$$

In Eq. (14.27), three quantities are functions of the slip, with I_2 increasing and with Φ_r and $\cos \varphi_2$ decreasing as the slip increases. Thus, there is what is known as the *critical slip*, s_{cr}, which corresponds to a maximum value of the driving torque.

In order to define the maximum driving torque in terms of the motor parameters, let us turn to Eq. (14.25) where I_2 can be deduced from the equivalent circuit of a stator phase, Fig. 14.19, on neglecting the no-load (open-circuit) current, $I_{1,oc}$:

$$T = (3p/\omega s) \frac{r'_{w_2} V_1^2}{(r_{w1} + r'_{w_2}/s)^2 + (x_{leak,\,1} + x'_{leak,\,2})^2} \qquad (14.28)$$

On taking the first derivative of Eq. (14.28) with respect to the slip and equating it to zero, we get

$$dT/ds = (3p/\omega) V_1^2 r'_{w_2} \frac{r'^2_{w_2} - s^2 [r^2_{w_1} + (x_{leak,\,1} + x'_{leak,\,2})^2]}{[(r_{w1} s + r'_{w_2})^2 + s^2 (x_{leak,\,1} + x'_{leak,\,2})^2]^2} = 0 \qquad (14.29)$$

In Eq. (14.29), only the numerator may be equal to zero, so the critical slip corresponding to a maximum driving torque is

$$s_{cr} = \pm r'_{w_2} / \sqrt{r_{w_1}^2 + (x_{leak,1} + x'_{leak,2})^2}$$

where the "—" sign applies when a machine is running as a generator. Since in actual machines $r_{w_1}^2$ is never more than 5% of the radicand, this term may be neglected. So the slip as expressed in terms of the rotor circuit parameters referred to the stator side will be

$$s_{cr} = \pm r'_{w_2} / (x_{leak,1} + x'_{leak,2}) \tag{14.30}$$

On replacing r'_{w_2} with kr_{w_2}, $x'_{leak,2}$ with $kx_{leak,2}$ and $x_{leak,1}/k$ with $x''_{leak,1}$, we obtain an expression for s_{cr} in terms of the rotor-winding resistance r_{w_2}, the rotor-winding leakage reactance $x_{leak,2}$, and $x''_{leak,1}$ which is the inductive leakage reactance of a stator phase winding referred to the number of phases, number of turns and winding coefficient of the rotor:

$$s_{cr} = \pm r_{w_2} / (x''_{leak,1} + x_{leak,2}) \tag{14.31}$$

The leakage inductance of the rotor windings is relatively large because the rotor conductors are placed in core slots. For this reason the maximum torque developed by an induction motor usually corresponds to fairly small values of slip, which is 4% for large-size motors and up to 14% for small-size motors.

A typical plot of the torque-vs-slip characteristics of an induction motor appears in Fig. 14.22. The peak value of torque divides the plot into a stable region between $s = 0$ and s_{cr}, and an unstable region from s_{cr} to $s = 1$ where the torque decreases with increasing slip.

In a running motor, a dynamic balance of the driving and load torques is maintained automatically with increasing slip so long as the opposing (load) torque remains smaller than the maximum torque developed by the motor. Once the load torque becomes equal to the maximum motor torque, any increase in the slip with a further increase in load will only lead to a decrease in the driving torque. In other words, the dynamic balance upset by an increase in load will no longer be restored and the motor will ultimately be brought to a stop by the prevailing load torque.

Now we will express the maximum driving torque in terms of the machine parameters. This can be done by substituting Eq. (14.30) into Eq. (14.28).

On neglecting r_{w_1} in comparison with $(x_{leak,1} + x'_{leak,2})$, we obtain an expression for the maximum torque of an induction motor in the following simple form:

$$T_{max} = (3p/2\omega)\, V_1^2 / (x_{leak,1} + x'_{leak,2}) \tag{14.32}$$

The maximum torque determines the overload capacity of an induction motor. Equation (14.32) shows that T_{max} is independent of the rotor-circuit resistance, whereas, as follows from Eqs. (14.30) and (14.31), the critical slip is proportional to that resistance. So, by increasing the resistance in the rotor circuit, we can raise the critical slip without causing any change in the maximum torque. This condition is utilized in order to improve the starting of wound-rotor induction motors.

The fact that the maximum torque is proportional to V_1^2 makes induction motors very sensitive towards a fall in the supply-line volt-

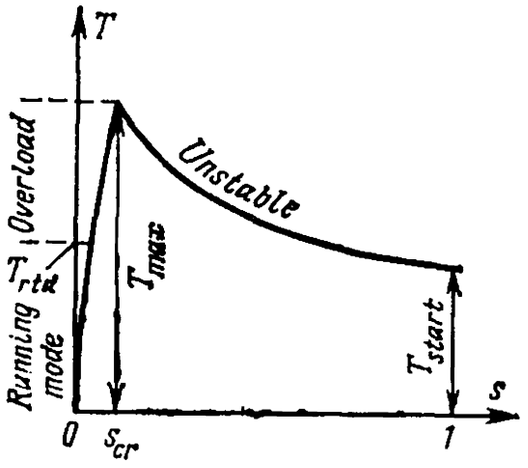


FIGURE 14.22

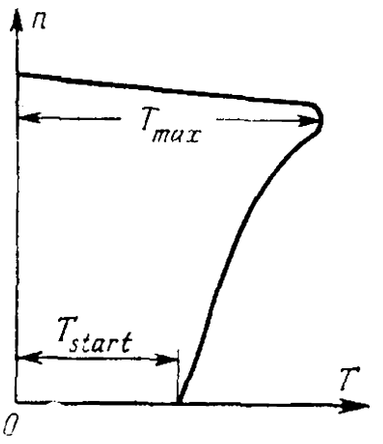


FIGURE 14.23

age. If V_1 happens to fall considerably, the motor may fail to start at all.

For conventional induction motors the maximum torque is 2-2.5 times the rated torque.

For purposes of electric drive, it is important to consider the *speed-vs-torque characteristic*, n (T), of an induction motor. It differs from the torque-slip curve only in that the coordinate axes are reversed.

As is seen from the torque-speed characteristic shown in Fig. 14.23, the speed of an induction motor only slightly falls as the torque increases from zero to its maximum value—this is what is known as a flat speed-torque characteristic. When the load exceeds T_{max} (which occurs when the load is twice or three times the maximum torque), the motor enters an instability region (if the load applied to the shaft remains constant) and comes to a stop. For this reason, the power rating of a motor should be chosen so that a momentary overload could not exceed its maximum torque; otherwise the drive must be fitted with a flywheel.

The speed-torque characteristic applying to the normal operating conditions is called the natural speed-torque characteristic in contrast to that of a wound-rotor induction motor in which the rotor circuit contains a rheostat.

14.14 Starting of an Induction Motor

The starting performance of induction motors is an important consideration in their evaluation. This performance is basically determined by the following quantities: the starting current I_{start}, the starting torque T_{start}, the starting time, and the economy of operation at starting. Makers' data sheets usually quote the starting-to-rated ratio of a given quantity (I_{start}/I_{rtd} or T_{start}/T_{rtd}).

The behaviour of an induction motor at starting is determined by its construction, notably the rotor arrangement.

Starting of wound-rotor induction motors. The starting of a wound-rotor induction motor (see its equivalent circuit in Fig. 14.24 a, and its circuit diagram in Fig. 14.24 b) can be improv-

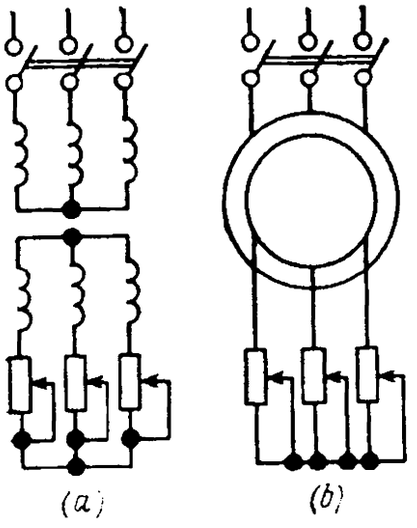


FIGURE 14.24

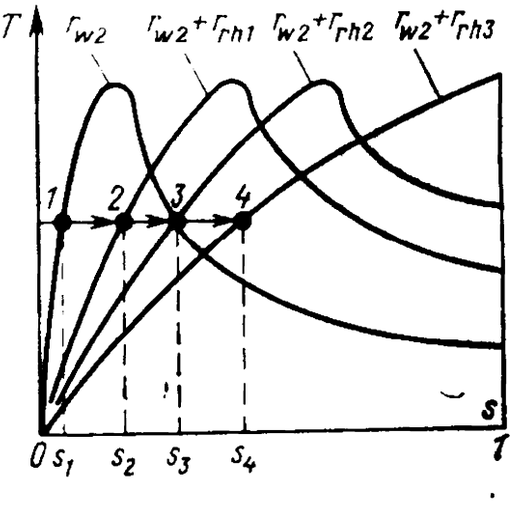


FIGURE 14.25

ed substantially at the cost of some complication in its design and operation.

If we put $s = 1$ in Eq. (14.28), we will obtain an expression for the breakaway torque or the torque that a motor is required to develop in order to break away its load from rest to rotation

$$T_{start} = (3p/\omega) \frac{r'_{w_2} V_1^2}{(r_{w1} + r'_{w_2})^2 + (x_{leak,\,1} + x'_{leak,\,2})^2} \tag{14.33}$$

If it is required that a motor should develop its maximum torque at starting, $T_{start} = T_{max}$, then, in accord with Eq. (14.30) the resistance of a rotor phase winding should be

$$r'_{w2} = x_{leak,1} + x'_{leak,2} \tag{14.34}$$

or

$$r_{w2} = x''_{leak,1} + x_{leak,2}$$

Since the resistance of a rotor phase winding is relatively small, the maximum breakaway torque can be obtained by placing in the rotor circuit a starting rheostat whose resistance per phase is

$$r_{rh} \approx x''_{leak.1} + x_{leak.2}$$

Then the torque-slip curve of an induction motor will have a peak at $s = 1$ (Fig. 14.25).

Just as the rotor begins rotating, the slip decreases, and so do the rotor emf and current with the result that the torque is reduced. For the motor to keep developing a torque close to the maximum one, the resistance of the starting rheostat must be gradually brought down. Finally, when the motor has come up to its rated speed, the starting rheostat must be shorted out.

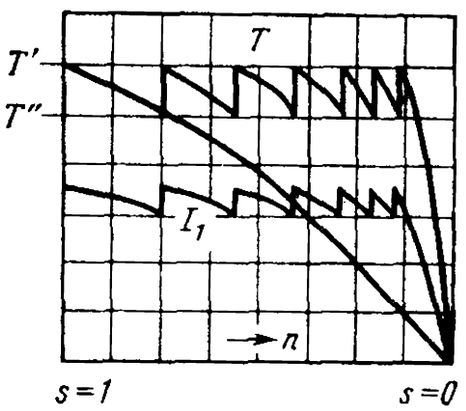


FIGURE 14.26

Induction motors are sometimes fitted with a device to lift the brushes and to short-circuit the slip-rings in order to reduce mechanical losses and the wear of the rings and brushes.

The inclusion of a rheostat in the rotor circuit affects the shape of the torque-slip curve. As follows from Eq. (14.32), this does not affect the maximum torque; an increase in r_{rh} only shifts the peak of the torque-slip curve towards the higher values of slip. For this reason all the torque-slip curves of an induction motor have their peaks at the same ordinate (see Fig. 14.25). As the starting rheostat is switched from step to step, the motor's performance moves from one curve to another. It is usual to bring out the sections of the starting rheostat to contacts so that at starting the torque and current of an induction motor could be varied in steps as illustrated by the curves in Fig. 14.26, the number of steps being the same as there are contacts in the starting rheostat. In the process, the starting torque varies from T'' to T'.

The higher the starting torque must be, the closer it is to the maximum torque and the higher the starting current. For this reason it is only for extremely adverse starting conditions that the rheostat is chosen such that the starting torque is equal to the maximum one.

The power rating of the *starting rheostat* is chosen to be approximately equal to that of the motor so as to avoid overheating. For large-size motors the starting rheostats use oil cooling. In other cases use is sometimes made of the cheaper and simpler water-cooled starting rheostats.

Of course, the use of a starting rheostat substantially improves the starting conditions for induction motors as it raises the starting

torque and reduces the starting current. On the other hand, wound-rotor induction motors are more expensive to build and less simple to maintain and operate, and have a low power factor and efficiency. For large-size induction motors this difference in efficiency and power factor is negligible, so the remaining limitations of wound-rotor motors are the higher cost, the longer starting time, and the more complicated starting procedure.

The starting of a squirrel-cage induction motor. In most cases squirrel-cage induction motors are started "direct-on-line". Direct-on-line starting is simple and takes up little time, but it calls for a simple switching device, such as a knife-blade switch or, in the case of high-voltage motors, an oil circuit breaker. In the case of direct-on-line starting the ratio of starting to rated current is high, being from 5.5 : 1 to 7 : 1 (for motors in sizes from 0.6 to 100 kW at synchronous speed, that is 750-3000 rpm). This short-duration starting current is relatively safe for the motor, but it brings about variations in the supply-line voltage which may adversely affect the other loads connected to the same supply line. For this reason the power rating of direct-on-line started induction motors depends on that of the available supply line. In the case of industrial supply lines, direct-on-line starting may be applied to squirrel-cage induction motors in sizes up to 1 MW or even bigger. In many cases, however, the power rating of motors may not exceed 100 kW.

There is one more drawback of direct-on-line starting to be kept in mind. For squirrel-cage motors the ratio of starting to rated torque ranges from 1 : 1 to 2 : 1 (see Fig. 14.22). This implies that at a high starting current the motor develops a relatively small starting torque.

The starting of induction motors can be facilitated by reducing the starting current where necessary while keeping the starting torque as large as practicable. One way to minimize the starting current is to bring down temporarily the voltage between the stator phase terminals by connecting a three-phase inductor in series with the stator winding. In such a case, the starting torque is reduced in proportion to the square of the stator voltage, Eq. (14.33). For example, halving the starting torque would reduce the starting torque to one-quarter of its full value. Unfortunately, this form of starting cannot be used to start a motor under load—it may so happen that the motor is not able to overcome the opposing mechanical moment at the output shaft.

As an alternative, the stator voltage can be brought down at starting by changing over from the delta connection normally used for the stator phases to the star connection temporarily. This form of starting is only applicable to squirrel-cage motors in sizes not over about 20 kW.

A much better starting performance is displayed by induction

motors with a double squirrel-cage rotor. As its name implies, the rotor is made up of two squirrel-cage structures, the outer cage being used for starting, and the inner cage for normal running. The outer cage is usually made of manganese brass which has a higher resistivity than copper. The inner cage is made of copper, and its bars are usually larger in cross-sectional area. For this reason the outer cage has a higher resistance than the inner cage (by a factor of 4 or 5). As is the case with the simple cage rotor, the double cage is shorted at the ends by end-rings.

The bars are located in the outer and inner parts of the slots. Owing to this arrangement there is a marked difference in leakage inductance between the outer and inner bars. The inner cage has a high leakage inductance because its bars are surrounded by a steel core with only a narrow slot at the top (Fig. 14.27 a and b). The outer cage has a lower leakage inductance because its leakage flux has its path mostly through the high-reluctance air gap between the rotor and stator and also across the slot width.

At the very beginning of the starting procedure (so long as $s = 1$), the currents in the rotor winding have the same frequency as the supply line; in the circumstances the impedance of the inner cage is mainly due to its high leakage inductance. Because of this, at starting the current is displaced from the inner cage. At the same time the impedance of the outer cage is primarily determined by its resistance. At starting, the current in the outer cage is almost in phase with the induced emf, and it gives rise to a high starting torque as is the case with a wound-rotor motor using a starting rheostat. The ratio of outer-cage to inner-cage currents depends on the ratio of their impedances; as a rule the current in the inner cage at starting is substantially lower than it is in the outer cage.

As the motor comes up to speed, the frequency of the currents in the rotor decreases, and so does the effect of the inductive reactance on the current distribution. At rated speed, the frequency of the rotor current is of the order of 1 Hz; in these conditions the leakage inductive reactance is very small and the current distribution between the two cages is determined by the ratio of their resistances. That is why the current in the outer cage is smaller than it is in the inner cage—its resistance and impedance are now small as they usually are in a single-cage induction motor.

The torque-slip characteristic of a double-cage induction motor may be looked upon as the sum of the characteristics of a motor in which the rotor winding has a relatively high resistance (see Fig. 14.27, curve 1) and of a motor in which the rotor winding has a relatively small resistance (see Fig. 14.27, curve 2).

Thus, the resistance of the rotor winding in a double-cage induction motor is a function of the slip—it is high at starting and low at the operating slip. This is the reason why a double-cage motor de-

velops a higher starting torque at a lower starting current than a single-cage motor.

In some designs, the two cages are made by casting molten aluminium directly into the rotor slots (*c* and *d* in Fig. 14.27). In such a case the aluminium also fills the neck between the top and bottom slots, and the top and bottom bars are joined together to form shaped bars.

A simplified modification of the double-cage type is the deep-bar type. In a deep-bar rotor, the conductors are rectangular bars of a small width and a large height, placed in the deep slots of a rotor core. In the general case, alternating current is distributed

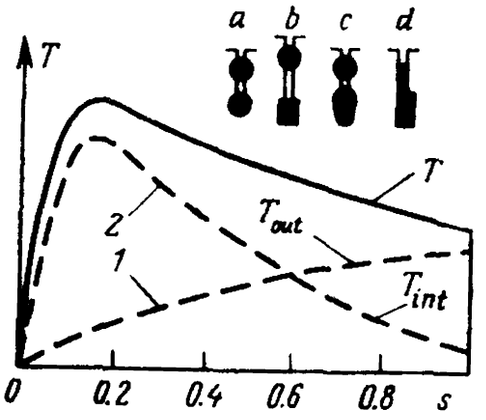


FIGURE 14.27

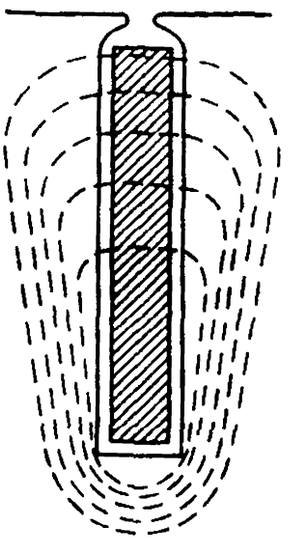


FIGURE 14.28

nonuniformly over the cross-section of a conductor—this fact is utilized in a deep-bar cage motor. The distribution of the leakage flux in the case of a deep-bar cage is shown in Fig. 14.28. The part of the bar lying deep in the slot links with a greater proportion of the flux than the part lying at the top of the same slot. At starting, the increased inductive reactance of the top part of the bar causes the rotor current to shift towards the outer portion of the bar. This is equivalent to an increase in the resistance of the rotor winding. This increase in rotor-winding resistance entails an increase in the breakaway torque of the motor, while the increase in inductive reactance due to the use of a deep bar serves to bring down the starting current. At the normal running speed the inductive reactance falls to a negligible value, the current is distributed over the cross-sectional area of each conductor almost uniformly, and the motor is running like an ordinary cage motor.

From the view-point of design, a deep-bar cage motor is simpler than its double-cage counterpart. On the other hand, a double-cage motor may be built to develop a wide range of breakaway torque and

various ratios of starting-to-rated current, so that it can be adapted to meet especially adverse starting conditions. Yet, deep-bar cage motors are used most frequently.

14.15 The Operating Characteristics of an Induction Motor

The speed-torque characteristic of an induction motor is a convenient indicator of its properties as a part of an electric drive. A deeper insight into its performance, however, can be gleaned from what we call its *operating characteristics* which relate its speed n, its torque T, its power factor $\cos \varphi$ and its efficiency $\eta = P_2/P_1$ to its output power (that is, the power available at its output shaft), P_2. The operating characteristics are plotted at the rated supply frequency f and at the rated stator terminal voltage $V_1 = V_{1.rtd}$.

Since $P_2 \approx P_m$, and $P_m = T\omega_{rot}$, it follows that the speed characteristic $n = f(P_2)$ differs but little from the speed-torque characteristic of the same motor, and it may likewise be called flat (Fig. 14.29).

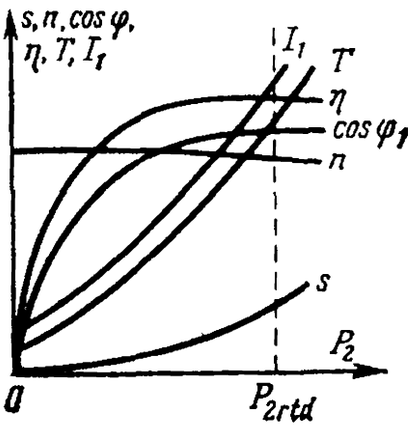


FIGURE 14.29

The torque T developed by an induction motor is the sum of the useful torque T_2 (utilized to overcome the load applied to its output shaft) and the no-load torque T_{nl} (which goes to make up for the mechanical losses in the motor). The no-load torque may be assumed to be independent of the load sustained by the motor. The useful torque is given by

$$T_2 = P_2/\omega_{rot}$$

and if ω_{rot} were always the same, the relation $T_2 = f(P_2)$ would be linear. Actually, however, the angular velocity of the motor slightly falls off with increasing P_2, so the plot of T_2 as a function of P_2 bends upwards as slightly. In consequence, the plot of T as a function of P_2, which is a combination of the plots of T_{nl} and T_2 as functions of P_2, intersects the axis of ordinates at a point corresponding to T_{nl}, past which it remains nearly flat and only slightly bends upwards.

As the load on an induction motor changes, the power factor $\cos \varphi_1$ varies due to the following causes. The magnetizing current of an induction motor depends slightly on load, because an increase in it only leads to an increase in the leakage flux linkages which are proportional to the currents in the stator and rotor windings, whereas

the main magnetic flux of the machine goes down a little as the load is increased. On the other hand, the current in the motor is proportional to its mechanical load. Thus, as the load on a motor is increased, the relative value of reactive current falls off at a high rate but the power factor cos φ_1 increases. At no load, the power factor of an induction motor is low, being about 0.2. As the load goes up, the power factor increases, too, and achieves its maximum value (0.7-0.9) at a load close to the rated one. As a result, the reactive current accounts for anywhere between 40 and 70% of the stator current even in the case of a motor running at full load.

The fact that induction motors are never loaded to their full capacity is one of the principal causes why industrial networks operate at a low power factor. A natural way to improve the power factor would be to operate induction motors at full rated load. The main magnetic flux of an induction motor is proportional to its stator voltage, Eq. (14.11b). Given a fixed value of flux, the magnetizing current exciting this flux is inversely proportional to the reluctance existing in the path traversed by the flux. The greater part of this reluctance is due to the air gap between the stator and the rotor. For this reason the designer usually strives to keep this air gap to a minimum just sufficient to assure proper operation of the bearings, to provide for the sag of the shaft, and to accommodate likely errors in centring. As a motor goes up in power rating, the necessary air gap increases more slowly than the power rating, so its power factor improves. As a motor goes down in rated speed, the main magnetic flux increases, because at a lower speed it induces a lower emf in the stator phases. As a consequence, low-speed motors have a relatively heavier magnetizing current and a substantially lower power factor.

The efficiency of an induction motor is defined as the ratio of the output power (that which is available at its shaft) P_2 to the power P_1 it draws from the supply line

$$\eta = P_2/P_1$$

The input power P_1 is the sum of the useful or output power P_2 and of all the losses

$$P_1 = P_2 + P_{loss}$$

The losses occurring in a motor may be divided into the fixed (or constant) losses which are independent of load, and the variable loss which is dependent on load.

The *fixed* (or *constant*) *losses* combine the power lost in the stator core due to hysteresis and eddy currents, and mechanical losses found from a no-load test.

The *variable loss* occurring in a motor is due to the power dissipated as heat in the stator and rotor windings

$$P_{v1} = 3r_{w1}I_1^2$$

and

$$P_{v2} = m_2 r_{w2} I_2^2$$

The efficiency of an induction motor is a maximum (65-95%) when the variable loss is equal to the fixed losses (see Sec. 8.9). For most motors, the maximum efficiency occurs at a load equal to 75% of the rated one, because motors are designed in view of the fact that they are not always operated at full load.

14.16 The Universal Torque-Slip Curve of an Induction Machine

As is the case with any electric machines, induction machines are reversible. This means that one and the same induction machine can operate as a motor, that is, convert electricity into mechanical energy (motoring action), or as a generator, that is, convert mechanical energy into electricity (generating action).

If we wish an induction machine to operate as a generator, we should apply an external mechanical force to its shaft and cause its rotor to revolve at a speed exceeding the synchronous one so that $n > n_1$. Then the rotor will overtake the rotating magnetic field and its conductors will cut the magnetic lines of force in a direction opposite to that in which they cut the flux in motoring. As a consequence, the rotor emf and the rotor currents will be reversed in direction. As a result, the force produced by the interaction of the rotating magnetic field with the rotor currents is reversed and acts against the rotation of the rotor. So that the rotor can keep rotating, mechanical energy must be transferred to it from an external source. The magnetizing current remains unchanged, because the conditions for a rotating magnetic field to be excited are the same in an induction motor and an induction generator. In the generating mode, however, the power developed by the machine is negative which means that it delivers energy to, rather than draws energy from, the supply line. Now the slip is

$$s = (n_1 - n)/n_1 < 0$$

The *negative slip* is an indication of generating action. As the slip increases in magnitude, the inductive reactance of the rotor winding increases, and so does the phase shift between the rotor emf and current. On the other hand, the rotor emf which is proportional to the slip increases and, since the opposing torque of an induction

generator is defined in the same way as the driving torque developed by an induction motor, Eq. (14.27),

$$T = \text{const} \times \Phi_r I_2 \cos \varphi_2$$

it follows that this torque will be a maximum when the slip, Eq. (14.31), is

$$s_{cr} = -r_{w2}/(x''_{leak,1} + x_{leak,2})$$

Any further increase in the negative slip brings about a decrease in the torque. To sum up, the torque-slip curve of an induction generator resembles, in general, that of an induction motor and is actually its extension in the third quadrant of the Cartesian coordinate system (Fig. 14.30).

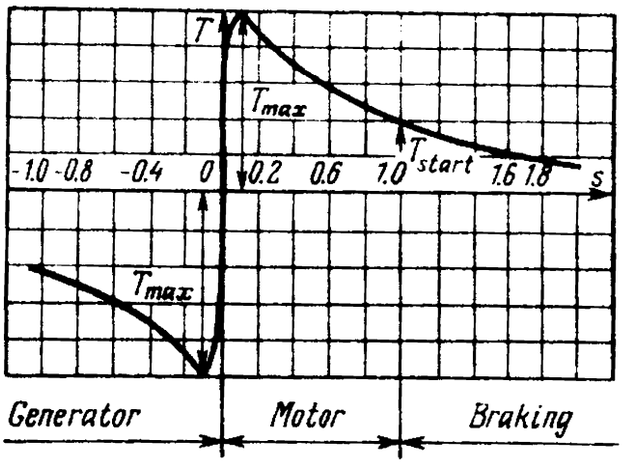


FIGURE 14.30

An induction generator draws an inductive reactive (magnetizing) current in much the same way as an induction motor does, and so it needs a source of reactive power. In other words, an induction generator cannot operate on its own. When connected to the supply line, an induction generator impairs the overall power factor of the system.

The situation is different and an induction generator can operate on its own if the required reactive current is supplied by capacitors connected in parallel with the generator. Now at starting the induction generator is self-excited due to the effect of the residual magnetization in its magnetic circuit.

An advantage of induction generators is that they are simple in design and to operate.

If an external mechanical force is applied to spin the rotor of an induction machine in a direction opposite to that of its rotating magnetic field, the speed n in the slip equation, Eq. (14.8), will take a "—" sign, so that

$$s = (n_1 + n)/n_1 > 1$$

The direction of current flow in the rotor winding will remain unchanged, and the rotor will develop a torque opposing the load torque applied to the shaft. Now the machine will be using the mechanical energy applied to the shaft and the electric energy drawn from the supply line. This will be *braking action*. The plot of torque as a function of slip at $s > 1$ is a direct continuation of the motor characterist-

ic, being the third portion of the *universal torque-slip curve* of an induction machine.

The braking mode is used to bring the machine to a rapid stop or when it is advisable to use an induction machine to brake a driven machine, such as cranes or hoists when the load is lowered.

An induction motor can be switched into the braking mode by what is called plugging (reverse-current braking). For this purpose one interchanges any two phases of the stator (see Fig. 14.7 a), and this causes the magnetic field to rotate in the opposite direction to the rotor. Now the slip is

$$s = (n_1 + n)/n_1 > 1$$

and the rotor moves against the field under the action of an external mechanical force (such as the weight of the load being lowered) or by inertia. When the rotor comes to a stop, it is essential to disconnect the machine from the supply line, or else it will go motoring.

14.17 Speed Control of Induction Motors

In many cases three-phase induction motors are used in drives where speed control is not required. However, induction motors have valuable advantages to offer, namely reliability, low cost, simple design, high efficiency, and a relatively small weight. For this reason, it is only natural that efforts have been made to adapt them to service in variable-speed drives.

In the case of squirrel-cage motors speed control is effected through frequency control. In this approach the speed at which the magnetic field rotates is varied by adjusting the frequency of the current in the stator windings, or by varying the number of pole pairs of the rotating magnetic field so that the speed of the field is changed stepwise.

In the case of wound-rotor motors their speed is controlled by varying the resistance of the phase windings with the result that the slip of the rotor is caused to change continuously.

Speed control with varying frequency. Of all the existing methods of speed control the most promising one is by *varying the frequency of the alternating current in the stator winding*. The angular velocity of the rotating field is

$$\omega_f = 2\pi f/p$$

Hence, a change in f causes the angular velocity of the field to vary in proportion. In practice, the voltage must also be changed. This is because, as follows from Eq. (14.10), the phase emf and, as a consequence, the supply voltage are proportional to the frequency and flux. Since the flux must be maintained unchanged in any mode of operation, the voltage must be made to vary in proportion to the

frequency (ignoring the voltage drops in the machine). Also this must be done so that the torque will not change as the speed is changed.

To evaluate the dependence of the torque on the current frequency in and the voltage across the stator, we will neglect the stator winding resistance r_{w1} and the leakage inductive reactances of the stator, $x_{leak,1}$, and of the rotor, $x_{leak,2}$, in Eq. (14.28) and use the expression for the slip frequency, Eq. (14.13):

$$T = 3psV_1^2/\omega r'_{w2} = (3pf_2/2\pi f^2 r'_{w2})\, V_1^2 \approx AV_1^2/f^2 \qquad (14.35)$$

where A is a constant.

Thus, if we wish to maintain the torque at a constant value as the current frequency is varied, we must vary the voltage across the stator winding. In other words, the condition for maintaining a constant torque as the frequency is varied takes the form

$$V_1/f = \text{const}$$

If we vary both the current frequency and the voltage so that the above condition is satisfied, the speed-torque characteristic of the motor will remain flat, and the maximum torque will be almost independent of frequency (it will change markedly only at relatively low frequencies). At the same time, the output power will vary in proportion to frequency, because

$$P_2 = T\omega_{rot}$$

For example, halving the frequency would cause the output power to fall to half its original value as well.

Speed control by varying the number of pole pairs. This form of speed control is possible over a wide range of values, but it involves a more sophisticated motor design and a higher cost.

So long as the supply frequency remains constant, the angular velocity of the rotating magnetic field is solely a function of the number of pole pairs that the stator-induced field has. Suppose that we have two independent windings on the same stator, one producing p pole pairs and the other, p' pole pairs. Then, when energized, the former will produce a field rotating at

$$n_1 = 60f/p$$

and the latter will produce a field rotating at

$$n'_1 = 60f/p'$$

Hence

$$n_1/n'_1 = p'/p$$

The speed of the rotor will be different in a similar way. The rotor winding must of course be of the squirrel-cage type.

The number of pole pairs in one winding is in no way related to that in the other, and both may be chosen according to the conditions in which the motor is to operate. Speed control reduces to a stepwise change in the rotational velocity of the magnetic field. However, the speed of the rotor cannot be changed stepwise due to the inertia of the entire drive. For the rotor speed to change, it is necessary to make changes in winding connections.

For a better insight into the process, let us construct two speed-torque characteristics for an induction machine with a variable number of pole pairs, one for p pole pairs and the other for $p' = 2p$ pole pairs, respectively (see Fig. 14.31 a and b). Suppose that as the speed of the rotating field is changed the output torque remains the same. As the speed is increased, that is, in going from p' to p pole pairs, the motor is first in a state close to the starting condition and the current undergoes a stepwise change. In going from p to p', however, that is, when the speed of the rotating field is decreased, the motor is first in the generating mode and delivers energy to the supply line. This mode is sometimes used to slow down an electric drive rapidly and economically.

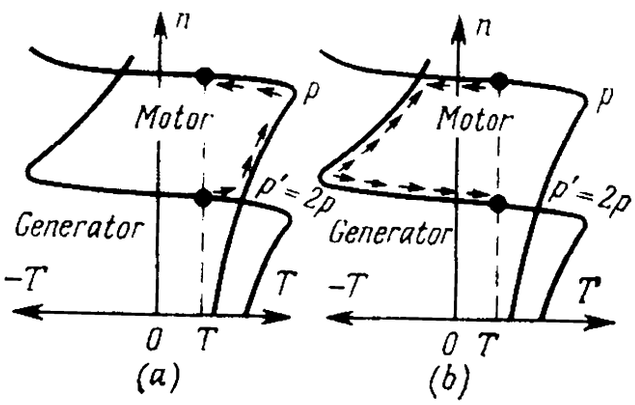


FIGURE 14.31

Two independent stator windings are provided only on small induction motors. In the case of large motors it is more advantageous to obtain a varying number of pole pairs by switching the coils of a single stator winding. The manner in which a single three-phase winding is switched from two to four poles is illustrated in Fig. 14.32. For pole-pair ratios other than 1 : 2 a more complicated switching arrangement would be necessary, so it is used but seldom. In most induction motors the stator carries two independent windings for each of which the number of pole pairs can be switched in the ratio 1 : 2 or any other. In this way, such a motor can be set to operate at 3000, 1500, 1000, and 500 rpm.

Speed control with changes of rotor-circuit resistance. This form of speed control (often called rheostat speed control) is applicable to three-phase wound-rotor induction motors. It is effected by placing a three-phase rheostat in the phase circuits in much the same way as at starting (see Fig. 14.24). In contrast to a starting rheostat, a speed-control rheostat must be designed to carry a sustained rather than a short-duration load current. The increase in the rotor-circuit resistance makes the torque-slip curve more drooping (see Fig. 14.25). Given the same torque at the output shaft, an increase in the rotor-

circuit resistance by increasing the setting of the speed-control rheostat ($r_{rh1} < r_{rh2} < r_{rh3}$) will cause the operating point to move from one torque-slip curve to the next which corresponds to the increased rotor-circuit resistance (points 1 through 4 in Fig. 14.25). Accordingly, the slip will increase too, and the speed of the motor will go down. In this way, the speed of the motor can be varied from the rated one to a full stop. Unfortunately this form of speed control involves relatively large power losses (see Sec. 14.11). If we neglect

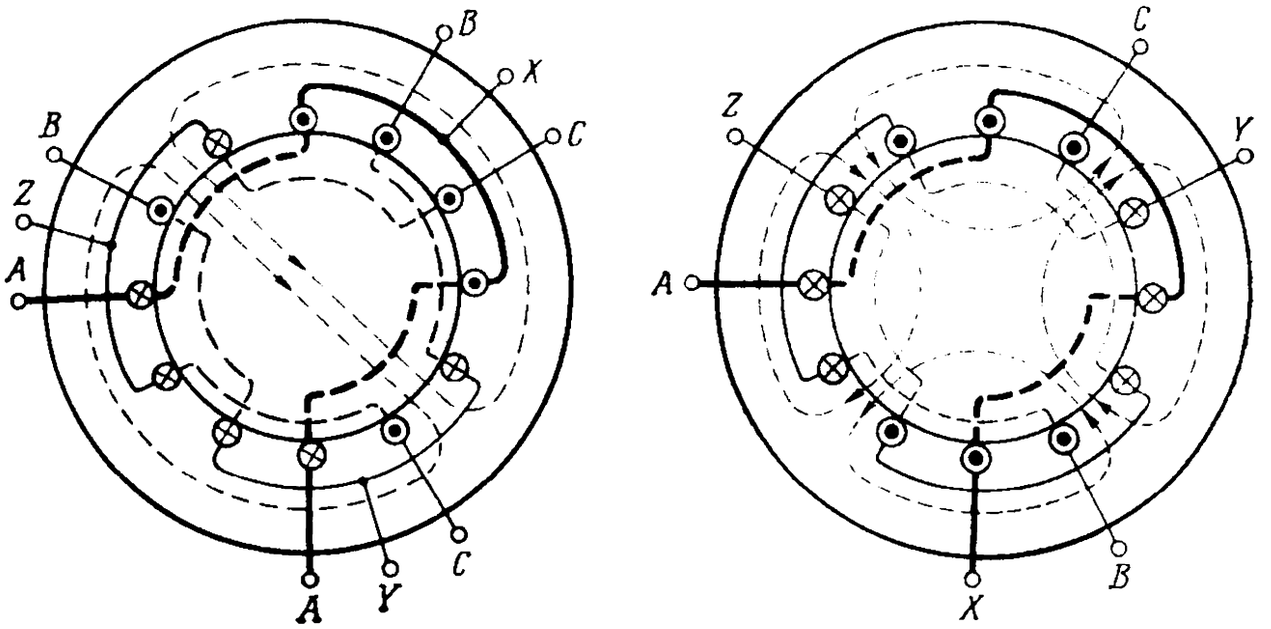


FIGURE 14.32

the power lost in the stator core (see Fig. 14.20), the power associated with the rotating magnetic field, P_r, is the sum of the power lost in the rotor conductors (see the equivalent circuit in Fig. 14.19)

$$P_{w2} = r'_{w2} (I'_2)^2$$

and the mechanical power

$$P_m = r'_{w2} (1 - s) (I')^2/s$$

The ratio

$$P_{w2}/P_m = s/(1 - s) = (n_1 - n)/n$$

indicates that the share of the mechanical power decreases in direct proportion to the decrease in the speed of the rotor, while the share of the power lost in the rotor-circuit resistance is increased. Hence, if we wish to reduce the speed of the motor by, say, 25%, the rheostat brought into the rotor circuit must be such that a quarter of the energy associated with the rotating magnetic field is wasted as heat. Another limitation of this form of speed control is the fact that provision of a rheostat in the rotor circuit reduces the flatness

of the speed-torque characteristic which implies that the speed of the motor becomes less stable. With the speed-control rheostat brought in, even minute changes in the load on the output shaft bring about substantial changes in the speed of the motor.

14.18 Two-Phase and Single-Phase Induction Motors

In an induction motor with only one single-phase winding on the stator (Fig. 14.33), the applied alternating current would excite an alternating magnetic field whose axis would remain stationary so long as its rotor remains stationary, too. This field would induce an

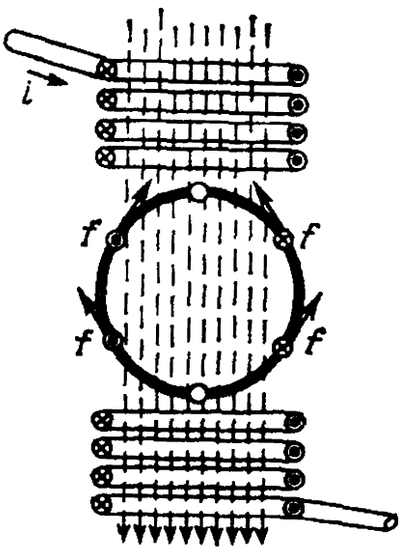


FIGURE 14.33

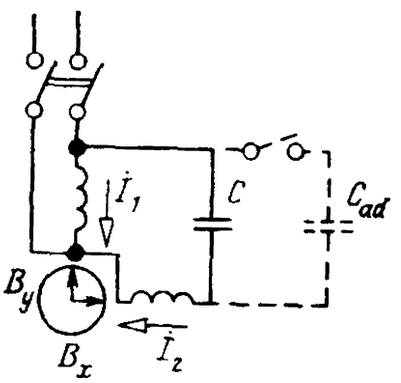


FIGURE 14.34

emf in the rotor winding, giving rise to rotor currents. The interaction of the rotor currents with the stator field would give rise to electromagnetic forces f acting in opposite directions in the right- and left-hand halves of the rotor. As a consequence, the resultant torque acting on the rotor would be zero. Thus, with only one single-phase winding, the breakaway (initial starting) torque in a single-phase motor would be zero which implies that such a motor could not start rotating on its own.

Two methods are used to produce the breakaway torque in motors powered from a single-phase supply line. Accordingly, it is usual to class them into two-phase and single-phase.

Two-phase induction motors. In addition to a winding connected directly to the supply line, there is a second winding which is connected to a phase-shifting device which may be a capacitor or an inductor. The better choice is a capacitor (Fig. 14.34) for which reason we have a *capacitor-start induction motor.* The stator slots of a capacitor-start motor hold two phase windings each of which takes

up half the total number of slots. This serves to produce the starting torque by the induction mechanism (see Sec. 12.9), that is, by use of two alternating magnetic fluxes displaced from each other in both space and time phase.

It is best to use a circular rotating magnetic field, and it can be obtained in a two-phase motor. With such an arrangement one has, however, to choose when to excite such a field and, as a consequence, a maximum torque—at starting or at rated load.

The point is that if the currents in the stator windings, $\dot{I}_1$ and I_2, have the same rms value and are in phase quadrature with each other, the magnetic field they set up will have two components, B_x and B_y, defined by Eq. (14.2) and Eq. (14.3), respectively. Now the resultant magnetic field will be a circular rotating one.

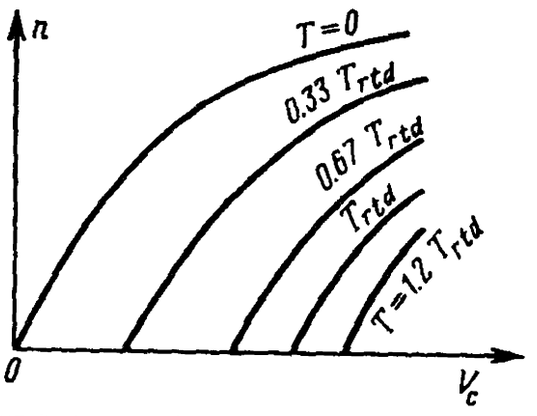


FIGURE 14.35

If the value of the capacitor is chosen so that a circular magnetic field is produced at starting, then at rated load variations in the second-winding current would cause a change in the voltage drop across the capacitor and, as a consequence a change in the magnitude and phase in the voltage across the second winding. As a result, the rotating magnetic field would turn elliptical (the flux would be pulsating during rotation), and this would cause the torque to decrease.

This limitation can be circumvented by disconnecting some of the capacitors in going from the start-up to normal running (the respective connections are shown dashed in Fig. 14.34). This reduction in capacitance can be effected automatically by a centrifugal switch when the motor has come to 75-80% of its rated speed, or by a time relay.

Two-phase induction motors are also used in automatic control applications as controlled motors: their speed or torque are varied by adjusting the rms value or phase of the voltage impressed on one of the windings. Instead of a conventional squirrel-cage rotor, such motors have a rotor in the form of a hollow thin-walled aluminium cylinder (called the drag cup) rotating in a narrow air gap between the stator and a stationary laminated-iron core. Such a *drag-cup motor* (as it is usually called) has a negligible inertia which is especially important in some process control applications. A plot of speed versus control voltage for a drag-cup motor is shown in Fig. Fig. 14.35.

Single-phase induction motors cannot develop any breakaway (initial starting) torque by themselves. If, however, the rotor is set

spinning by some external force in any direction, the motor will then keep rotating on its own and develop a considerable torque.

Similar conditions are produced in a three-phase motor when the fuse in any one phase has blown—the motor will keep running on a single-phase supply. The only precaution that needs to be made is to keep the load to not over 50-60% of the rated value so as to avoid overheating in the remaining two phase windings.

The operation of a single-phase motor may be explained by considering its alternating magnetic field as one produced by the superposition of two magnetic fields rotating in opposite directions at

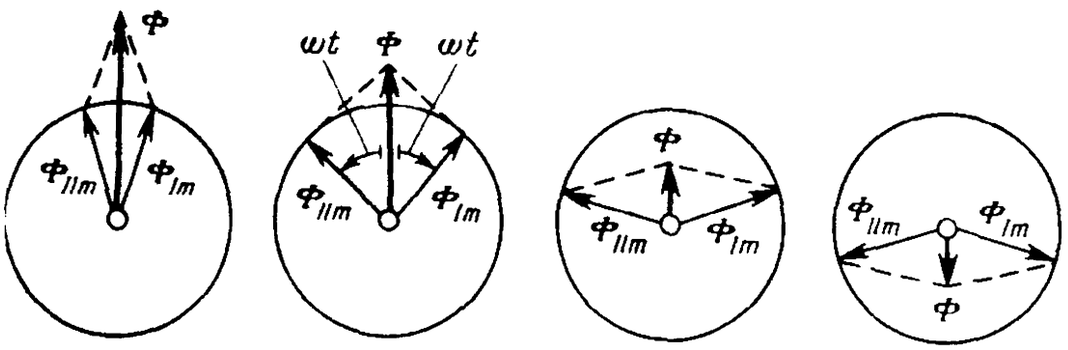


FIGURE 14.36

a constant angular velocity ω/p. These fluxes have the same peak value, $\Phi_{Im} = \Phi_{IIm}$, equal to half the alternating magnetic field of the machine

$$\Phi_{Im} = \Phi_{IIm} = \Phi_m/2$$

As is seen from the simple graphic construction in Fig. 14.36, the superposition of the two identical fluxes, Φ_{Im} and Φ_{IIm}, rotating in opposite directions, results in a magnetic flux which varies sinusoidally

$$\Phi = \Phi_m \sin \omega t$$

In a single-phase motor this condition holds so long as the rotor is stationary.

From the assumption that the resultant alternating magnetic field arises from the superposition of rotating fields, we may conclude that these two fields produce identical currents in the rotor winding. Interacting with the rotating fields, the rotor currents develop two identical torques acting in opposite directions and cancelling each other.

This equality of two torques is upset when the rotor is set spinning by an external force in any direction. Now the torque produced by the forward field (that is, one rotating with the rotor) substantially exceeds the torque developed by the reverse field (that is one rotat-

ing against the rotor). As a result, the rotor can not only rotate on its own, but also drive some mechanism.

The reduction in the opposing torque when the rotor is revolving is due to the weakening of the reverse field. With respect to the reverse field the slip of the rotor is

$$s_{II} = (n_1 + n)/n_1 = [n_1 + n_1 (1 - s_I)]/n_1 = 2 - s_I \qquad (14.36)$$

where s_I is the slip of the rotor relative to the forward field.

Equation (14.36) shows that the frequency of the currents induced in the rotor by the reverse field is close to twice the supply-line frequency. At such a frequency, the inductive reactance of the rotor is many times its resistance, so the currents induced by the reverse field become nearly purely reactive. Referring to Fig. 14.21, the field set up by these currents has a strong demagnetizing effect on the field that induces them, that is, the reverse field. For this reason when s_I is small, the resultant magnetic field of the machine becomes a nearly circular rotating field and the opposing torque due to the reverse field is negligible.

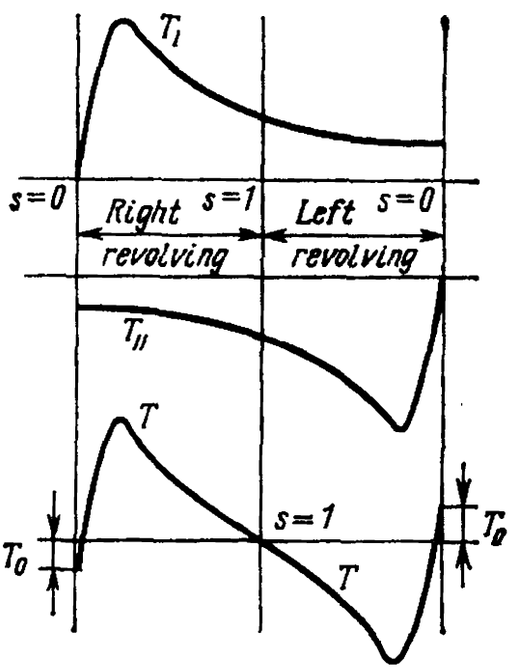


FIGURE 14.37

For each of the two fields, we may apply the already known torque-slip curves of the usual three-phase induction motor and find the resultant torque T as the difference between the forward torque, T_I, and the reverse torque, T_{II} (Fig. 14.37). An important property of a single-phase motor is that it develops a small negative torque T_0 when the rotor is revolving at synchronous speed relative to the forward field.

In a single-phase motor, an increase in the forward-field slip s_I brings about not only an increase in the current I_I induced by the forward field, but also an increase in the opposing torque associated with the reverse field. Because of this, a single-phase induction motor is less stable in operation than a three-phase one, and its maximum torque is substantially lower. Also the efficiency of a single-phase induction motor is lower than that of a three-phase unit due to a number of stray losses.

One needs an auxiliary device in order to start a single-phase induction motor. Most often, this is an auxiliary winding similar to the second winding of a two-phase motor, but disconnected from the supply line once the motor has been started because it is designed

for a short-duration load current. This auxiliary winding is connected in series with a phase-shifting device.

Shaded-pole induction motors. In one particular case the starting device of a single-phase motor may be left energized after starting and during normal running. This case refers to shaded-pole induction motors which may be looked upon as standing midway between single-phase and two-phase induction motors (Fig. 14.38).

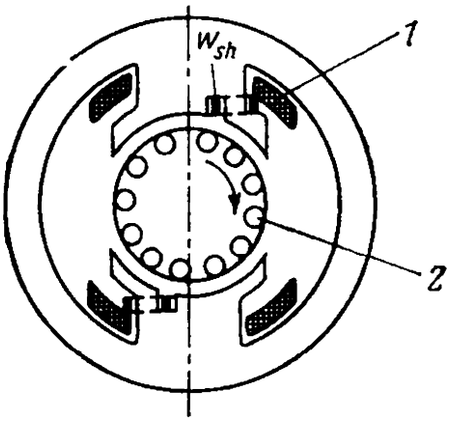


FIGURE 14.38

In this type of motor, one side of each stator pole carrying the main (primary) winding 1 is enclosed by a solid ring of copper or brass called a shading ring or coil w_{sh}. The current I_1 in the stator winding 1 which is energized from the supply line sets up a magnetic flux Φ_1. Some of this flux threads the shading coil and induces in it a current, I_2, substantially lagging in phase behind I_1. This current excites a second magnetic flux so that a system of two alternating fluxes exists in the motor, displaced from each other both in space and time phase. This situation is not unlike that existing in the induction type of instrument movement (see Fig. 12.23). This gives rise to a rotating magnetic field which interacts with the cage rotor 2 and develops a torque. Shaded-pole motors come in fractional-horsepower ratings (0.5-30 W) and are widely used to actuate a large variety of final control elements.

14.19 The Induction Regulator and the Phase Shifter

An induction regulator is essentially a booster transformer constructed in the same way as a three-phase wound-rotor induction motor except that the rotor is not free to rotate; it can, however, be set in a range of orientations with respect to the stator.

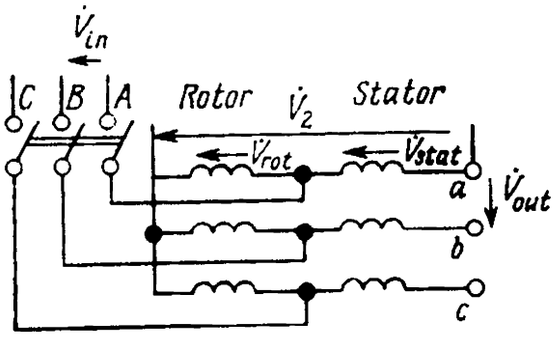


FIGURE 14.39

The winding connections in an induction regulator (also called a *rotary autotransformer*) are shown in Fig. 14.39. The stator and rotor windings in all the three phases are connected in series, with the supply voltage impressed on the rotor winding.

The basic resemblance between an induction regulator and a three-phase autotransformer will be readily seen if we compare

the circuit arrangement of the two. The rotor winding serves as the primary winding of an autotransformer, and the three free leads of the stator windings are the secondary (or output) leads. In an autotransformer, however, the primary to secondary voltage ratio is constant and determined by the primary to secondary turns ratio. In an induction regulator the voltage ratio is further dependent on the relative spatial position (relative orientation) of the stator and rotor windings. If the axes of the two windings are in line, the emfs induced in them will, too, be in line. If, on the other hand, the rotating magnetic field cuts the stator winding at a time different from the instant when it cuts the rotor winding, the spatial shift β between the winding axes will cause a proportional time-phase shift α (see Sec. 14.5) between the stator and rotor emfs and, in consequence, between the stator and rotor voltages, V_{stat} and V_{rot}.

The phase voltage on the secondary side of an induction regulator is the sum of the stator and rotor voltages

$$\dot{V}_2 = \dot{V}_{rot} + \dot{V}_{stat}$$

The rotor is locked, but it can be moved by a worm transmission so that the phase angle between $\dot{V}_{rot}$ and $\dot{V}_{stat}$ is adjusted from 0 to 360° (Fig. 14.40). As a result, the secondary voltage can be varied from $V_2 = V_{rot} - V_{stat}$ to $V_2 = V_{rot} + V_{stat}$.

The rotor phase voltage is equal to the supply-line phase voltage, $V_{rot} = V_{ph}$, and the stator winding of an induction regulator usually has as many turns as the rotor phase winding has, so $V_{stat} = V_{rot} = V_{ph}$. It follows then that the secondary voltage (the voltage between the stator terminals), $V_{out} = \sqrt{3}V_2$, of an induction regulator can be varied from 0 to $2V_{in} = 2V_{line}$, twice the line voltage of the supply. In cases involving heavy blocks of powers it is valuable that this voltage adjustment can be effected without having to make or break any contacts. This materially improves the reliability of operation.

The interaction of the rotor currents with the magnetic field might produce a substantial torque in an induction regulator. If no measures were taken to the contrary, this would complicate the arrangement of the rotating mechanism (which is usually a self-braking worm transmission) and the control of rotor orientation. As a way of minimizing this undesirable torque, heavy-duty induction regulators are usually made as a combination of two induction machines on a common shaft, with the electromagnetic torques that act on their rotors being oriented in opposite directions and cancelling each other. In such an induction regulator, the rotor windings are connected in parallel and the stator windings in series.

As compared with a conventional transformer, an induction regulator has a markedly lower efficiency due to the presence of an air gap in the magnetic circuit. In comparison with a conventional

induction motor, an induction regulator has a fairly large loss in the rotor core in addition to the stator core loss. The cooling conditions in an induction regulator are worse than they are in an induction motor because the rotor is at standstill. To rectify the situation, medium- and large-size induction regulators are fitted with oil coolers.

The fact that the rotation of the locked rotor can be used so as to continuously adjust the relative phase of the emfs induced in the rotor is utilized in *phase shifters* (or, more accurately, phase-shifting

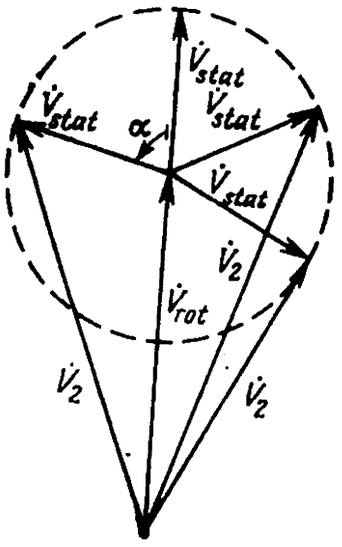


FIGURE 14.40

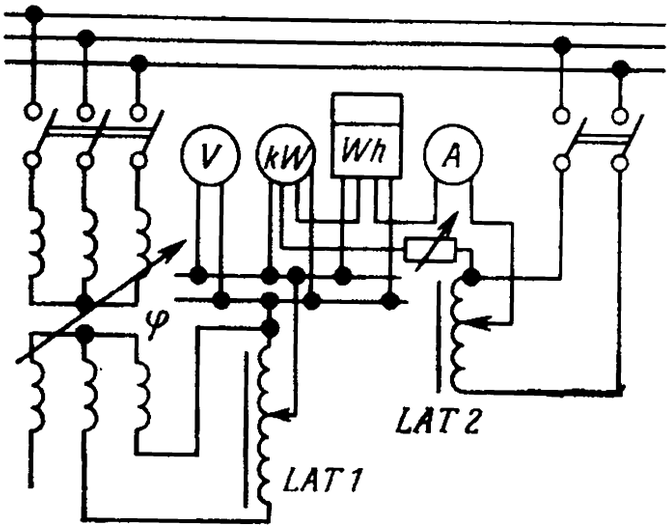


FIGURE 14.41

transformers). The three-phase stator winding of such a phase shifter, φ, is connected to the supply line, and the locked-rotor winding serves as the secondary winding of a rotary transformer (Fig. 14.41). The rms value of the emf induced in the rotor winding is independent of the rotor orientation because the rotating magnetic field of the machine induced by the stator currents is constant in magnitude. However, the relative phase of the stator and rotor emfs does depend on the rotor orientation. Therefore, by rotating the rotor it is possible to adjust the relative phase anywhere between zero and 360°. Phase shifters (or phase-shifting transformers) are used to test a.c. electricity meters and wattmeters (Fig. 14.41), to control rectifier converters, etc.

14.20 The Induction Tachogenerator

Induction tachogenerators are employed in automatic control systems and electric drives as angular-rate transducers. One form of induction tachogenerator known as the drag-cup type is shown in Fig. 14.42.

When the rotor is at a standstill ($\omega_{rot} = 0$), the alternating magnetic flux Φ_t excited by the current traversing a field coil, w_f energized from the supply line induces alternating currents i_{tr} in the drag-cup rotor as it were the short-circuited secondary winding of a transformer. These alternating currents set up an alternating magnetic flux Φ_y. As this happens, no emf is induced in a measuring coil, w_{meas}, whose axis is at right angles with the axis of the field coil, w_f.

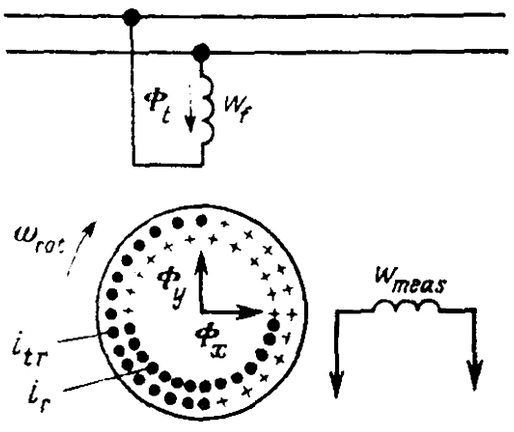


FIGURE 14.42

When the rotor of the tachogenerator is spinning at an angular velocity ω_{rot}, a second set of alternating currents, i_r, will be induced in the drag-cup rotor cutting the excitation flux Φ_t, in addition to the transformer alternating currents i_{tr}. The alternating magnetic flux Φ_x set up by i_r will be in line with the axis of the measuring coil and will induce in it an alternating emf whose amplitude is proportional to the angular velocity of the rotor.

Simple and reliable, induction tachogenerators are widely used in automatic control systems.

15 Synchronous Machines

15.1 General

Synchronous machines are called "synchronous" because in a steady state their rotors are running at the angular velocity of the rotating magnetic field set up by the currents in the phase windings of a stator similar to that of an induction machine. This synchronism is achieved owing to the fact that the rotor of a synchronous machine is usually an electromagnet or, more seldom, a permanent magnet with as many pole pairs as there are of them in the rotating magnetic field. The interaction between the poles of the rotating magnetic field and those of the rotor assures that the rotor is spinning always at the same angular velocity, irrespective of the torque applied to or developed at its shaft. This property of synchronous machines makes them a natural choice as drive motors for mechanisms that must be driven at a constant angular velocity. Synchronous motors are not so popular as induction motors, but there are applications, such as metal-making, where they are indispensable. Synchronous motors come in ratings of several megawatts.

The principal field of application for synchronous machines is in the form of generators at electric power stations. Their rating may be as high as 1500 MVA per unit.

15.2 Construction of a Synchronous Machine

A synchronous machine consists essentially of a stator and a rotor, the stator being similar to that of an induction machine (see Fig. 14.1). The stator core is built up of electrical-sheet steel laminations insulated from one another and is secured within a substantial frame. On its inner surface the stator has slots which receive an a.c. winding, mostly three-phase.

The rotor of a synchronous machine is a salient-pole electromagnet (Fig. 15.1 where the poles are at 1, the pole coils at 2, the rotor core at 3, and slip-rings at 4). Alternatively, it may be a cylindrical or nonsalient-pole type (such as in Fig. 15.2 where the rotor core is at 1, the slots are at 2, and the slip-rings at 3). Current from an external source, called the exciter, is conveyed to the rotor winding via the slip-rings and brushes.

In a multipole synchronous machine the rotor has p pole pairs, and the currents in the stator winding likewise produce p pole pairs of a rotating magnetic field (as in the case of an induction machine). The rotor must revolve at the same velocity as the magnetic field, so its synchronous speed is

$$n = 60f/p \tag{15.1}$$

At a standard power (commercial) frequency of 50 Hz, the maximum speed for a two-pole machine ($p = 1$) is 3000 rpm. This is the

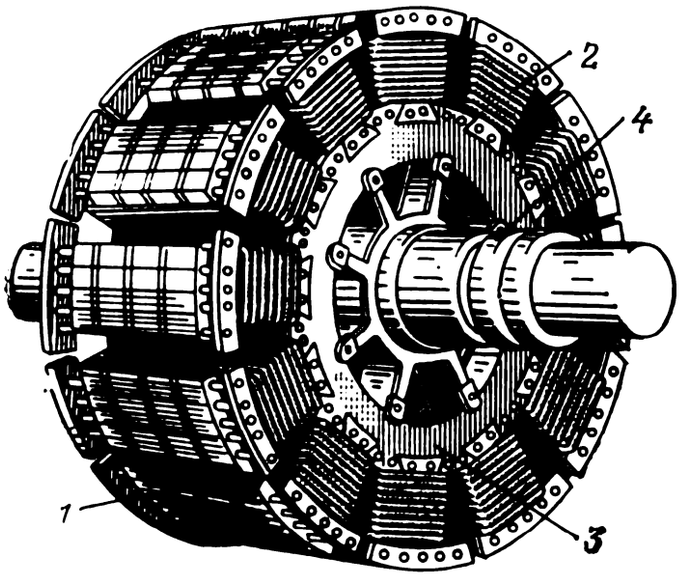


FIGURE 15.1

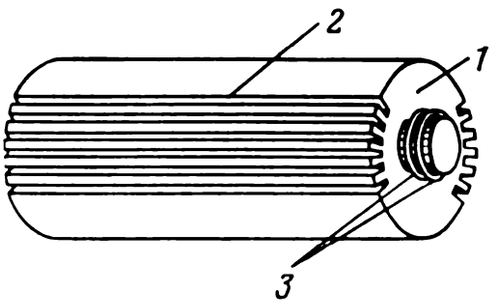


FIGURE 15.2

speed of a present-day steam-turbine generator set consisting of a steam turbine as the prime mover and a nonsalient-pole synchronous generator.

In a hydraulic-turbine generator set the turbine runs at a relatively low speed. Because of this, the associated generator has to be made of the multipole, salient-pole type, mostly with a vertical shaft. The speed of such generators ranges from 60 to several hundred revolutions per minute, so there must be several tens of pole pairs. Since they run at relatively low speeds, hydraulic-turbine-driven generators have a substantially greater mass per unit power (over 8 kg kVA^{-1}) than steam-turbine-driven generators (less than 2.5 kg kVA^{-1}).

15.3 Motoring and Generating Action of a Synchronous Machine

Like other electrical machines, synchronous machines may be operated either as generators or as motors. Which action of the two does take place depends on the interaction between the magnetic fields set up by the currents in the stator and rotor windings.

Consider the operation of a two-pole machine. The superposition of the magnetic fields due to the currents in the phase windings of the stator excites in a synchronous machine (in much the same way as this happens in an induction machine) a magnetic field (see Sec. 14.3) which rotates at an angular velocity ω. An approximate distribution of the rotating magnetic field in the magnetic circuit of a synchronous machine running as (a) a generator and (b) a motor is shown by dashed lines in Fig. 15.3. As is seen, it may approximately

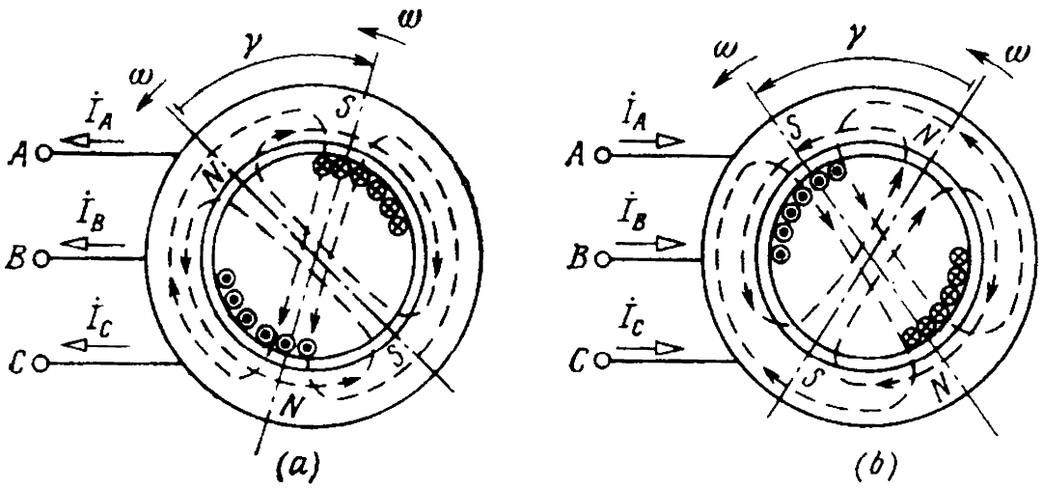


FIGURE 15.3

be visualized as a pole pair on the stator, rotating at an angular velocity ω.

Similarly, the magnetic field set up by current in the winding of the revolving rotor may be approximately visualized as a rotating pole pair on the rotor.

If we neglect any losses in a synchronous machine, then, with no torque developed at the shaft, the axis of the rotor poles will be in line with that of the stator poles.

If we wish a synchronous machine to operate as a generator and supply electric energy to the line, we must increase the mechanical torque applied by its prime mover to the shaft. Then the increased driving torque will cause the axis of the magnetic poles on the rotor to move through an angle γ relative to the axis of the stator poles in the direction of rotation (Fig. 15.3 a). This will also cause a change in the resultant magnetic field produced by the superposition of the stator and rotor magnetic fields and, in consequence, the current in the stator windings. The interaction of this current with the magnetic field of the rotor produces the braking torque acting on the rotor. This is nothing other than the conversion of the mechanical power supplied by the prime mover into the electric power by the generator connected to the electric system. The magnetic poles on the rotor appear to pull along with them the magnetic poles on the stator.

If now, instead of a driving torque, we apply to the shaft a mechanical load, the axis of the rotor poles will move through an angle with respect to the axis of the stator pole against the direction of rotation (Fig. 15.3 b). As before, there will appear currents in the stator winding and there will appear electromagnetic forces due to the interaction between the stator currents and the rotor field, but now these forces will tend to pull the rotor along in the direction of rotation. This is nothing other than the conversion of the electric energy drawn from the supply line into mechanical power available at the output shaft. In this way, a synchronous machine changes to operation as a motor.

A synchronous machine goes over from acting as a generator to acting as a motor depending on the mechanical action on its shaft, with electromagnetic forces acting as an elastic coupling between the rotor and the stator.

15.4 Production of a Sinusoidal EMF in a Synchronous Generator

The stator winding of a synchronous machine is constructed in the same fashion as that of an induction machine. However, in an induction machine the emfs in the stator phase windings are induced by a rotating magnetic field. In a synchronous machine the emfs in the stator phase windings are induced by the magnetic field that is set up by the current in the winding of a revolving rotor.

At no-load the magnetic flux linking with one turn of the phase winding in the stator of a synchronous machine ranges between $+\Phi_{0m}$ and $-\Phi_{0m}$, where Φ_{0m} is the maximum magnetic flux linking one turn, as produced by the magnetic field of the rotor revolving at an angular velocity ω.

It has been shown in Sec. 2.5 that in such a case the emf induced in one turn of a phase winding varies sinusoidally with an amplitude

$$E_{0m} = \omega\Phi_{0m}.$$

Since the coils making up each phase of the stator winding are distributed among several slots, their emfs are combined vectorially, which fact is accounted for in calculating the phase emf by applying the winding factor $k_w < 1$ (see Sec. 14.5).

Thus, the rms value of phase emf in a synchronous machine with w turns per phase is

$$E_0 = 4.44\, fwk_w\Phi_{0m} = \omega\Psi_0 \tag{15.2}$$

where Ψ_0 is the rms flux linkage of the rotor magnetic field with the stator phase winding.

If we wish to obtain a sinusoidal emf in the stator phase turns, it is essential that the magnetic induction produced in the air gap

by the magnetic field of the rotor be distributed sinusoidally all the way round the rotor. In a salient-pole-rotor machine the air gap gradually increases from the middle of a pole towards its tips, so the magnetic induction is distributed in inverse proportion to the reluctance of the magnetic path, that is, other than sinusoidally. A sinusoidal distribution is assured through the use of suitably shaped pole pieces.

Obviously, this device is inapplicable to high-speed cylindrical-rotor machines. The usual practice in such cases is to dispose the field coils all the way round the rotor in such a way that the distribution of magnetic induction is trapezoidal which is a good approximation to a sinusoidal distribution. The higher harmonics in the emf waveform are suppressed by utilizing the special properties of the distributed stator winding.

15.5 The Equation of Electric State for a Phase of a Synchronous Generator

The magnetic field of a loaded synchronous machine is excited not only by the excitation current in the rotor winding, but also by the currents in the phase windings of the stator (armature). This implies that, similarly to d.c. machines, synchronous machine experiences armature reaction, that is, the effect of the armature mmf on the main magnetic field of the machine. The waveform of the emf induced in a phase of a loaded generator depends not only on the distribution of the rotor field around the circumference of the machine, but also on the distribution of the stator field. The latter must likewise be as close to sinusoidal as practicable, so that the resultant field and the emf waveform of the loaded machine could be sinusoidal.

The current in the stator phase windings of a loaded synchronous generator produces voltage drops across the inductive reactance and the resistance of the phase windings. The inductive reactance of a stator phase winding is determined by the leakage flux linkage and the magnetic flux due to the armature reaction.

The inductive leakage reactance of a stator phase winding is determined by the stator *leakage flux linkage* Ψ_{leak}. As is the case with an induction machine, this flux linkage is that part of the overall flux linkage which is due to the magnetic lines of force that have their path outside the rotor, across the slots, at the tips of the teeth, and around the stator coil ends (see Fig. 14.13). The greater proportion of the path traversed by the leakage flux is through air. Therefore it is legitimate to think that the leakage flux linkage is proportional to the stator current and is in phase with it. In the stator phase winding, the leakage flux linkage induces a leakage emf, E_{leak}, which is in quadrature lagging with the leakage flux linkage. The voltage which balances the leakage emf in terms of the stator

current I and the inductive leakage reactance x_{leak}, Eq. (2.33)

$$\dot{E}_{leak} = -jx_{leak}\dot{I} = -j\omega\dot{\Psi}_{leak}$$

At rated load, the voltage drop across leakage inductance in synchronous machines accounts for 10-15% or even 20% of the rated phase voltage. The leakage inductive reactance responsible for this voltage drop serves a useful purpose because it limits the current in the case of a short circuit between the generator terminals. The resistance of a stator phase winding is very small. As a rule, given the rated load, the voltage drop across the resistance of a phase winding accounts for 1-2% of the rated phase voltage in the case of large generators. Therefore it may be neglected in many calculations; we will consider it only in some special cases.

As already noted, the phase emf of a synchronous machine is affected by armature reaction. Physically, only one resultant magnetic field exists in a synchronous machine; it is the sum of the magnetic field due to the rotor, the leakage magnetic field of the stator, and the magnetic field due to armature reaction. It is convenient, however, to consider them as existing independently of one another, producing independent flux linkages with the stator phase windings, and inducing in them the respective emfs. Thus, in a loaded synchronous generator the rotor flux linkage Ψ_0 induces in a stator phase winding an emf, E_0, equal to the no-load (open-circuit) emf; the leakage flux linkage Ψ_{leak} induces the leakage emf, E_{leak}, and the armature-reaction flux linkage $\Psi_{a.r}$ induces an armature-reaction emf, $E_{a.r}$, in the stator winding. The simplest relations can be derived on neglecting the effect of hysteresis and core saturation on the armature (stator) flux linkage, that is, on assuming that the armature flux linkage is proportional to the armature current I and is in phase with it. Actually this assumption holds only when the magnetic circuit of a synchronous machine is unsaturated. Under the assumption made, the armature-reaction emf, $E_{a.r}$, may be expressed in the same way as the leakage emf E_{leak}, that is, in terms of the armature current I and the armature inductive reactance x_{arm} so that

$$\dot{E}_{a.r} = -jx_{arm}\dot{I} = -j\omega\dot{\Psi}_{a.r}$$

The term $jx_{arm}\dot{I}$ is the voltage required to overcome the emf due to armature reaction.

With the above approach, the emf $\dot{E}_0$ induced in a phase winding by the flux linkage $\dot{\Psi}_0$ is equal to the sum of the voltage $jx_{leak}\dot{I}$ and $jx_{arm}\dot{I}$ existing across the inductive reactances x_{leak} and x_{arm} of the phase winding, the voltage $r_w\dot{I}$ across the resistance of the phase winding conductors, and the terminal phase-winding voltage $\dot{V}$.

Hence, the equation of electric state for a phase of a synchronous generator takes the form

$$\dot{E}_0 = \dot{V} + [r_w + j(x_{leak} + x_{arm})]\dot{I} = \dot{V} + (r_w + jx)\dot{I} \qquad (15.3)$$

where $x_{leak} + x_{arm} = x$ is the inductive reactance of a stator phase winding, called the *synchronous inductive reactance*. In a synchronous machine with an unsaturated magnetic circuit it is a relatively constant quantity.

15.6 The Equivalent Circuit and a Simplified Phasor Diagram for a Phase of a Synchronous Generator

The equivalent circuit answering the equation of electric state of a stator phase of a synchronous generator is shown in Fig. 15.4 a. Now we will construct a phasor diagram for a phase of a synchronous generator. To do this, we choose as reference the main flux linkage vector $\dot{\Psi}_0$ which is directed to the left along the axis of abscissae (Fig. 15.4 b). The phasor of $\dot{E}_0$ induced by $\dot{\Psi}_0$ is in quadrature lagging with $\dot{\Psi}_0$. The phasor of the stator (armature) current $\dot{I}$ is lagging $\dot{E}_0$ by an angle φ_0 determined by the relative magnitudes of the reactances and resistances

$$\varphi_0 = \arctan \frac{x + x_{load}}{r_w + r_{load}} \qquad (15.4)$$

where x_{load} and r_{load} are the reactance and resistance of the generator load.

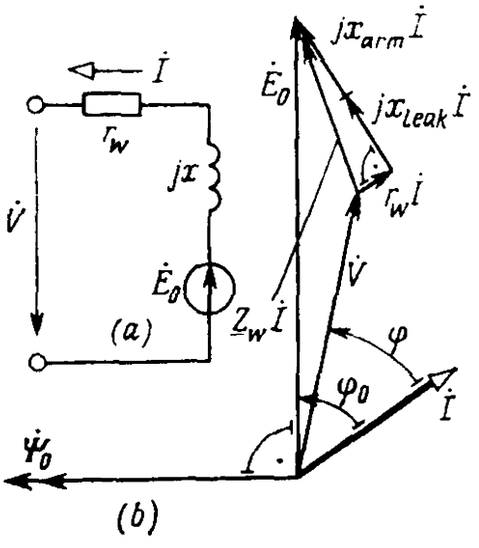


FIGURE 15.4

The phasor of voltage $r_w\dot{I}$ is aligned with the current phasor $\dot{I}$, and the voltage phasor $jx\dot{I}$ is in quadrature leading with it. The phasor of the terminal voltage per phase, $\dot{V}$, can be located by subtracting the sum of the voltage phasors across the resistance and reactance per phase from $\dot{E}_0$:

$$\dot{V} = \dot{E}_0 - jx\dot{I} - r_w\dot{I}$$

On joining the tips of the phasors $\dot{E}_0$ and $\dot{V}$, we obtain a triangle of voltages across the resistance and inductive reactance per phase, with $\underline{Z}_w\dot{I}$ as hypotenuse.

The phasor $r_w\dot{I}$ is shown on an exaggerated scale to make the diagram more instructive.

15.7 The Power Rating and Efficiency of a Synchronous Generator

The power balance around a synchronous generator may be explained by reference to its phasor diagram (see Fig. 15.4 b). By projecting the $\dot{E}_0$ phasor and its components onto the current phasor $\dot{I}$, we obtain the active component of the emf

$$E_0 \cos \varphi_0 = r_w I + V \cos \varphi \tag{15.5}$$

By multiplying Eq. (15.5) by the rms current I we transform Eq. (15.5) into an equation defining the power per phase of a synchronous generator

$$E_0 I \cos \varphi_0 = r_w I^2 + VI \cos \varphi \tag{15.6}$$

It shows that the electric power developed by the stator, $P_{e.s}$, is the sum of the power dissipated in the armature conductors, P_w, and the electric power, P, which the generator delivers to the line. In addition to the power dissipated in the conductors (traditionally called the copper loss), there is also the power lost mechanically (through windage and friction), P_m, and the power lost in the iron of the stator and pole shoes due to hysteresis and eddy-currents, collectively called the core (or iron) loss, P_c. As follows from Eq. (15.6), these losses are made up for not by electric power, but directly by the mechanical power coming from the associated prime mover. The overall power break-down is shown in Fig. 15.5. There is also some power lost on field excitation. It is equal to the power delivered by a d.c. exciter, P_{ex}, and accounts for about 0.3-1% of the power rating of a large generator.

The total losses associated with a synchronous generator may be classed into fixed (or constant) losses which are nearly independent of load, and the variable loss which depends on load. The fixed losses, P_{fixed}, is the sum of the mechanical losses, the excitation losses, and the core loss. The variable loss, P_{var}, is equal to the copper loss, that is, the power dissipated in the conductors.

Given the same current, the electric power of a synchronous generator, as expressed in terms of phase voltage and current, is

$$P = 3VI \cos \varphi$$

and is a function of the load power factor. However, the cross-sectional area of winding conductors is chosen for a particular value of current, whereas its insulation and the cross-sectional area of the magnetic circuit are chosen for a particular value of voltage. Hence,

these quantities are chosen irrespective of the load power factor. For this reason, the *power rating of a synchronous generator* is defined, like that of a transformer, as

$$S = VI$$

and is expressed in kilovolt-amperes. It would be ill-advised to couple a synchronous generator to a turbine designed for its total power S (divided by its efficiency), because cos φ is less than unity nearly always. That is why the turbine chosen to drive a particular syn-

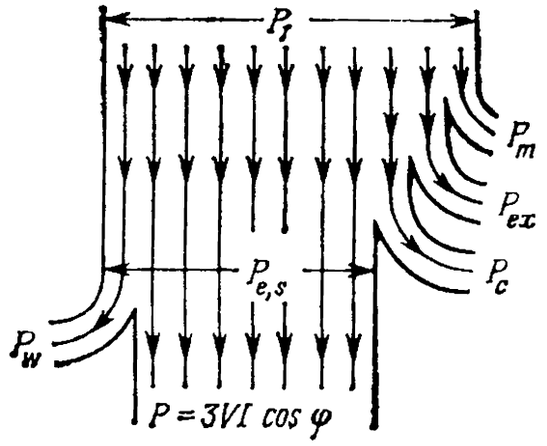


FIGURE 15.5

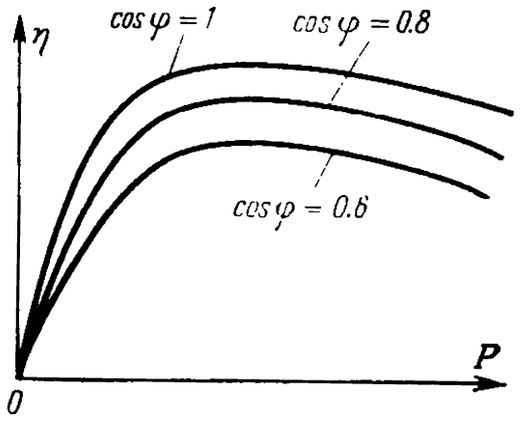


FIGURE 15.6

chronous generator usually has a somewhat lower power rating than the total power of the generator (usually it is taken on the assumption that cos φ = 0.8).

The power output of a synchronous generator is proportional to its volume, so an increase in the power rating of a generator leads to a decrease in the cooling area per unit power and to a poorer cooling. To maintain proper cooling, one has to employ forced cooling and ventilation. In large turboalternators the quantity of air needed for ventilation is very large. The quantity of air expended to cool a generator every hour is about equal in weight to that of the machine itself.

For generators in sizes over 25 MVA it is usual to employ hydrogen cooling. Hydrogen cooling is more attractive because the weight of hydrogen is 1/14th that of air, its heat capacity is 14 times as great, its thermal conductivity is 7 times that of air, and the heat transfer per unit area is 1.35 times as great.

The *efficiency of a synchronous generator* is defined as the ratio of the power that the generator delivers to an electric network, to the power supplied by its prime mover. It is convenient to express the latter as the sum of the power output of the generator and all forms of power losses in the machine, that is,

$$\eta = 3VI \cos \varphi / (3VI \cos \varphi + P_{\text{fixed}} + P_{\text{var}}) \qquad (15.7)$$

Equation (15.7) shows that a decrease in load leads to a decrease in efficiency. Plots of generator efficiency as a function of load for several values of the power factor appear in Fig. 15.6. As the power rating of generators goes up, the efficiency of both the generator and of its prime mover improves.

15.8 Operation of a Synchronous Generator in a Large Power System

A large three-phase power system combines a great number of sources and of three-phase loads operating in parallel. It may be taken that a minor change in the number of sources and loads will not affect the operation of a large power system. Therefore, the rms voltage across the system busbars and the system frequency may be assumed to be constant. An equivalent circuit per phase of a large system comprising an infinite-power source $\dot{E}$ and the system load Z_{load} is shown in Fig. 15.7. The figure also shows an equivalent circuit for a phase of a synchronous generator, neglecting the phase-winding resistance, connected to the system busbars.

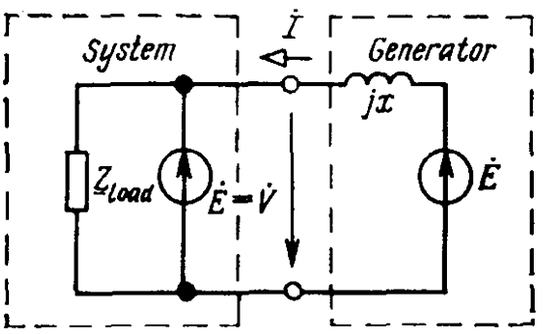


FIGURE 15.7

On neglecting the resistance of a phase winding, the electric state per phase of a synchronous generator may be described by the following equation:

$$\dot{E}_0 = \dot{E} + jx\dot{I} = \dot{V} + jx\dot{I} \tag{15.8}$$

The events taking place in a synchronous generator connected to a large power system are illustrated in the phasor diagram of Fig. 15.8. It is convenient to begin constructing this diagram by directing the phasor of the large-system terminal voltage, $\dot{E} = \dot{V}$, upward along the axis of ordinates. This voltage is balanced by a part of the stator phase emf E_0 induced in the stator winding by the flux linkage $\dot{\Psi}_0$. Adding $jx\dot{I}$ which is at right angles to $\dot{I}$ to the phasor $\dot{V}$ yields the $\dot{E}_0$ phasor. On the phasor diagram, $\dot{\Psi}_0$ is in quadrature leading with the phasor of the emf $\dot{E}_0$ it induces. Since

$$jx\dot{I} = -\dot{E}_{leak} - \dot{E}_{a.r}$$

that is, equal to the sum of the emfs induced by the leakage flux linkage and the armature reaction flux linkage, it follows that the

equation defining the electric state per phase of a synchronous generator, Eq. (15.8), may be re-written as

$$\dot{E}_0 + \dot{E}_{leak} + \dot{E}_{a.r} = \dot{V} = \dot{E}$$

The phase voltage of a synchronous generator is equal to the sum of the emfs induced in each phase winding by three flux linkages, namely $\dot{\Psi}_0$, $\dot{\Psi}_{leak}$, and $\dot{\Psi}_{a.r}$. Physically, however, these three flux linkages form a single resultant flux linkage with the phase winding

$$\dot{\Psi} = \dot{\Psi}_0 + \dot{\Psi}_{leak} + \dot{\Psi}_{a.r}$$

In consequence, it may be thought that the terminal voltage per phase of a synchronous generator is equal to the emf induced by the

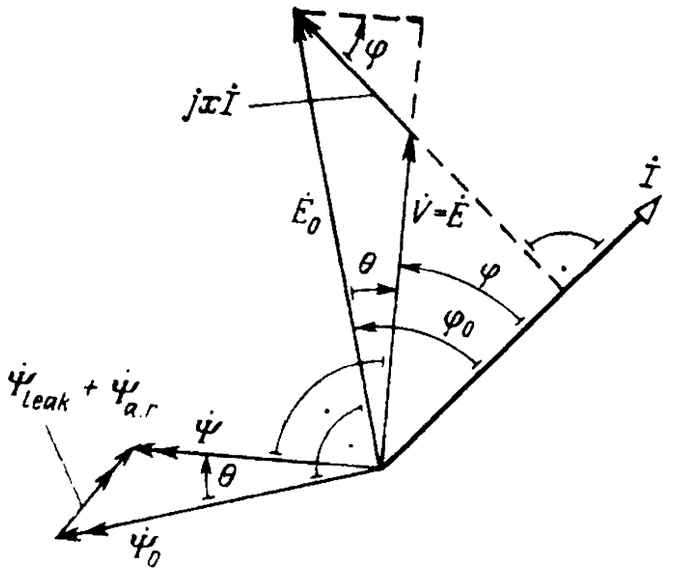


FIGURE 15.8

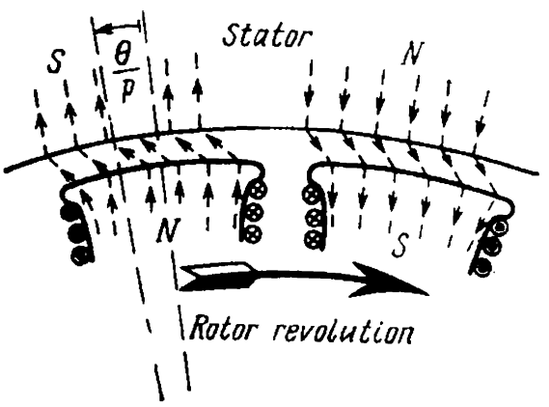


FIGURE 15.9

resultant flux linkage $\dot{\Psi}$ with a phase winding. This determines the direction of the $\dot{\Psi}$ phasor which must be in quadrature leading with the phasor $\dot{V} = \dot{E}$. The phasors $\dot{\Psi}_{leak}$ and $\dot{\Psi}_{a.r}$ are in line with the phasor $\dot{I}$. The phase angle θ between $\dot{E}_0$ and $\dot{V}$ is equal to that between $\dot{\Psi}_0$ and $\dot{\Psi}$. It is customary to reckon the phase angle θ from the phasors $\dot{E}_0$ and $\dot{\Psi}_0$. For a synchronous machine in the generating mode this angle is always less than zero ($\theta < 0$).

The phase angle θ between the flux linkage phasors corresponds to a mechanical angle of θ/p between the axes of the rotor poles and the direction of the resultant magnetic field in a synchronous generator (Fig. 15.9).

As follows from the phasor diagram, the rms value of the resultant flux linkage with a phase winding of a synchronous generator connected to a large power system with $V = \text{const}$ is a constant quantity ($\Psi = \text{const}$), independent of load.

15.9 The Electromagnetic Torque and the Load-Angle Characteristic of a Synchronous Generator

Let us see how the power P and the electromagnetic torque T_{em} of a synchronous generator depend on the load angle θ ($\theta < 0$). To this end, we will refer to the phasor diagram in Fig. 15.8.

The electric power supplied by the three phases of a synchronous generator is given by

$$P = 3VI \cos \varphi = 3E_0 I \cos \varphi_0 = 3E_0 I \cos (\varphi - \theta) \tag{15.9}$$

Taking the $\dot{E}_0$ phasor as hypotenuse, we construct a right triangle in which $\dot{V}$ is a part of one of the sides. The other side of the triangle, opposite the angle θ, is

$$xI \cos \varphi = E_0 \sin |\theta|$$

On the basis of the same diagram,

$$E_0 \cos \varphi_0 = V \cos \varphi$$

and so we may express the electric power of a synchronous generator as

$$P = 3VI \cos \varphi = 3E_0 V \sin |\theta| / x \tag{15.10}$$

The electromagnetic torque developed by the interaction of the armature current with the magnetic field of the machine is connected to its electric power by a well-known simple relation

$$T_{em} = P/\omega_{rot}$$

where the synchronous angular velocity of the rotor is

$$\omega_{rot} = 2\pi n/60 = 2\pi f/p$$

Hence,

$$T_{em} = (3p/2\pi f)\, E_0 V \sin |\theta| / x \tag{15.11}$$

Since in a large power system the voltage V and the frequency f are constant, it follows that, given a constant excitation current, the power and the electromagnetic torque of a synchronous generator depend solely on the load angle $|\theta|$. This sinusoidal relation is known as the *power/angle or torque/ angle characteristic* of a synchronous generator (Fig. 15.10). For power and electromagnetic torque, it only differs in scale.

The power-vs-angle and torque-vs-angle characteristics give an insight into what happens in a synchronous generator as its load is varied.

The work done by the prime mover is converted to electric energy, and this is delivered to the line. As the torque developed by the prime

mover increases, $T > T_{pm1} = T_{em1}$ (point 1), the rotor is accelerated, and the load angle $|\theta|$ increases. After several oscillations about the synchronous angular velocity the balance between the driving torque supplied by the prime mover and the opposing electromagnetic torque developed by the generator is restored, $T_{pm2} = T_{em2}$ (point 2) at a new value of the load angle $|\theta| : |\theta_2| > |\theta_1|$.

A synchronous generator will give a stable performance if $|\theta|$ varies from 0 to $\pi/2$. At $|\theta| = \pi/2$, the electric power is a maximum

$$P_{max} = 3E_0V/x \qquad (15.12)$$

and the electromagnetic torque is likewise a maximum

$$T_{em.max} = 3pE_0V/\omega x \qquad (15.13)$$

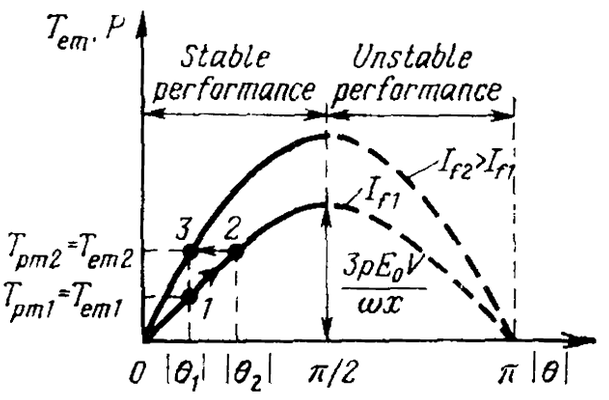


FIGURE 15.10

The range of values $\pi/2 - |\theta|$ defines the margin of stability for a synchronous generator. At $|\theta| > \pi/2$ a synchronous generator tends to be unstable. Now the driving torque supplied by the prime mover exceeds the opposing electromagnetic torque of the driven synchronous generator. The excessive torque ($T_{pm} > T_{em}$) accelerates the rotor still more, thereby leading to an increase in $|\theta|$ and a further decrease in the torque of the generator, and so on, until the generator drops out of synchronism. The margin of stability $\pi/2 - |\theta|$, of a synchronous generator at an increased load can be restored by increasing the excitation current (point 3).

15.10 The V-Curve of a Synchronous Generator

A valuable property of a synchronous generator operating into a large power system is that its reactive current can be controlled by varying its excitation current. For a better understanding of the matter, let us turn to the phase phasor diagram of a synchronous generator (see Fig. 15.8) and analyse it from this point of view (Fig. 15.11). If the power developed by a synchronous generator, $P = \omega_{rot}T_{em}$, and the voltage across the system busbars, V, are constant, then the products of the terms in Eq. (15.10)

$$\begin{aligned} I\cos\varphi &= I_{act} = \text{const} \\ E_0\sin|\theta| &= \omega\Psi_0\sin|\theta| = \text{const} \end{aligned} \qquad (15.14)$$

are independent of the excitation current. However, a change in the excitation current brings about a change in the associated flux

linkage with the stator phase winding, $\dot{\Psi}_0$, and in the emf $\dot{E}_0$ induced by this flux linkage in the phase winding.

It follows from the equation of electric state of a stator phase, Eq. (15.8), that this can happen only if there is a proportionate change in the phase current

$$\dot{I} = \dot{I}_{act} + \dot{I}_{reac}$$

or, to be more exact, in the reactive current, $\dot{I}_{reac}$.

When the excitation current is less than some limiting value, $I_f < I_{f.\,lim}\,(P)$, the current in a synchronous generator contains a capacitive reactive component, $I_{reac.\,C}$. When the excitation current is greater than some limiting value, $I_f > I_{f.\,lim}$, the current in a synchronous generator contains an inductive reactive component, $I_{reac.\,L}$ (Fig. 15.11). The phase angle respectively is $\varphi < 0$ and $\varphi > 0$. In consequence, underexcitation makes the reactive power of a synchronous generator capacitive in its effect,

$$Q_C = 3VI_{reac.\,C}$$

and overexcitation makes it inductive in its effect

$$Q_L = 3VI_{reac.\,L}$$

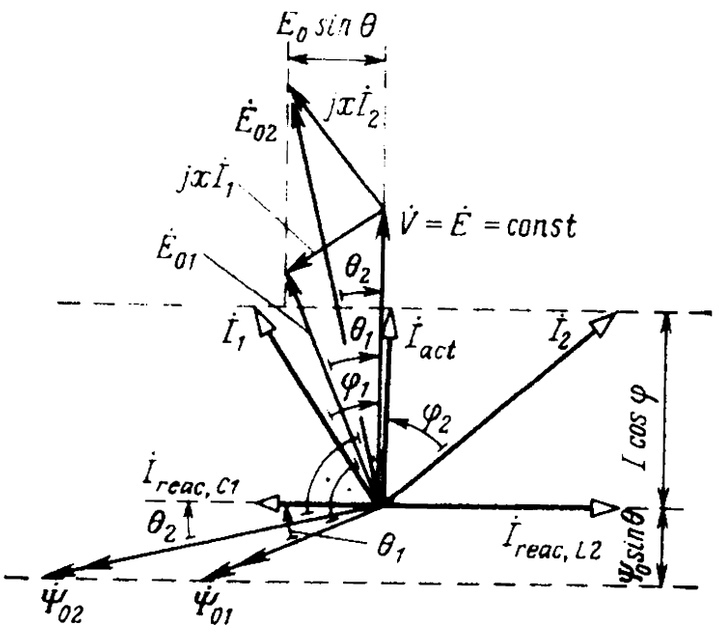


FIGURE 15.11

If a synchronous generator operates into a large power system with $V = \text{const}$, its equivalent circuit may be drawn up as a parallel combination of two current sources, one being a source of active current which depends on the torque developed by the prime mover, $I_{act} = f\,(T_{pm})$, and the other being a source of reactive current which depends on the torque developed by the prime mover and the excitation current, $I_{reac} = f\,(I_f,\ T_{pm})$.

In the special case of a zero prime-mover torque, $T_{pm} = 0$, the phase equivalent circuit of a synchronous generator operating into a large system will contain no active current source, and the reactive current will be solely dependent on the excitation current, $I_{reac} = f\,(I_f)$.

The plots of the current of a synchronous machine operating into a large power system with $V = \text{const}$, as a function of the exci-

tation (field) current, $I = f(I_f)$, at a constant prime-mover torque, $T_{pm} = \text{const}$, are called the *V-curves of the machine* (Fig. 15.12). At some low value of excitation current the angle $|\theta|$ (see Fig. 15.10) may exceed $\pi/2$ in value, and the stability of the machine may be upset. The greater the active power of a synchronous generator, the greater the excitation current at which the machine may lose its stability. In Fig. 15.12 the boundary of stable operation for a synchronous generator is shown by a dashed line.

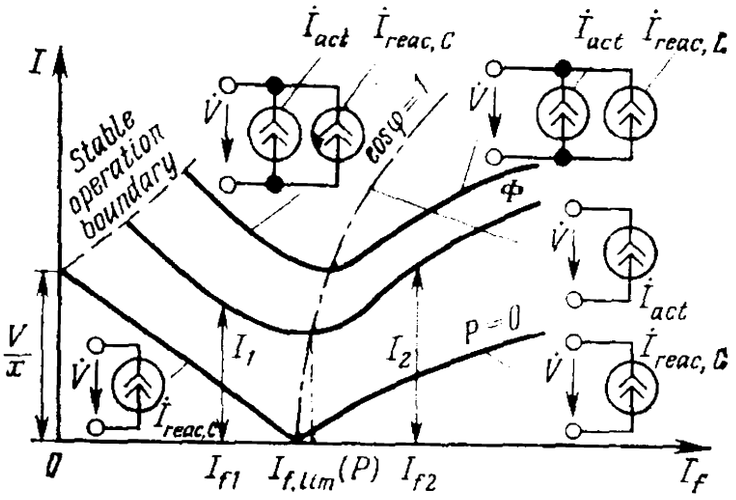


FIGURE 15.12

The torque developed by the prime mover is zero, $T_{pm} = 0$, then, on neglecting all forms of loss, we may take it that the current in a synchronous generator is purely reactive (see Fig. 15.12, $P = 0$):

$$\dot{I}(I_f) = \dot{I}_{reac}(I_f) = (\dot{E} - \dot{V})/jx = (j\omega\dot{\Psi}_0(I_f) - \dot{V})/jx \qquad (15.15)$$

Now the generator current is a linear function of the excitation current. The function $I = f(I_f)$ ceases to be linear only at high values of excitation current owing to the saturation of the magnetic circuit of the machine.

15.11 Control of the Active and Reactive Power of a Synchronous Generator

The *active power*

$$P = 3VI_{act}$$

of a synchronous generator operating into a large power system ($V = \text{const}$) can be controlled by adjusting the power delivered by its prime mover

$$P_m = \omega_{rot} T_{pm}$$

An increase in the power (or, which is the same, the torque) supplied by the prime mover, T_{pm} (a steam or hydraulic turbine), brings about an increase in the active component of the generator current, $I_{act} = f(T_{pm})$. At the same time, the load angle $|\theta|$ goes up, and this reduces the margin of stability, $\pi/2 - |\theta|$, of the generator. If the synchronous generator is not to lose its stability as its active power is increased, it is necessary to raise the excitation current. Industrial synchronous generators are fitted with suitable control-

lers which respond to a change in active power by maintaining the requisite margin of stability.

The *reactive power*

$$Q = 3VI \sin \varphi$$

of a synchronous generator operating into a large power system (V = const) at constant active power (P = const) is controlled by varying its excitation current, I_f. When I_f is equal to $I_{f,\,11m}$, the reactive power in the machine is zero. When I_f is greater than $I_{f,11m}$, the reactive power is inductive in its effect:

$$Q_L = 3VI_{\text{reac},L}$$

When I_f is less than $I_{f,11m}$, the reactive power is capacitive in its effect:

$$Q_C = 3VI_{\text{reac},C}$$

As a rule, the excitation of synchronous generators is maintained at a level which results in the prevalence of inductive reactive power essential for operation of induction motors.

15.12 Parallel Operation of a Synchronous Generator with a Power System

A synchronous generator can be brought in for parallel operation with a power system in any one of two methods, exact synchronizing and self-synchronizing. We will consider *exact synchronizing*. In this case (Fig. 15.13), the instantaneous emf of a generator phase, say, e_{A1}, must be equal in magnitude to, and be in the same direction as, the instantaneous voltage of the respective system phase, e_A, at any time. To meet these requirements, it is essential that the phase emf of the incoming generator must have the same rms value as the phase voltage of the system, the same frequency, the same phase and, finally, the phase sequence in the generator must be the same as that of the system.

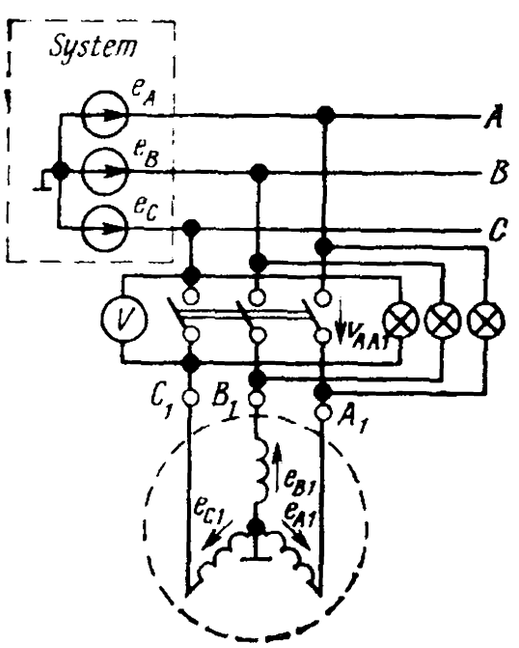


FIGURE 15.13

Preliminarily, an incoming generator is synchronized with the system as follows. The rotor of the generator is brought up approximately to the synchronous angular velocity, and its excitation is

adjusted so that a voltmeter reads a generator terminal voltage equal to the system voltage, with the phase sequence being the same in both the generator and the system. Before a generator is to be finally switched in for parallel operation, a more exact adjustment is needed for the frequency and, especially, the phase emf of the incoming machine. This final stage in synchronizing is effected with a *synchroscope*. In simpler cases, synchronizing is done with the aid of two synchronizing lamps.

In Fig. 15.13 the synchronizing lamps are connected between the system busbars and the like generator terminals (A-A_1, B-B_1, and C-C_1). This connection is used for what is known as "synchronizing dark".

Figure 15.14 shows the waveforms of the instantaneous system phase voltage $v_A = e_A$, the instantaneous phase emf e_{A1} of the in-

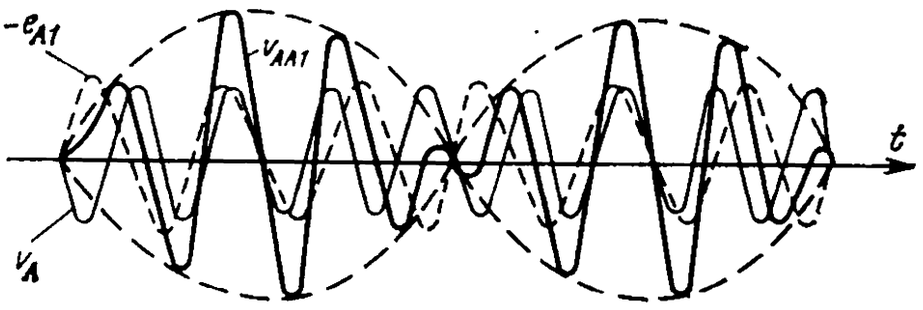


FIGURE 15.14

coming generator, and the instantaneous resultant voltage $v_{AA1} = e_A - e_{A1}$. So long as there is a difference in frequency between the incoming generator and the system, the rms resultant voltage between the switch contacts will now fall to zero and now rise to twice the phase voltage of the system, and the lamps will now go on and now go out. The closer the generator frequency to the system frequency, the lower the rate at which the synchronizing lamps pulsate; they will go on and out for longer intervals of time. The objective is to make the generator frequency be maximally equal to that of the system, so that the intervals when the lamps are dark are long enough (at least 3-5 s). When the lamps go dark completely, the operator should cut in the synchronizing switch.

Self-synchronizing consists in that the incoming generator is brought in the system unexcited, when its speed differs from the synchronous one by not more than 2-3%. In the meantime the rotor (field) winding must be connected in series with a resistor or short-circuited. After the generator has been switched into the system, the rotor winding is connected to a d.c. excitation source, and the generator is brought into synchronism by electromagnetic forces.

Since at the time when the generator is brought into the system it is rotating at a speed other than synchronous, a current in-

rush takes place in the stator winding; this short-duration current may be many times the rated current of the generator. As experience shows, however, neither this current inrush nor the accompanying mechanical forces are detrimental to the turbine-driven generator. Self-synchronizing is used for generators in sizes up to 50 MV A.

Any chance deviations from the synchronous speed in a generator brought in for parallel operation with a large power system give rise to circulating currents in the stator winding. Their interaction with the magnetic field of the rotor produces forces which tend to restore synchronous rotation.

15.13 The Equation of Electric State, Equivalent Circuit and Phasor Diagram for a Phase of a Synchronous Motor

In contrast to a synchronous generator, the axis of rotor poles in a synchronous motor lags behind that of the poles produced by the rotating magnetic field of the stator (see Fig. 15.3 b). The elec-

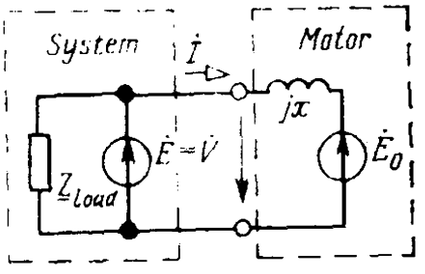


FIGURE 15.15

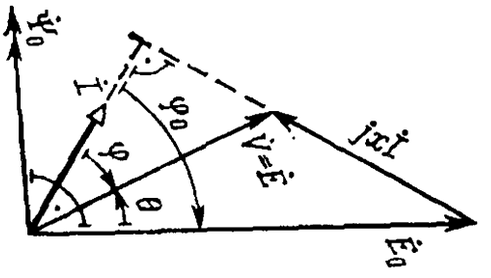


FIGURE 15.16

tromagnetic torque which results from this action is equal to the opposing (load) torque applied to the motor shaft, $T_{em} = T_{load}$. In this way a synchronous motor converts electric energy into mechanical energy.

The equivalent circuit of a phase of a synchronous motor connected to a large power system (V = const) is shown in Fig. 15.15. As is seen, it is similar to the equivalent circuit of a phase of a synchronous generator operating into a large power system (see Fig. 15.7), except that the motor draws electric power from the system while the generator delivers it to the system.

From the equivalent circuit for a phase of a synchronous motor the following equation of electric state may be derived:

$$\dot{E}_0 = \dot{V} - jx\dot{I} \tag{15.16}$$

where

$$\dot{E}_0 = j\omega\dot{\Psi}_0$$

and

$$x = x_{leak} + x_{arm}$$

similar to their counterparts in a synchronous generator (see Sec. 15.5).

The phasor/vector diagram answering Eq. (15.16) appears in Fig. 15.16. Here the load angle θ corresponds to the mechanical angle between the axis of the rotor poles and the axis of the resultant magnetic field of the machine divided by the number of pole pairs, similarly to Fig. 15.9. For a synchronous motor the load angle θ is always greater than zero ($\theta > 0$).

15.14 The Electromagnetic Torque and the Load-Angle Characteristic of a Synchronous Motor

Similarly to Eq. (15.9) defining the electric power developed by a three-phase synchronous generator, that taken by a three-phase synchronous motor is given by

$$P = 3VI \cos \varphi = 3E_0 I \cos \varphi_0 = 3E_0 I \cos (\varphi - \theta) \qquad (15.17)$$

where $\theta > 0$.

Rearranging Eq. (15.17) in the same manner as Eq. (15.9) and noting that $\theta > 0$, we derive an expression for the electromagnetic torque developed by a synchronous motor, similar to that defining the electromagnetic torque of a synchronous generator, Eq. (15.11):

$$T_{em} = (3p/2\pi f)\, E_0 V \sin \theta / x \qquad (15.18)$$

Since the mechanical power developed by a synchronous motor is

$$P_m = P = T_{em}\omega_{rot}$$

then, subject to Eq. (15.18), we get

$$P_m = P = 3VI \cos \varphi = 3E_0 V \sin \theta / x \qquad (15.19)$$

In a large power system the system voltage V and the system frequency f are constant.

In view of this observation we may conclude that, at a constant field (excitation) current, the electromagnetic torque T_{em} and the power P developed by a synchronous motor operating as part of a large power system depend solely on the load angle θ. The relations $T_{em} = f(\theta)$ and $P = f(\theta)$ are called the *torque-angle* and *power-angle characteristics* of a synchronous motor and have the shape similar to that of the load-angle characteristics of a synchronous

generator (see Fig. 15.10). These characteristics give insight into what happens in a synchronous motor as its load is varied. As the load (retarding) torque at the shaft of a synchronous motor increases

$$T_{\text{ret.2}} > T_{\text{ret.1}} = T_{\text{em.1}}$$

the rotor of the motor slows down, and the load angle θ and the electromagnetic torque T_{em} begin to increase. The retarding and electromagnetic torques strike a balance, $T_{\text{ret.2}} = T_{\text{em.2}}$, some time later, at a new value of the load angle, $\theta_2 > \theta_1$. If we wish to retain the necessary margin of stability, $\pi/2 - \theta$, at an increased retarding (load) torque, we should increase the field (excitation) current.

15.15 The V-Curves of a Synchronous Motor

The magnitude of excitation current affects both the margin of stability and the reactive current of a synchronous motor. Let us analyse this relation by referring to the phasor/vector diagram of a phase of a synchronous motor connected to a large power system

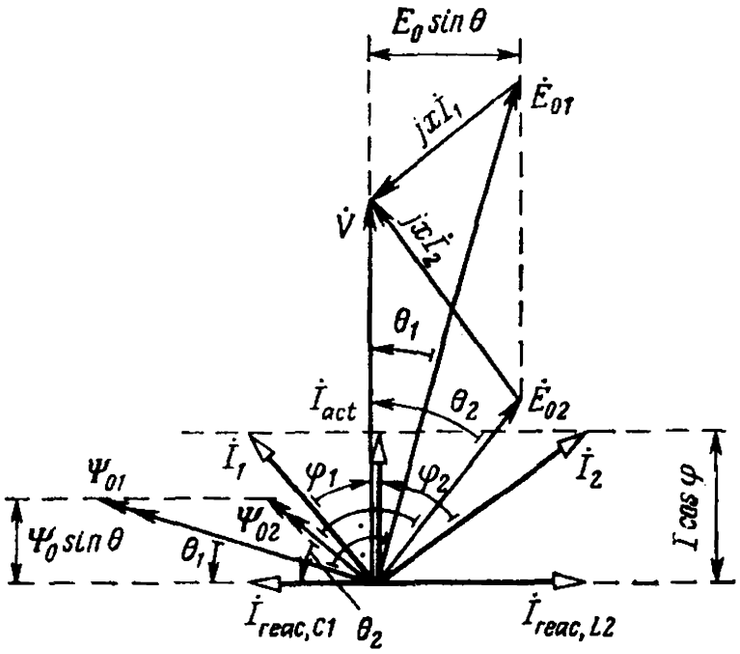


FIGURE 15.17

(V = const), as shown in Fig. 15.16. Given a constant retarding torque at the motor shaft, $T_{\text{ret}} = T_{\text{em}}$, its power $P = T_{\text{em}}\omega_{\text{rot}}$ will be likewise constant, so, as follows from Eq. (15.18) and Eq. (15.19), the products

$$E_0 \sin\theta = \omega\Psi_0 \sin\theta = \text{const}$$

and

$$I\cos\varphi = I_{\text{act}} = \text{const}$$

are always the same and independent of the excitation current. A combined phasor/vector diagram of a phase of a synchronous motor for $T_{ret} = \text{const}$ and for several values of the excitation (field) current $I_f = \text{VAr}$ is shown in Fig. 15.17. As I_f (excitation flux linkage Ψ_0) is reduced, the angle θ increases until the synchronous motor loses stability.

As follows from the phasor/vector diagram, the magnitude and effect of the stator current of a synchronous motor

$$\dot{I} = \dot{I}_{act} + \dot{I}_{reac}$$

depend on the excitation (field) current I_f. When the field current is less than some limiting value, $I_f < I_{f,\,lim}\,(P)$, the stator current I has an inductive reactive component $I_{reac,\,L}$ ($\varphi > 0$). When the field current exceeds some limiting value, $I_f > I_{f,\,lim}$, the stator current has a capacitive reactive component $I_{reac,\,C}$ ($\varphi < 0$). Hence, in the case of underexcitation the reactive power of a synchronous motor is inductive in its effect

$$Q_L = 3VI_{reac,\,L}$$

In the case of overexcitation, it is capacitive in its effect

$$Q_C = 3VI_{reac,\,C}$$

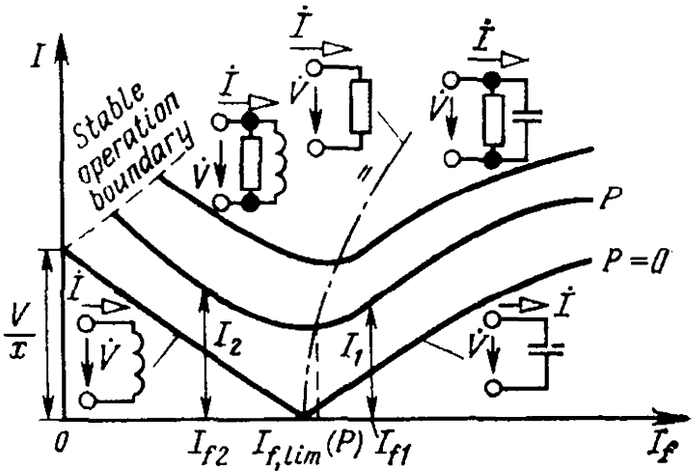


FIGURE 15.18

Accordingly, each phase of a synchronous motor connected to a large power system may be represented by an equivalent circuit consisting of a parallel combination of an equivalent resistive element whose resistance depends on the load torque, $r = V/I_{act} = F\,(T_{load})$, and an equivalent inductive (capacitive) element whose inductance (capacitance) is a function of the load torque and the field current:

$$L = V/\omega I_{reac,\,L} = F\,(I_f,\ T_{load})$$

or

$$C = I_{reac,\,C}/\omega V = F\,(I_f,\ T_{load})$$

If the load torque of the motor is zero, $T_{load} = 0$, then the equivalent circuit for a phase of a synchronous motor connected to a large power system will contain no resistive element, and the magnitude of the inductive (capacitive) element will be solely dependent on the field current.

The plots relating the stator current of a synchronous motor connected to a large power system (V = const) to the field current for a constant load torque, T_{load} = const, are called the *V-curves of the motor* (Fig. 15.18).

When no load torque is applied to the shaft of a synchronous motor ($T_{load} = 0$), then, on neglecting all forms of loss, it may be deemed that the stator current of a synchronous motor is purely reactive (see Fig. 15.18, $P = 0$):

$$\dot{I} = \dot{I}_{reac} = \frac{-\dot{E}_0 + \dot{V}}{jx} = \frac{\dot{V} + j\omega\dot{\Psi}_0}{jx} = F(I_f)$$

15.16 Control of the Active and Reactive Power of a Synchronous Motor

The *active power*

$$P_m = P = 3VI_{act} = \omega_{rot} T_{load}$$

of a synchronous motor connected to a large power system (V = const) can be varied by varying the load torque applied to its shaft (T_{load} = VAr). As the load torque is increased, the power of a synchronous motor also increases, leading to a large angle θ and to a reduction in the margin of stability, $\pi/2 - \theta$. If a synchronous motor is not to lose its margin of stability with increasing active power, one has to increase the field current at the same time. Large-size synchronous motors are equipped with suitable apparatus which maintains the necessary margin of stability in response to a change in the active power of the motor.

The *reactive power*

$$Q = 3VI \sin \varphi$$

of a synchronous motor connected to a large power system (V = const) at constant active power P is adjusted by varying the field current I_f. When I_f is less than some limiting value, $I_{f,11m}(P)$, the reactive power of the motor is inductive in its effect, $Q_L = 3VI_{reac,L}$. When I_f is greater than some limiting value, $I_f > I_{f,11m}$, the reactive power of the motor is capacitive in its effect, $Q_C = 3VI_{reac,C}$.

As a rule, synchronous motors are operated at an excitation level which results in a capacitive reactive power, so that the inductive reactive power of the induction motors connected to the same system is balanced out and the system is relieved of reactive current.

A case of practical importance is the use of a synchronous motor as a synchronous condenser which is an adjustable capacitive element (see Fig. 15.18, $P = 0$). Synchronous condensers serve to improve the power factor of the system (see Sec. 2.20).

15.17 Starting of a Synchronous Motor

The resultant torque that a synchronous motor develops due to the interaction of the stator magnetic field with the excited rotor at standstill is zero very nearly. Therefore, for starting, the rotor of a synchronous motor has to be brought up to a nearly synchronous speed in one way or another. Until quite recently this purpose was served by a small induction motor, and the synchronous motor would then be brought in synchronism with the system in the same way as a synchronous generator for parallel operation. That was a very complicated procedure which greatly limited the wide acceptance of synchronous motors. At present, this difficulty is no longer existent—*a synchronous machine is now started as an induction motor.* For this purpose the pole faces of a salient-pole rotor have slots which receive heavy copper or brass bars shorted together at both ends of the rotor, thus forming a squirrel-cage structure which occupies only a part of the rotor circumference. In one design of a synchronous motor, there is no starting squirrel-cage structure. Its purpose is served by the substantial rotor core, the metal wedges embedded in the rotor slots, and the binders insulated electrically from the rotor core.

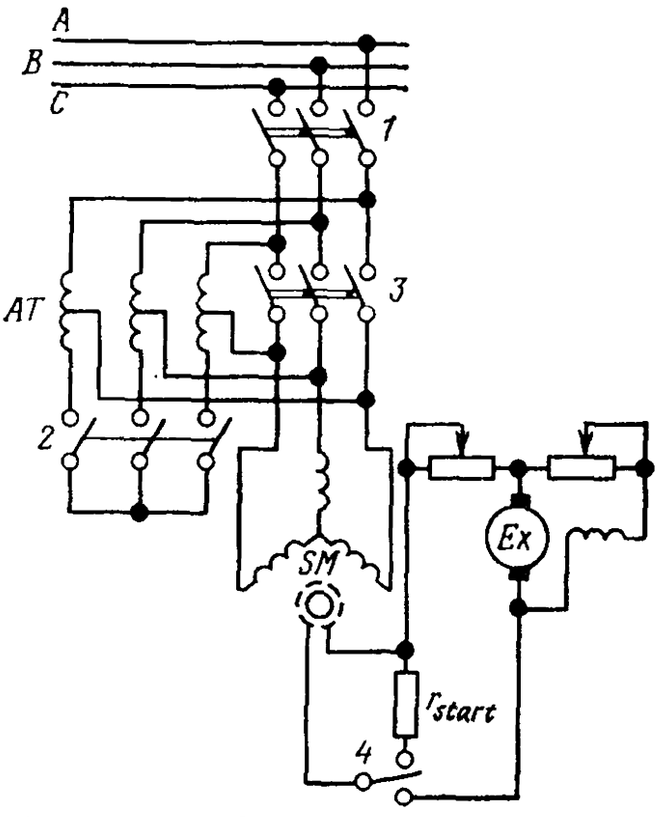


FIGURE 15.19

Starting a synchronous motor by induction-motor action is in two steps. During the first step, induction-motor action brings the synchronous motor to nearly synchronous speed with no d.c. excitation applied. During the second step, the direct field current is turned on and the synchronous motor is caused to pull into synchronism. As a rule, the field winding terminals are shorted through a resistor, r_{start} (Fig. 15.19) whose resistance is 8-10 times that of the field winding. The field winding ought not to be left open-circuited as the rotating magnetic field might induce in it a heavy emf dangerous to the insulation. Nor would it be well advised to short-circuit the field winding as the relatively heavy single-phase current induced in it might retard the rotor once it has come up to half the synchronous speed.

In most cases, synchronous motors are of heavy power ratings, and this necessitates a reduction in their starting currents by bringing down their starting voltage. This is often done by connecting the motor to be started to the power system via a starting autotransformer or an inductor. The starting procedure commences by closing switch 2 (see Fig. 15.19) so that the three phase windings of an autotransformer, AT, are connected in a star. Closure of switch 1 applies the line voltage to the input terminals of the autotransformer. In this way, the voltages applied between the stator-winding terminals of a synchronous motor, SM, are the line voltages of the three-phase system stepped down by the autotransformer. The synchronous motor is started as a squirrel-cage induction motor. When the slip of the rotor becomes sufficiently small, switch 2 is opened, so that the voltage applied to the synchronous motor is somewhat raised since now only a part of each phase winding of the autotransformer is acting as an inductor connected in series with the phase winding of the motor and its resistance somewhat limits the starting current. During the next step the full supply-line voltage is applied to the synchronous motor by closing switch 3. However, until the direct current is turned on the rotor keeps rotating as an induction motor. The starting procedure is completed by applying the direct current to excite the field with the aid of switch 4. The resultant electromagnetic forces bring up the motor to the synchronous speed, and the motor develops the required torque. This starting procedure does away with synchronizing and may be readily automated.

15.18 Fractional-Horsepower Synchronous Motors

The property of synchronous motors to run at constant speed despite variations in the load torque is achieved by connecting a d.c. source to the rotor winding via sliding contacts. Instead of a d.c.-excited electromagnet, fractional-hp synchronous motors use a permanent magnet fabricated from a magnetically-hard material and mounted on the rotor shaft. With this arrangement there is no need for an auxiliary d.c. source, and the motor is usually started by connecting its stator phase windings directly to the supply line. The pole pairs of the permanent magnet carry the bars of a short-circuited (damper) winding intended to produce the starting torque by induction action.

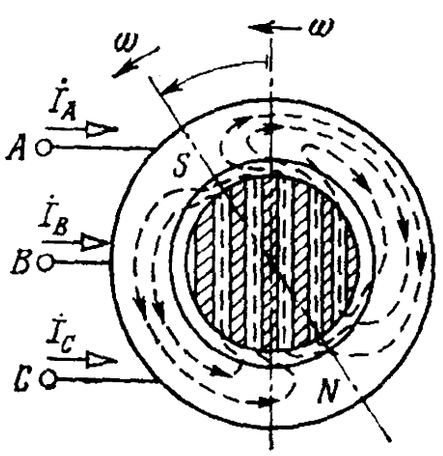


FIGURE 15.20

There is a further variety of fractional-hp synchronous motors known as *reluctance synchronous motors*. Their distinction is that the

rotor has the property of magnetic anisotropy which means that its reluctance is different in different directions. A sectional view of a two-pole anisotropic rotor is shown in Fig. 15.20. As is seen, the rotor is an assembly of electrical-sheet steel laminations interleaved with a layer of aluminium (the shaded areas). At synchronous speed, the blocks of rotor laminations are magnetized in the easy direction, and this determines the magnetic flux pattern set up by the stator currents. With load applied to the motor shaft, the magnetic flux due to the stator currents is bent, and this gives rise to a driving torque which balances the load torque.

A limitation common to all fractional-hp synchronous motors is the fact that there is no way of controlling their reactive power and the margin of stability.

16 Motor Drives

16.1 General

A *motor drive* refers to an electromechanical system consisting of a drive motor (or motors), a driven machine, a connection from the drive motor to the driven machine, and associated switchgear and controls.

This chapter will be mainly concerned with matters bearing on the selection of a drive motor for a particular application.

Historically, the form of driving that appeared first was *group drive* (or *driving*). In it, a single motor operated a line shaft, and this transmitted motion to the driven machines by means of ropes, V-belts, gears or some other members varying in number. Obviously, this form of drive was inevitably wasteful of power (in some cases as much as 50% of the input power would be lost on its way from the drive motor to the driven machines). Also, the installation would ordinarily take up quite a lot of floor area.

With advances in drive-motor design, group driving gave way to *individual drive* — in this case each machine was equipped with a drive motor of its own, often coupled to each other directly.

The next step forward was the reduction in the power lost within the driven machine itself. With only one motor per driven machine, one has to provide gear trains, V-belts, eccentrics and similar elements so that the machine could perform the various of its functions. With multimotor drive, each function (or, rather, each tool) of a driven machine is supplied by power by a separate drive motor. In other words, each tool has an individual drive of its own.

The advent of variable-speed drive motors has solved the problem of speed variation of the associated tool — now this can be done without resort to sophisticated mechanical devices (such as gearboxes and the like).

Direct connection of a drive motor to a driven machine makes it advantageous to adapt one to the other, so that in a wide range of present-day equipment it is difficult, if at all possible, to draw a boundary between its electric and mechanical sections.

In simplified form, the history of drive for a radial drilling machine is shown in Fig. 16.1 where the use of group driving is illustrated at 1, improvements in individual drive at 2 through 4, and multimotor drive at 5.

A further advantage of motor drive for machine tools is that their control with the aid of clutches, flat and V-belts and the like may be replaced with purely electric controls. With electric control, the operator spends less time on the necessary adjustments, speeds up the production process, and turns out more products per unit time. In cases involving operations that follow one another at a

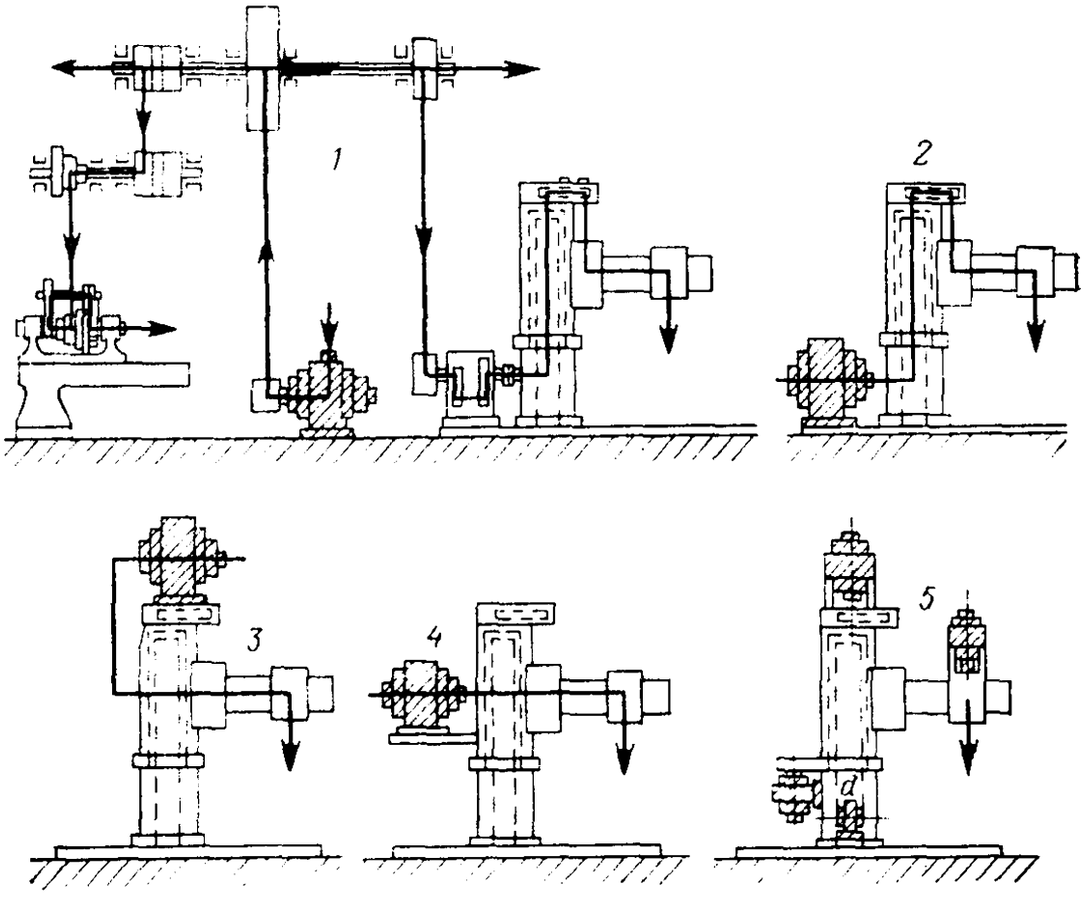


FIGURE 16.1

high rate, it has become necessary in many cases to replace the human operator with automatic facilities in the control function.

Automatic control is now applied to both simple machines such as squirrel-cage induction motors with ratings less than 1 kW, and to large units such as are used to drive blooming mills and totalling between them several megawatts of power. Automatic control may be limited to starting, stopping and reversing a motor. In other cases it may effect a sequence of operations in machining a part to a template. Automatic action is also used to protect driven machinery against overloads or breakdowns.

16.2 The Equation of Motion for a Motor Drive

In designing a motor drive, one has to know the mechanical arrangement, horsepower and torque requirements, and special conditions of operation of the driven machine. All of these factors will

determine, in one way or another, the static and dynamic loads applied to the shaft of the drive motor.

Static load refers to that arising from friction, the force(s) exerted in machining a component, the self-weight of the motor and driven machine, and the like. Dynamic load arises from variations in the kinetic energy of the drive system in the wake of changes in the speed of its various parts or members. Accordingly, the torque developed by the drive motor may be defined as

$$T_{mot} = T_{st} + T_{dyn} \tag{16.1}$$

Here T_{st} is the *static torque* due to all opposing forces. It may not depend on speed (curve 1 in Fig. 16.2) if it is due to friction, the opposing force which results from machining a metal part, and the like, or it may depend on rotational speed to some extent. In the case of a centrifugal pump, for example, feeding a constant-head system, the static torque has a constant component and a component proportional to the square of rotational speed (curve 2 in Fig. 16.2). This dependence on speed may be linear (curve 3) or nonlinear (curve 4).

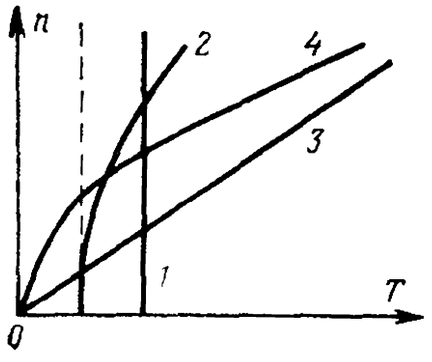


FIGURE 16.2

The second term, T_{dyn}, is the *dynamic torque* defined as $T_{dyn} = J\,d\omega/dt$. It may be positive or negative. It is proportional to J, the *moment of inertia*. For a rotating solid body the moment of inertia with respect to a given axis of rotation is the limit of the sum of the products of the masses m_k of each of the elementary particles into which the body may be conceived to be divided and the square of their distance, R_k, from the given axis of rotation

$$J = \sum_{k=1}^{n} m_k R_k^2 \tag{16.2}$$

Usually it is convenient to express the moment of inertia as the product of the mass of the body by the square of the *radius of inertia*, also called the *radius of gyration*, R_{iner}, that is

$$J = \sum_{k=1}^{n} m_k R_k^2 = m R_{iner}^2 \tag{16.3}$$

where R_{iner} is the distance from the axis of rotation at which all the mass of the body must be concentrated in order to obtain a moment of inertia equal to the actual one when the mass is distributed. The radii of inertia (radii of gyration) for simple solid bodies are given in reference tables.

In the solution of problems dealing with rotating bodies, notably drives, it was customary at one time to use the concept of flywheel effect to which the moment of inertia of a solid body is connected by a simple relation of the form

$$J = GD^2/4g \tag{16.4}$$

where G is the weight of the body, $D = 2R_{iner}$ is the diameter of inertia (diameter of gyration), g is the acceleration due to gravity, and GD^2 is the flywheel effect.

The moments of inertia for the rotors and armatures of electric motors are usually given in data sheets.

The best proposition would be to connect a drive motor to the driven device (say, the cutter of a machine-tool) directly, avoiding the use of any intermediate gearing or belting. In most cases, however, this cannot be done because the tool has to be driven at a relatively low speed (50-300 rpm), whereas the available motor is a high-speed type. It would not pay to design and build a special-purpose low-speed motor as it would be too large and too heavy. An optimum decision is to connect a normal motor (with a speed of 750-3000 rpm) to the driven tool via a step-down gearbox.

In the solution of problems dealing with driven machinery incorporating elements which execute rotational or translational motions at different speeds it is advantageous to replace the actual system with what may be called a *reduced system*, that is, a simplified one which consists of an only member driven at the speed of the drive motor. In replacing an actual system with a reduced one, an important step is to re-scale all the torques in such a way that the power (or energy) relations remain unaffected.

Suppose there is a drive motor rotating at ω_{mot}, which is coupled via a single-step gear transmission to a machine tool whose angular velocity is $\omega_{m.t}$ (Fig. 16.3, in which $J_m\omega_m = J_{mot}\omega_{mot}$). If we neglect the losses in the transmission (they are included in the reduced system), then from the condition of the conservation of energy we may write

$$T_{st}\omega_{mot} = T_{m.t}\omega_{m.t}$$

or

$$T_{st} = T_{m.t}\omega_{m.t}/\omega_{mot} = T_{m.t}/k_{g.r} \tag{16.5}$$

where T_{st} is the unknown static torque of the driven machine referred to the motor shaft (that is, to the angular velocity of the motor shaft), $T_{m.t}$ is the actual static torque of the driven machine available at its shaft, and $k_{g.r} = \omega_{mot}/\omega_{m.t}$ is the gear ratio from motor to driven machine.

If, when acted upon by a force $F_{m.t}$, the tool executes a translational (rather than rotational) motion at a speed $v_{m.t}$, then on the basis of the conservation of energy we may write

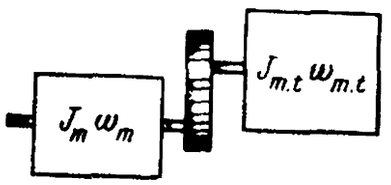


FIGURE 16.3

$$T_{st}\omega_{mot} = F_{m.t}v_{m.t}$$

Hence, the sought reduced static torque is

$$T_{st} = F_{m.t}v_{m.t}/\omega_{mot} \tag{16.6}$$

The reduced (or referred) system should also include *reduced (referred) moments of inertia*, J_{rd}. A reduced (referred) moment of inertia is that of a system made up solely of the elements rotating at the motor speed ω_{mot}, but storing a kinetic energy equal to the kinetic energy of the actual (prototype) system. From the equivalence of kinetic energy it follows that for a system consisting of a motor, a single gear transmission, and a machine tool which rotates at $\omega_{m.t}$ and has a moment of inertia $J_{m.t}$, we may write

$$J_{rd}\omega_{mot}^2/2 = J_{mot}\omega_{mot}^2/2 + J_{m.t}\omega_{m.t}^2/2 \tag{16.7}$$

and the sought reduced (referred) moment of inertia of the system is

$$J_{rd} = J_{mot} + J_{m.t}(\omega_{m.t}/\omega_{mot})^2 = J_{mot} + J_{m.t}/k_{g.r}^2 \tag{16.8}$$

Thus, for a complex drive system Eqs. (16.1) and (16.4) involve the reduced (referred) static moments of inertia. If the designer knows the torque T in newtons × metre, and the speed n in revolutions per minute, the required power P in kilowatts will be

$$P = Tn/9550 \tag{16.9}$$

where the coefficient $9550 = 60 \times 10^3/2\pi$ is a dimensionless number.

16.3 The Basic Forms of Duty for Drive Motors

The term "duty" refers to a statement of loads including no-load, rest and de-energized periods, to which the machine or apparatus is subjected, including their duration and sequence in time. With respect to motor drive, three basic forms of duty may be singled out. They are continuous duty, short-time duty, and intermittent duty.

Continuous duty refers to a duty that demands operation at a substantially constant load for a time sufficiently long for all the components of the motor drive to achieve a steady-state temperature (Fig. 16.4 a). Examples of machines operating in continuous duty are centrifugal pumps at pump-houses, fans, blowers, compressors, conveyors, smoke exhausters, paper-making machines, and textile-finishing machines.

Short-time duty normally refers to a duty at constant load during a time less than that required to obtain the steady-state temperature in continuous duty at the same load, followed by a rest of sufficient duration to re-establish equality of temperature with the surroundings (Fig. 16.4 b). Examples include motor drives for navigation locks, drawbridges, and aircraft landing gear.

Intermittent duty defines a requirement of service that demands operation for alternate periods of load and no load; load and rest; load, no load and rest. These intervals are such that during load periods the drive motor does not reach a steady-state temperature, whereas during no load or rest periods it has no time to re-establish equality of temperature with the surroundings (Fig. 16.4 c). Examples are cranes, rolling mills, oil drilling rigs, and metal-working machine-tools. Data sheets for electric machinery operating in intermittent duty include a very important variable called the duty cycle which is defined as the ratio of a load period to the sum of the load, no-load (and rest) periods, expressed as a percentage.

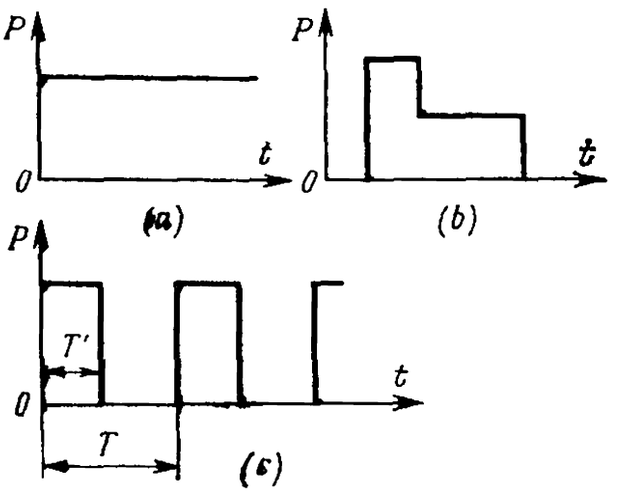


FIGURE 16.4

Knowledge of the expected form of duty is important as it governs the selection of a drive motor in terms of its power rating. Also the conditions in which a drive motor operates in intermittent duty will obviously differ from those applicable to a motor intended to run in continuous duty. For example, the cooling conditions for the field winding of a shunt-wound d.c. motor remain practically constant both during the load and the rest periods, while the cooling of the armature at rest is gravely impaired. For this reason a d.c. motor designed for continuous duty under unvarying cooling conditions would be a poor choice for use in intermittent duty: at the maximum safe temperature of the armature winding and commutator the field winding would be heated too much below its safe temperature.

For this and some other reasons it does pay to build motors specifically designed for intermittent duty. In fact, crane motors come in three modifications each adapted to a particular form of duty: continuous, short-time, or intermittent. Information to that effect is stated on the nameplate of each motor. In each case, the power rating of a motor refers to the rated power in the respective duty and subject to the duty cycle stated.

Every motor is designed so that it is capable of developing more than its rated power for some time without any damage to itself. This is referred to as overload capacity. It may be classed as instantaneous or as short-time. The *instantaneous overload capacity* is mostly determined by the electrical properties of the motor (the maximum torque in the case of induction motors or the commutation conditions in the case of d.c. motors). Sometimes, the mechanical design of a motor may also be a factor in this respect. It is customary to define the instantaneous overload capacity of a motor in terms of the *overload torque ratio*, λ_T, which is the ratio of the maximum instantaneously safe torque, T_{max}, to the rated torque, T_{rtd}, of the motor. For most electric motors λ_T is about 2 (for special-purpose motors, $\lambda_T = 3\text{-}4$).

The short-time overload capacity is an important consideration for motors operated under certain service conditions. This may be the *maximum power output* considered safe for 5, 10, 15, or 30 min, whichever may be the case. At the end of the specified overload period the motor must be de-energized for a time sufficiently long to re-establish equality of temperature with the surroundings (or the cooling medium). For the same type of motor the relation between its continuous-duty, instantaneous-overload and short-time overload ratings depends on its characteristics and design.

16.4 The Selection of Power Rating for a Drive Motor

The primary considerations in the selection of the power rating for a drive motor are economy, efficiency, and reliability. The use of an overrated motor would entail a waste of power, unwarranted capital outlays, and an increase in the required floor area. An underrated motor would reduce the efficiency of the driven machine, impair overall reliability, and might be readily damaged.

A drive motor must be chosen so that its power output is utilized to the best advantage. In operation, it should be heated to about the maximum safe temperature, but never exceed it. Also a drive motor should give normal performance in the case of likely short-time overloads and develop the starting torque required by the driven machine. Accordingly, the power rating of a drive motor is mostly chosen with regard to heat capacity (or temperature rise), following which a check is run to see if the motor thus chosen has an overload capacity sufficient to meet the expected starting conditions and to sustain short-time overloads. Sometimes (when the short-time load is very large), a drive motor would be chosen to meet the maximum power requirements. In such a case, it often happens that the continuous-duty power of the motor is not fully utilized.

The selection of the power rating for a drive motor to be used in continuous duty is the simplest of all. In the circumstances the

power output of the motor should be equal to the load power, and no check needs to be made for heat capacity and overload under running conditions. However it is important to see if the motor thus chosen can develop sufficient torque at starting in order to start the load satisfactorily under the most severe starting conditions that may be encountered in that particular application.

For a wide range of traditional loads the power requirements are determined on the basis of theoretical calculations that have stood the test of practice. For example, for a fan the power rating of its drive motor, in kW, must be

$$P = v\,\Delta p/1000\eta_{\mathrm{fan}}\eta_{\mathrm{trans}}$$

where v is the quantity of air handled by the fan, m^3s^{-1}; Δp is the pressure differential, Pa; η_{fan} is the efficiency of the fan (for propeller fans the figure is 0.2-0.35 and for centrifugal fans it is up to 0.8); and η_{trans} is the efficiency of the transmission from motor to fan.

In the above equation the product $v\,\Delta p$ is the useful power output of the fan, and 1000 is a factor used to convert power to kilowatts.

In many cases the power rating of a drive motor for use in continuous duty is found by empirical formulas tested in many years' practice.

For poorly explored applications the continuous-duty rating of a drive motor is determined by measuring specific energy consumption in actual service or by putting the drive to a trial operation.

The selection of the power rating for drive motors to be used in short-time and intermittent duty. In any class of duty it is important to know how the temperature rise, ϑ, of the drive motor varies with time. From the view-point of heat build-up an electric machine is an extremely heterogeneous body. Yet, in cases which do not call for an especially high accuracy it is legitimate to treat an electric machine as a homogeneous body.

Accordingly, we may use the following simplified equation of heat build-up:

$$Q_0\,\mathrm{d}t = C\,\mathrm{d}\vartheta + H\vartheta\,\mathrm{d}t \qquad (16.10)$$

where C is the heat capacity of the machine, H is its heat transfer, and Q_0 is the heat generated within the machine per unit time.

At t tending to infinity,

$$Q_0\,\mathrm{d}t = H\vartheta_{\mathrm{max}}\,\mathrm{d}t$$

and

$$\vartheta_{\mathrm{max}} = Q_0/H$$

Hence,

$$\vartheta_{max}\, dt = (C/H)\, d\vartheta + \vartheta\, dt$$

On solving the above equation by separation of variables, we get

$$A \exp(-t/\tau) = \vartheta_{max} - \vartheta$$

where $\tau = C/H$ is the time constant of heat build-up found by experiment.

At $t = 0$, the initial temperature rise is

$$\vartheta = \vartheta_0$$

so that

$$A = \vartheta_{max} - \vartheta_0$$

and the temperature rise of the machine can be written as

$$\vartheta = \vartheta_{max} - (\vartheta_{max} - \vartheta_0) \exp(-t/\tau) \qquad (16\ 11)$$

That is, in the above case the temperature rise will increase exponentially, tending to ϑ_{max} in much the same way as it does when $\vartheta_0 = 0$.

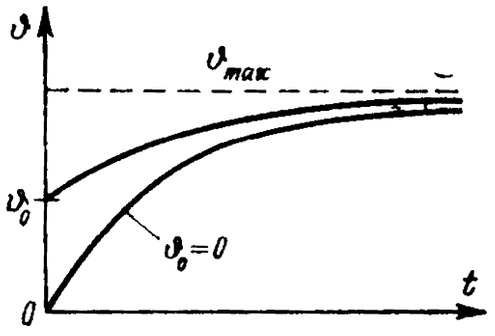


FIGURE 16.5

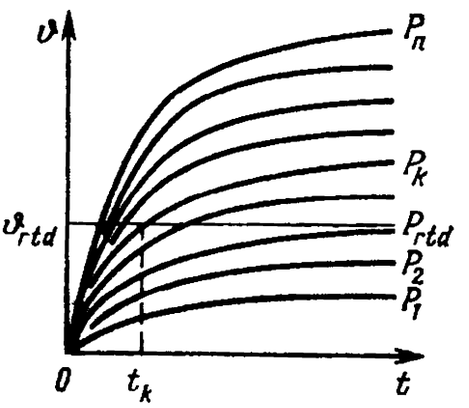


FIGURE 16.6

The value of the initial temperature rise only varies the rate of temperature change, but does not affect the nature of the process (Fig. 16.5).

Given various values of continuous-duty load, the plots of ϑ as a function of time of one and the same machine will only differ in ordinates (Fig. 16.6). The maximum safe temperature rise for a given machine is ϑ_{rtd}, shown by a line, $\vartheta = \vartheta_{rtd}$, parallel to the axis of abscissae. It intersects the $\vartheta = f(t)$ curves at various points corresponding to various loads sustained by the motor. The abscissa of the intersection defines the time interval, t_k, during which the power delivered by the motor may temporarily be equal to P_k representing the overload over its continuous-duty power rating. The temperature-rise curve asymptotically approaching ϑ_{rtd} corresponds to the power rating, P_{rtd}, of the motor. At loads less than P_{rtd}, the

power of the motor is not utilized fully. This also goes for cases where a motor is loaded to its rated power for a relatively short time. In fact, it is advisable to overload a motor for a short time, with the overload being progressively increased as the duration of overload is decreased. The limit to which the overload may be increased as the duration of a load period is decreased is determined by the instantaneous overload capacity of the motor which in turn depends on its electric parameters (the maximum torque in the case of induction motors and the commutation conditions in the case of d.c. machines).

In intermittent duty, a drive motor alternatively builds up and dissipates heat. Variations in its temperature during each cycle depend on its heat balance during the previous cycle. A plot of heat

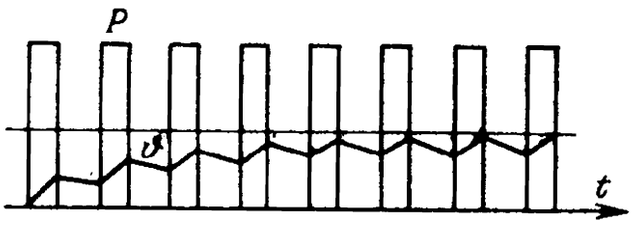


FIGURE 16.7

build-up and cooling as a function of time for an intermittent-duty motor is shown in Fig. 16.7. The final temperature rise in a given part of the cycle is the initial temperature rise for the next part of the cycle. In the case of a marked change in the cooling conditions during a particular part of a cycle (the motor is stopped or its speed is changed substantially), this will cause a change in $\tau = C/H$, the time constant of heat build-up, and this change must be reflected in the construction of the plot.

The selection of a power rating on the basis of temperature rise and heat build-up plots (or through the use of an applicable analytical approach) is very time-consuming and does not yield accurate results. We have included the above graphs only in order to illustrate how the temperature of a motor varies with changes in its load. In most cases, power requirements are determined by simpler methods, notably by the equivalent current method.

The *equivalent current method* is based on the assumption that the average losses of a motor carrying a variable load should be equal to the losses occurring in the case of continuous (rated) load. As has been noted more than once, the losses of a motor are the sum of the fixed (or constant) losses, P_{fixed}, and the variable loss, P_{var}. The fixed losses include windage and friction losses, core losses (in the case of induction and shunt-wound d.c. motors), and excitation losses (in the case of synchronous and shunt-wound motors). The fixed losses may be taken to be proportional to the square of the

running current I of the motor and the resistance r of the corresponding winding, of which the latter may be assumed to be constant. If the current changes during some time intervals, then the overall loss of power in a motor in the time $\Sigma t = T$ will be

$$(P_{\text{fixed}} + rI_1^2)\, t_1 + (P_{\text{fixed}} + rI_2^2)\, t_2 + \ldots + (P_{\text{fixed}} + rI_k^2)\, t_k + \ldots + (P_{\text{fixed}} + rI_n^2)\, t_n$$

Assuming an equivalent load current, I_{eq}, the power lost in a motor over the same time T would be

$$(P_{\text{fixed}} + rI_{\text{eq}}^2)\, T$$

In either case the same amount of power must be dissipated as heat. Hence the equivalent current may be defined as

$$I_{\text{eq}} = \sqrt{\frac{I_1^2 t_1 + I_2^2 t_2 + \ldots + I_k^2 t_k + \ldots + I_n^2 t_n}{t_1 + t_2 + \ldots + t_k + \ldots + t_n}} \tag{16.12}$$

Once we know the equivalent current, the rated voltage and the rated power factor, we can determine the power rating of the motor $P_{\text{rtd}} \geqslant V_{\text{rtd}} I_{\text{eq}} \cos \varphi_{\text{rtd}}$.

The equivalent-current method is applicable only if the core loss, the windage and friction losses, and the winding resistances remain constant through the total operating time. This requirement will not, for example, be satisfied by a series-wound motor in which a change in load brings about marked variations in the flux and speed (and, in consequence, variations in the core loss and windage and friction losses).

In some applications it is necessary to determine the required torque directly, rather than current. This is done by the use of the *equivalent-torque method*: in all electric motors the torque is proportional to the product of the current and flux. For a.c. motors, the power factor may be assumed to be approximately constant. By the same token, the magnetic flux may be deemed constant in all motors except series-wound and compound motors. Under the above simplifying assumptions we may define the torque as

$$T = kI$$

where k is a constant. Then, on the basis of Eq. (16.12), it follows that

$$T_{\text{eq}} = \sqrt{\frac{T_1^2 t_1 + T_2^2 t_2 + \ldots + T_k^2 t_k + \ldots + T_n^2 t_n}{t_1 + t_2 + \ldots + t_k + \ldots + t_n}} \tag{16.13}$$

Knowing the equivalent torque and the rated angular velocity ω_{rtd}, we can readily find the power rating of the motor

$$P_{\text{rtd}} = T_{\text{eq}} \omega_{\text{rtd}}$$

After the power rating of a motor has been selected by the equivalent-current or equivalent-torque method it is important to verify that the maximum torque T_{max} of the motor is sufficient to meet the short-time overload requirements in a particular application. In other words, the designer should check to see that the following requirement is satisfied: The overload factor λ_{rtd} must be greater than the ratio of the maximum load torque T_{max} to the rated motor torque, that is,

$$\lambda_{rtd} > T_{max}/T_{rtd}$$

16.5 The Selection of Motor Design and Enclosure

The selection of a motor design (d.c., induction, synchronous) is a relatively simple task in the case of a continuous, unvarying load. This application does not call for speed control, so the best choice is a synchronous motor. The point is that a synchronous motor is started in the same manner as an induction motor, while it is smaller in size and more economical in operation than an induction motor of the same power rating (it has a higher power factor and a higher maximum torque).

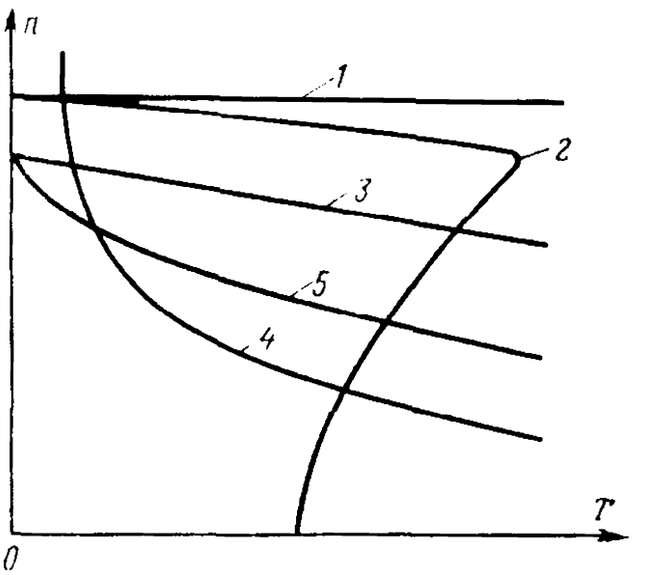


FIGURE 16.8

In applications involving speed control, frequent starting and stopping, a varying load, and so on, however, the decision as to which motor design should be preferred must be taken by considering the operation of the drive and the speed-torque characteristics of various motors. Speed-torque characteristics may be classed into natural and controlled. A natural speed-torque characteristic is that which is obtained when a motor is started under its rated conditions, is connected in the circuit in the normal way, and there are no additional elements in the motor circuits. A controlled speed-torque characteristic is that which is obtained when the motor circuits include some additional elements and the circuits are connected in some special manner. The natural speed-torque characteristics of several motors are compared in Fig. 16.8.

An important consideration with regard to the speed-torque characteristics of motors is their flatness

$$\alpha = \Delta T/\Delta n \qquad (16.14)$$

The flatness of a speed-torque characteristic may vary from region to region of operation.

In terms of flatness, it is customary to class the speed-torque characteristics of motors (Fig. 16.8) into absolutely flat (1) for which $\Delta n = 0$ and $\alpha = \infty$ (such as in the case of synchronous motors); flat for which $\alpha = 10\text{-}40$ (the linear portion of the speed-torque characteristic of an induction motor (2), the speed-torque characteristic of a shunt-wound motor (3)); and drooping for which $\alpha \leqslant 10$ (such as the speed-torque characteristic of a series-wound motor (4), the controlled speed-torque characteristic of a wound-rotor induction motor (5), and the controlled characteristic of a shunt-wound d.c. motor).

In some cases the degree of flatness of the speed-torque characteristic is a controlling consideration in the selection of a motor. For example, the motors used to drive hoisting and conveying machinery should preferably have a drooping speed-torque characteristic, while the drive motors for cold-rolling steel mills should have a very flat characteristic.

In applications involving frequent starting and a varying load, preference is given to squirrel-cage induction motors as they are most reliable, simple to service, and are relatively inexpensive. Wound-rotor induction motors are more expensive, involve more servicing, are large in size, and have a lower power factor (because of the greater air gap). The advantages of wound-rotor induction motors in terms of starting torque are insignificant when compared with a double-cage motor. This is the reason why wound-rotor induction motors are used only in cases of special requirements for starting torque or starting current. (These requirements usually arise because the transformer substation can supply a limited power or because it also serves some special loads.) All in all, for loads in ratings not over 100 kW, with no speed control, the best choice is a squirrel-cage motor. For higher ratings, if a cage motor is ruled out, preference is given to wound-rotor motors.

Until quite recently, the speed of induction motors has been usually controlled by placing a resistance in the rotor circuit and by varying the number of pole pairs on the stator winding, both methods being fairly imperfect. The former is warranted only when the speed needs to be controlled between narrow limits, with the torque at the motor shaft remaining constant. The latter provides only stepped speed control and is practically limited to low-power metal-cutting machine-tools.

A more recent addition is speed control by thyristors. One of their advantages is that they can change the frequency of alternating current, so that the angular velocity of the rotating magnetic field can be varied continuously and between broad limits. As a

corollary, the speed of induction and synchronous motors can likewise be controlled at will.

Direct-current motors are more expensive, take more labour and time for servicing and wear out earlier than a.c. motors. Still, preference in many cases is given to d.c. motors because their speed can be varied between broad limits (in the ratios 3-to-1, 4-to-1 and more) by simple means. But even in such cases the Ward-Leonard system assuring a wide range of speed control is giving way to the more economical and compact thyristor speed control units.

Examples of applications where a d.c. motor drive is preferable are reversible rolling mills, adjustable nonreversible multidrive rolling mills, machinery operating in intermittent duty, blast-furnace skip hoists, and special-purpose lathes.

Types of motor enclosure. The basic types of motor enclosure are as follows*.

An *open motor* is one having ample openings in the end shields and the frame to permit passage of external cooling air over and around the windings of the machine. It is only suitable for few applications, because the machine might easily be fouled by dirt in an industrial environment and attending personnel runs the risk of an electric shock because the live parts of the machine are readily accessible.

A *screen-protected* (or *guarded*) *machine* is one in which all openings are covered by wire-mesh screens so constructed that rain drops, foreign matter, filings, saw-dust and the like cannot enter the machine, but particles of dust can. Motors with this type of enclosure may be installed outdoors.

A *totally enclosed machine* is one so enclosed as to prevent the free exchange of air between the inside and the outside of the case. For better cooling, air is forced through the machine, for which purpose inlet and outlet air conduits may be provided. Totally enclosed machines are installed in locations where the atmosphere may contain dust, vapours, corrosive fumes, and the like.

An *explosion-proof machine* is a totally enclosed machine whose enclosure is designed and constructed to withstand an explosion of a specified gas or vapour which may occur within it and to prevent the ignition of the surrounding medium by sparks, flashes or explosions occurring inside the machine case.

Motors intended for use in damp locations have damp-proof or moisture-proof insulation.

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* It is to be stressed that the classification of motor enclosure types adopted in the USSR is different from those in force in the UK or the USA.—*Translator's note.*

16.6 Thyristor-Controlled Motor Drives

Several factors have served to boost the use of thyristor-controlled motor drives, notably the high efficiency of thyristor converters (95-97%), the relatively small size, mass and time lag of thyristors, and the small power needed for the operation of thyristor controllers. A further advantage is that thyristor control offers a wide choice of control characteristics and dynamic behaviour for both a.c. and d.c. motors.

Thyristor speed control for a.c. motors. The speed of an induction motor is controlled by placing thyristors in the stator or rotor circuit.

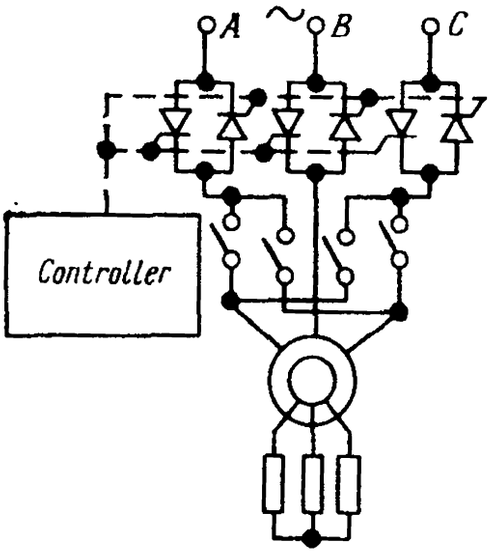

FIGURE 16.9

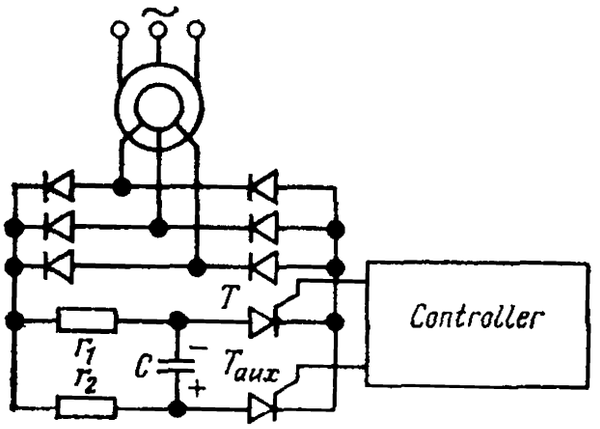

FIGURE 16.10

When connected in the stator circuit, they control the amplitude of the stator terminal voltage (phase control) or the frequency of the stator terminal voltage (frequency control) and, in consequence, the torque developed at the output shaft of the motor, Eq. (14.35). When connected in the rotor circuit, the thyristors vary the resistance in the rotor circuit, thereby varying the rotor speed (see Fig. 14.25).

The arrangement of a *thyristor controller* for the stator voltage of an induction motor is shown in Fig. 16.9. As is seen, each phase of the controller contains two anti-parallel thyristors. The controller synchronously turns on the thyristors in the same order as is the sequence of phases. By varying the instant when a thyristor is fired (see Fig. 10.17), it is possible to control the amplitude of the fundamental voltage. The thyristors are turned off automatically as the voltage of the respective phase is reversed in polarity. This is *natural commutation.*

In contrast, any one of the thyristors may be turned off by

applying a reverse current to its terminals from some external source. This is *controlled commutation*. An example is offered by a *thyristor controller operating in the rotor circuit* of an induction motor (Fig. 16.10). A controlled-commutation unit operates the main thyristor T which completes the path for the rectified current through a resistor r_1. The controlled-commutation unit contains an auxiliary thyristor T_{aux}, a resistor r_2, and a capacitor C. Let initially the main thyristor T be conducting and the auxiliary thyristor T_{aux} be turned off. In this condition, the capacitor C is charging via the resistor r_2 as shown in Fig. 16.10. Now we apply a turn-on signal from the controller to the gate of the auxiliary thyristor. The latter will fire, and the capacitor will begin discharging through the two thyristors. Since the discharge current of the capacitor opposes the forward current of the main thyristor, the latter is turned off. After that the capacitor begins to re-charge through the resistor r_1 and the conducting auxiliary thyristor. It takes another turn-on signal from the control unit in order to drive the main thyristor into conduction. As this happens, the discharge current of the capacitor turns off the auxiliary thyristor, and the capacitor itself recharges and regains the condition shown in the figure. The next cycle of operation begins by application of a turn-on signal from the control unit to the gate of the auxiliary thyristor.

The equivalent resistance of the rotor circuit depends on the ratio of the ON period to the OFF period of the main thyristor. By varying this ratio it is possible to adjust the average rotor current and, as a consequence, the torque developed by the induction motor.

Speed control of induction motors by varying the frequency of the stator voltage is a more promising approach to motor speed control. This purpose is widely served by self-contained *thyristor inverters*, that is, devices which convert a direct voltage to an alternating voltage with any number of phases.

Consider the circuit of a simple single-phase self-contained inverter (Fig. 16.11) in which a d.c. source E is connected to the centre tap in the primary winding of a transformer. When a signal from a control unit turns on thyristor T_1 while thyristor T_2 remains off, the d.c. voltage source E is connected to the left-hand half of the transformer primary. The current in that half-winding sets up a magnetic flux in the transformer core, so that an emf is induced in the transformer secondary, and the capacitor C is charged as shown in the figure.

When a control signal is applied to T_2, the capacitor C recharges so that T_1 is turned off in the same manner as shown in Fig. 16.10, the d.c. voltage source E is connected to the right-hand half of the transformer primary, so that the magnetic flux set up in the transformer core is in the opposite direction, as is the emf induced in the

transformer secondary, w_2. The frequency of the alternating voltage supplied by the inverter is determined by the frequency (or rate) at which the control generator supplies the signals that turn on the thyristors.

The above inverter may be used for the continuous speed control of single-phase and two-phase induction motors (see Sec. 14.18).

Inversion of a direct voltage into a three-phase or a polyphase system of voltages is effected in a similar way. Customarily, a d.c. voltage source E is replaced with a rectified voltage from an a.c.

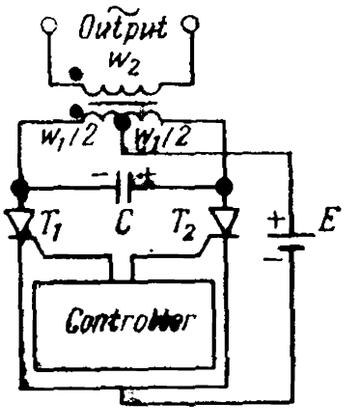

FIGURE 16.11

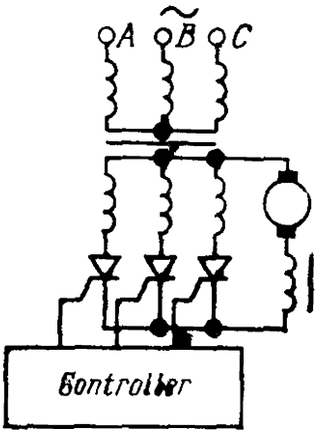

FIGURE 16.12

supply line. If thyristors are used for both inversion and rectification, the rectified voltage can be adjusted over a wide range of values (see Fig. 10.17), so that a more flexible control can be applied to the motor.

In addition to a combination of a thyristor inverter and a thyristor rectifier, motor control utilizes more sophisticated combinations, such as a thyristor rectifier + inverter + rectifier.

With thyristor control of one form or another, the speed of induction motors can be varied in the ratio up to 50-to-1 or even greater.

Thyristor speed control for d.c. motors. Apart from speed control, thyristor controllers may be used to modify the characteristics of a given drive motor. With them, a d.c. motor can be operated off an a.c. supply line.

A simple way of connecting a d.c. motor to a three-phase supply line is shown in Fig. 16.12. The controller synchronously turns on the thyristors in the same order as is the sequence of phases during the positive half-cycles of the phase voltages (see Fig. 10.17). During the negative half-cycles of the phase voltages natural commutation takes place, and the thyristors of the respective phases are turned off.

The average rectified voltage of the three phases is equal to the armature voltage of a d.c. motor, Eq. (10.10)

$$V_{\text{arm,av}} = (3/2\pi)\, V_m\, (1 + \cos \alpha)$$

where α is the firing angle of the thyristors.

By varying the firing angle with the aid of a control unit, it is possible to adjust the direct voltage across the motor armature, that is, the direct current in the armature circuit, I_{arm}. In turn, this leads to a change in the torque available at the output shaft of the motor, Eq. (13.2). The ripple in the rectified currents is minimized by an inductor placed in the armature circuit.

17 Fuses, Switches, Relays

17.1 General

The operation of electric equipment involves the use of a large variety of devices such as fuses, switches, circuit breakers, relays, etc.

In their permanent locations these devices are interconnected by means of bare buses set up on insulators or by means of power cables. Bare buses are simpler and less expensive to install and maintain, and they can operate at higher current densities. In most cases, buses are made of aluminium. Steel buses are permitted for installations operating at currents not over 200 A.

Aluminium buses may be single-strip and multi-strip, of rectangular cross section. For heavy alternating currents it is usual to install buses in the form of channels (Fig. 17.1). They are mechanically strong and easy to install. Also, because the conducting material is located round the periphery of the overall cross section, channel-shaped buses are almost immune to skin and proximity effects which tend to reduce the effective cross-sectional area of a conductor. At appropriate intervals the straight portions of buses supported by insulators are joined together by means of expansion compensators. As their name implies, these devices make up for the increase in the length of the buses on heating.

The reliable operation of an electric installation is to a large extent dependent on the quality of contact. The most commonly used forms of contact are permanent such as when leads are joined to the terminals of a machine or some other apparatus; sliding such as made between the stationary and moving parts of a device; and switching such as used in switches and similar devices. Sliding contacts have to stand up to most adverse effects, especially when they have to interrupt short-circuit currents.

In any form of contact, two conductors actually touch each other at two elementary areas produced as the microscopic irregularities or asperities on the touching surface are crushed by pressure (Fig. 17.2). This crushing action tends to reduce the cross section of the current path. The resultant increase in resistance is called the *contact resistance.* It decreases with increasing contact pressure, since the asperities are crushed more.

The contact resistance between two mating surfaces may grow tens or even hundreds of times due to their oxidation. This often

happens when the contacting surfaces are heated to over 70-75°C. For this reason it is important to take precautions against heating and oxidation at contact areas.

As a rule, it takes some time to open ("interrupt") an electric circuit. The interruption of a circuit is inevitably associated with the production of an emf of self-induction varying in magnitude. The joint action of this emf and the supply voltage breaks down the air gap between the parting contact surfaces and an electric arc

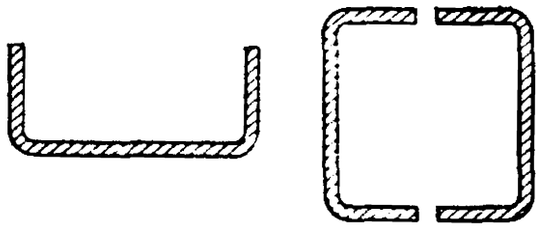

FIGURE 17.1

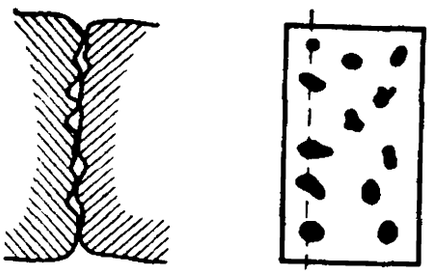

FIGURE 17.2

strikes. As it has a very high temperature, the arc may rapidly destroy or weld together the contacts. An electric arc is an especially hazardous occurrence in high-voltage switches or breakers intended to interrupt short-circuit currents.

It is a far simpler matter to open a.c. circuits because an alternating current goes through value zero every half-cycle, and this serves to extinguish the arc. An a.c. switch must be designed and constructed so as to prevent an arc from re-striking after it has been extinguished at the first zero-crossing.

Direct-current circuits are more difficult to interrupt. Switches for d.c. applications must be capable of dissipating the very large energy released by a sustained electric arc.

17.2 Thermal Protection of Electric Circuits

The conductors of electric circuits must be protected against the overheating that might be caused by short-circuits and sustained overloads.

The term 'short-circuit' refers to an abnormal connection of relatively low resistance between two points of different potential in a circuit. A short circuit may arise inadvertently when, say, two (or more) uninsulated live parts touch each other (such as in the case of two conductors of an overhead power line) or it may result from damage to the insulation due to ageing, wear, puncture, and the like. A short circuit leads to an abrupt increase in current and, since the heat dissipated in conductors is proportional to the square of the current they carry, the thermal action of a short circuit may

lead to the destruction of the insulation and a fire. Also, a short circuit gives rise to excessive electrodynamic forces between the conductors. Last but not least, a short circuit brings about a substantial change in the system voltage with the result that the motors and other machines supplied by the system may slow down or even come to a complete stop.

The limits of current for conductors and cables, as set up by a relevant Soviet code, are given in Table 17.1.

TABLE 17.1 **Limits of Load Current for Wires and Cords in PVC and Rubber Insulation and with Aluminium Conductors**

Cross-sectional area, mm², per conductor	Load current, A					
	Open wiring	Wires laid in common conduit				
		Two single-conductor	Three single-conductor	Four single-conductor	One two-conductor	One three-conductor
2.5	24	20	19	19	19	16
4	32	28	28	23	25	21
6	39	36	32	30	31	26
10	60	50	47	39	42	38
16	75	60	60	55	60	55
25	105	85	80	70	75	65
35	130	100	95	85	95	75
50	165	140	130	120	125	105
70	210	175	165	140	150	135
95	255	215	200	175	190	165
120	295	245	220	200	230	190
150	340	275	255	—	—	—
185	390	—	—	—	—	—
240	465	—	—	—	—	—
300	535	—	—	—	—	—
400	645	—	—	—	—	—

The simplest way to interrupt short-circuit currents is to utilize their thermal effect in a protective device called a *fuse*. The element that interrupts a fault current is called a fuse link; it will melt when the current in the protected circuit exceeds a definite rated value. In fact, a fuse link is a part of the protected circuit, readily destroyed by the thermal action of current. For this to happen, the fuse link must have a reduced thermal durability which is achieved by making it from a material having a high resistivity (such as an alloy of tin and lead) or from a well conducting material (such as silver or copper) but with a small cross-sectional area.

The blowing of a fuse link ought not to be accompanied by an arc along the fuse. To avoid this, fuse link ought not to be used on

voltages in excess of their rated value. For this reason it is customary to stamp both the rated current and the rated voltage on each fuse.

Fuses come in a large variety of types and designs. For voltages up to 250 V and currents to about 60 A, wide use is made of plug fuses (Fig. 17.3). A plug fuse consists of a wire or strip, 3, of a fusible alloy mounted in a porcelain container, or plug, 2. The container is fitted with a screw base corresponding to the thread in a receptacle,

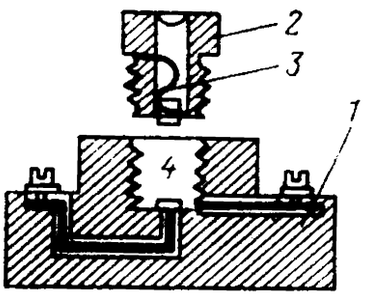

FIGURE 17.3

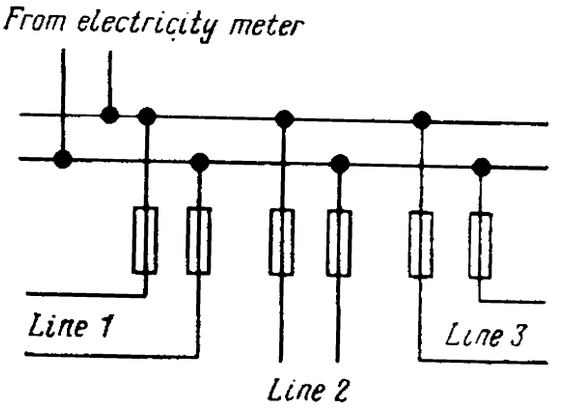

FIGURE 17.4

1, and bearing upon a fixed central contact, 4. The porcelain container has two metal contacts to which the fuse link is soldered.

Plug fuses are ordinarily clustered on *group fuse boards* from which individual lines run to the respective loads. On a fuse board, every wire of each line must be protected by a fuse of its own (Fig. 17.4). This arrangement provides for ease of supervision and replacement of blown fuses.

High-voltage lines are usually protected by several types of cartridge fuses (Fig. 17.5) in which a fusible wire is enclosed in a porcelain tube and runs for a considerable length. The molten metal is contained within the cartridge, and the flow of air through the tube rapidly quenches the arc. Fuses are rated for the maximum current that they will carry indefinitely, and the respective value is indicated on the fuse link. The time necessary for a fuse link to blow can approximately be found from the dependence of the melting time on the ratio of the actual to the rated current of the fuse.

Short-circuit protection is not the same as sustained-overload protection. The current ratings of fuses intended to protect individual portions of a line or a circuit should be chosen so as to allow for a short-time overload such as may occur at starting a motor.

The thermal principle of protection is also utilized in the more sophisticated devices called *thermal relays.* In such relays the overload protection is obtained by a *bimetallic element.* As its name implies, the element is made up of two metal plates differing in thermal expansion, laminated together and pressed into a concave

disc, 1 (Fig. 17.6). When heated to a predetermined temperature by a heater element, 2, carrying the same current as the protected circuit, the more rapid expansion of one metal increases disc tension until it suddenly snaps to its convex position and disengages a trip

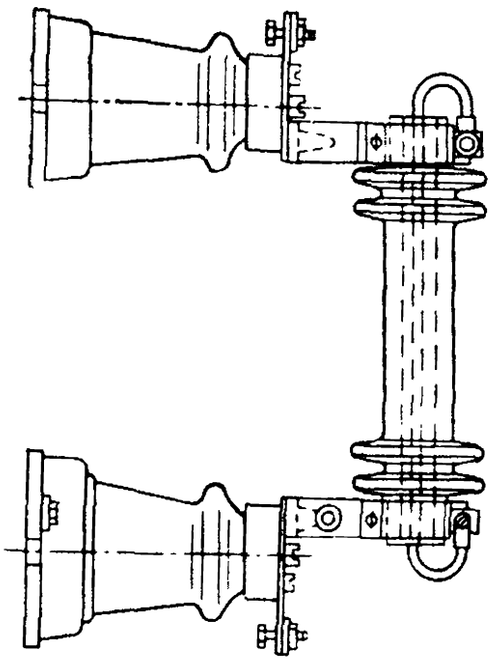

FIGURE 17.5

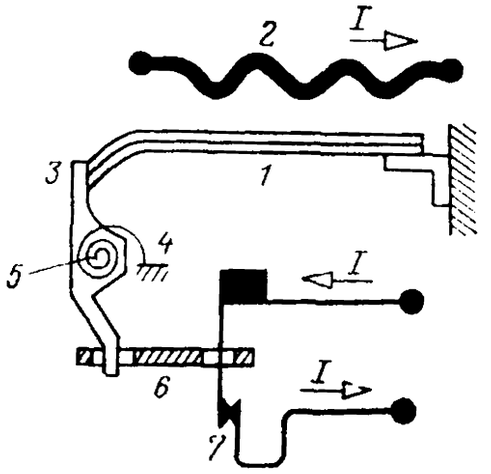

FIGURE 17.6

lever 3. Acted upon by a spring 4, the trip lever rotates about on its shaft 5 clockwise and causes a tie-rod 6 to break the contacts 7 thereby disconnecting the overloaded line.

Because of a considerable thermal lag, thermal relays cannot provide short-circuit protection. Therefore, they are usually employed in combination with fuses.

17.3 Low-Voltage Automatic Air Circuit Breakers

Fuses can protect electric motors and other electric loads against short-circuit currents only, but cannot give a reliable protection against sustained overloads. This is the reason why high-power electric installations are additionally equipped with automatic circuit breakers.

In electric installations operating at not over 1 kV, the simplest devices that can serve this purpose are automatic air circuit breakers*. In fact, they provide not only overload protection. Some of them are designed to de-energize a machine or a circuit in the case of an overvoltage, an overcurrent, an undervoltage, an undercurrent, or a reverse current. There are also a large number of special-purpose circuit breakers.

* Nonautomatic types are discussed in Sec. 17.4.

The actuation of an automatic circuit breaker in the case of an abnormal condition (whichever it may be) can be performed by means of electromagnetic, thermal or combination electromagnetic and thermal tripping mechanisms. In the case of a combination tripping mechanism, each of the two elements may open the circuit breaker independently. An electromagnetic tripping mechanism acts instantaneously, so no additional fuses need be provided.

The choice of a tripping mechanism for a circuit breaker depends on the function it is expected to perform. Several forms of electromagnetic tripping mechanisms are shown in Fig. 17.7.

The most commonly used type of air circuit breaker is the *overcurrent circuit breaker* (Fig. 17.7 a). When the current in the pro-

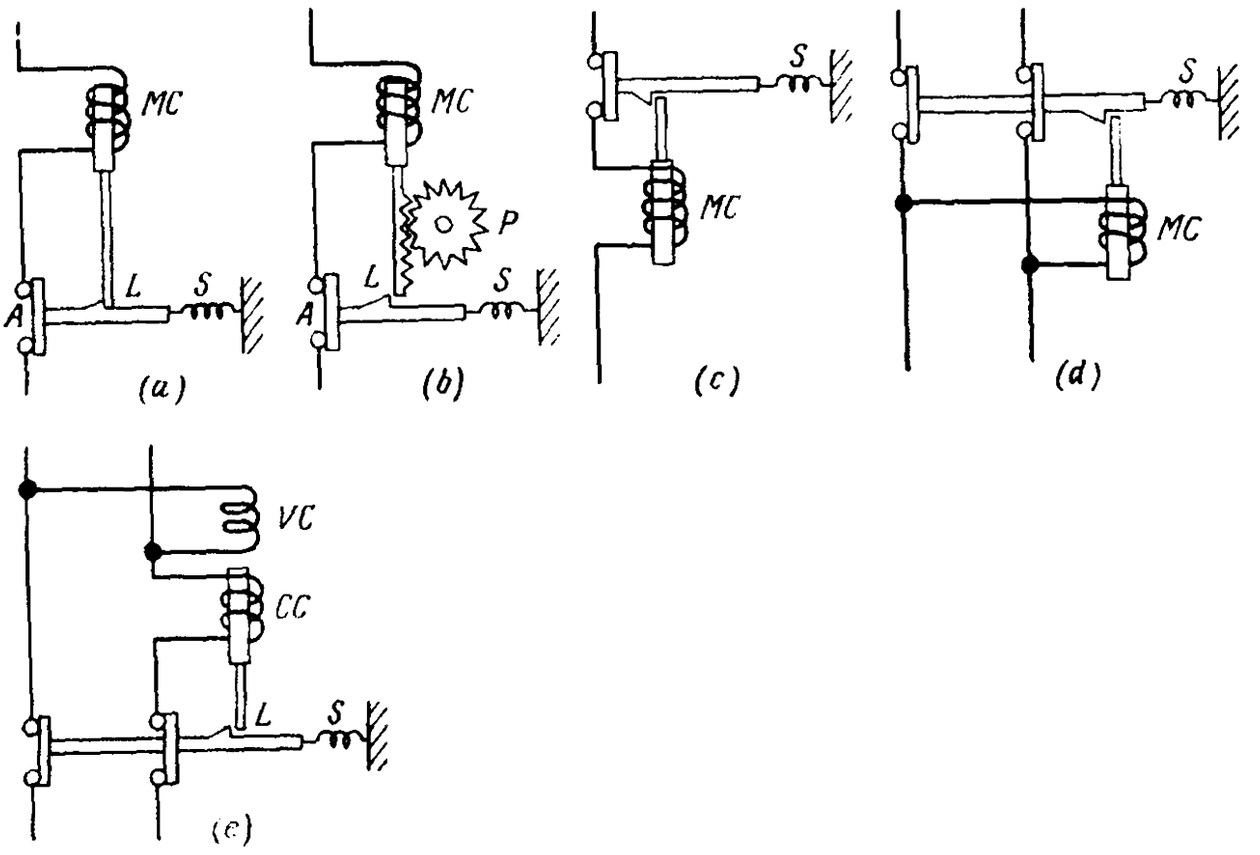

FIGURE 17.7

tected circuit reaches its limiting value, the magnet coil MC draws in an iron plunger, and a latch L releases a spring S; the latter pulls the contacts A apart and de-energizes the circuit. The figure illustrates only the basic features of the circuit breaker; actual designs and arrangements are extremely many and diverse.

Overcurrent circuit breakers are used in the lighting circuits of residences instead of fuses. An operated circuit breaker will usually be re-closed by hand. To avoid reclosure on a faulted circuit,

it is customary to equip overcurrent circuit breakers with a release-free (or trip-free) mechanism. Its function is to open the closing circuit of the circuit breaker even though the operator is holding the operating handle in the closed position. As often as not, automatic circuit breakers are fitted with a device to set the value of current at which the breaker should open. Obviously, this offers a more accurate means for setting the current limit than with the aid of fuses, which is a major advantage.

Sometimes automatic circuit breakers include *a time-delay feature*—with it, the automatic circuit breaker is allowed to open a definite time after an abnormal condition has occurred in the associated circuit or machine. In this way, one avoids de-energizing the circuit or a machine on a short-duration increase in current not dangerous to the installation (such as the starting current of a motor). The arrangement of a time-delay feature is shown in Fig. 17.7 b. A pinion P engages a rack R so that the magnet coil MC cannot draw in the plunger and release the latch L at once—before this can happen, the pinion has to turn through a definite angle. The time delay thus provided can be adjusted at will.

If the rise in current ceases before the time-delay assembly releases the latch, the plunger will regain its original position, and the circuit breaker will not open.

Apart from a rack-and-pinion combination, the desired time delay can be provided by an oil or an air dash-pot or some other means.

An *automatic undercurrent circuit breaker* is used in cases where a circuit must be de-energized whenever the current in the circuit or in any one of its arms (or branches) has dropped below a predetermined value. The operation of an undercurrent circuit breaker can be understood from reference to Fig. 17 7 c. The magnetic coil MC holds the plunger and latch until the current in the coil drops to a preset value. When this happens, the plunger falls by gravity, the latch releases the spring S, the spring pulls the contacts apart, and the circuit breaker opens.

An *automatic undervoltage circuit breaker* (Fig. 17.7 d) operates on about the same principle as an undercurrent circuit breaker. One of its uses is to protect motors fitted with a starting rheostat; an undervoltage circuit breaker will de-energize the associated motor(s), should the terminal voltage fall below a predetermined value. In the absence of an undervoltage circuit breaker a reduction in, or complete failure of, supply voltage would cause the motor to come to a stop, but the restoration of supply voltage would, with the starting rheostat brought out of circuit, give rise to a heavy starting current undesirable for proper operation of the supply system and detrimental to the motor itself. As often as not, it is so arranged that, on opening, the undervoltage circuit breaker acts to bring the starting rheostat into circuit automatically.

An *automatic discriminating* or *reverse current circuit breaker* may be used, for example, to protect generators operating in parallel when one of them tends to go motoring. How it works will be clear from reference to Fig. 17.7 e. When power transfer is in the normal direction, the current coil *CC* of the circuit breaker sets up a magnetic field which opposes that due to the voltage coil *VC*; as a result, neither of the coils can draw in the core and release a latch *L*. The situation is different when power transfer to or from one of the paralleled machines is reversed. Now the direction of current flow in *CC* is reversed, the two magnetic fields are combined, the core is drawn in, the latch is released, the contacts are parted, and the faulty generator is disconnected.

17.4 Isolators. High-Voltage Circuit Breakers

Isolators (also known as *disconnectors*, *isolating* or *disconnecting switches*) are mechanical switching devices used primarily for the purpose of isolating equipment from the source of power, for sectionalizing buses or circuits, and also for transfer, testing, and

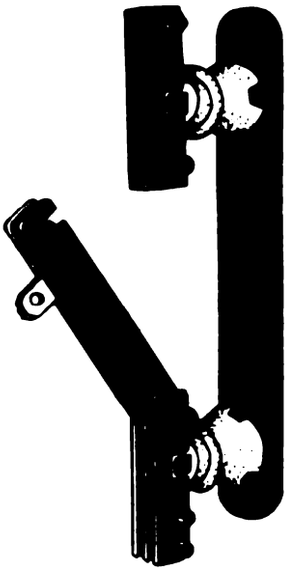

FIGURE 17.8

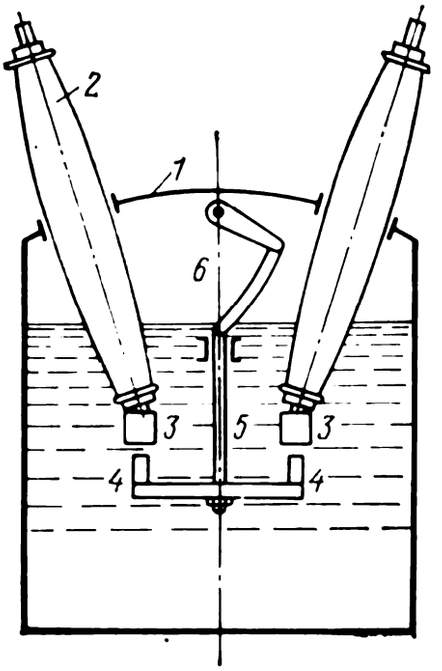

FIGURE 17.9

grounding purposes. They are not usually intended to break load currents, except under special conditions and in limited amounts.

An isolator has fixed and movable contacts set up on insulators. The contacts may be of the flat (knife-blade) type (Fig. 17.8) or of the turning type (usually employed in outdoor switchgear units for 35-500 kV).

Isolators may be operated manually, by motor or by air-actuated devices. For manual operation, the movable contacts have hooks

or rings which are engaged with an insulated operating rod for opening or closing. Heavy-duty isolators are usually motor- or air-operated from a remote position.

If an isolator were used to break a circuit before load has been thrown off by some other means, an electric arc would strike between the contacts and destroy them. To avoid this, the operating mechanisms of isolators are customarily fitted with an interlock which would not let the isolator be opened before the circuit has been opened by a circuit breaker.

In many cases the need exists to disconnect low-power loads at currents which do not exceed the normal operating value. It would be unwarranted to install an expensive and bulky circuit breakers (oil or air-blast) capable of breaking short-circuit currents. A better choice is to use a *load-breaking isolator*. In design, a load-breaking isolator is similar to a knife-blade off-load isolator (or disconnecting switch), but is additionally equipped with an arc-extinguishing device of some kind. They are used on voltages of 6 kV and 10 kV and can break currents not more than twice the operating current. For short-circuit protection, it is usual to connect load-breaking isolators in series with fuses.

High-voltage circuit breakers. Of course, it may seem a bit artificial to consider these switching devices separately from those taken up in Sec. 17.3. But the objective of the separation is simple—the circuit breakers grouped here are, firstly, not automatic and, secondly, they may be used for opening and closing electric circuits under abnormal and normal conditions.

We will take up two designs of circuit breakers, namely *oil circuit breakers* and *air circuit breakers*. Since they have often to operate on short circuits, it is important to know the short-circuit current that is likely to be handled. On this basis, the designer can then specify the electromagnetic and thermal endurance of a prospective circuit breaker in terms of the *rated power breaking capacity* and the *rated current breaking capacity* (the latter being the maximum short-circuit current that the device can reliably break at a specified transient recovery phase-to-phase voltage).

In *oil circuit breakers* the arc-extinguishing medium is mineral oil. They may be of the plain-break type and of the arc-controlled type. All oil circuit breakers, except the simple plain-break type, employ some form of arc control device (such as oil pots, cross-jets, or turbulators).

Plain-break oil circuit breakers are made for 6-10 kV and with a power breaking capacity of 150 MVA. As is shown in Fig. 17.9, a plain-break oil circuit breaker consists of an oil-filled tank 1 through the cover of which are carried porcelain or composition terminal bushings 2. The fixed contacts 3 at the bottom of the bushings are bridged by a pair of movable contacts 4 set up on a conduc-

ting crosshead which is carried by an insulated lift rod 5. The lift rod can be lifted by a lever system 6 in turn coupled to a long shaft rotated by a handwheel or some other device.

In common designs, the crosshead carrying the movable contacts drops by gravity, thus opening the breaker. Often, accelerating springs are used to increase the rate of opening. In the closed position the crosshead is held locked by a latch incorporated in the operating mechanism. When the latch is released, the lift rod and the crosshead move (or are moved) downwards and give two breaks per pole. More often, a plain-break oil circuit breaker will have more contacts and more crossheads so as to give four or even six breaks per pole. As the current circuit is interrupted, an electric arc strikes between the contacts. The high temperature of the arc causes some oil to evaporate and to form a gas bubble around the parting contacts. Thus, the contacts in a plain-break circuit breaker separate and the arc exists in a gas medium under a high pressure. This speeds up arc extinction because the rise in pressure improves the electric strength of the gas medium. As the contacts move apart, the length of the arc is increased, and this calls for a higher voltage to maintain the arc. In a.c. circuits where plain-break oil circuit breakers are mostly used, the current in the interrupted circuit crosses zero every half-cycle, and this speeds up arc extinction still more.

In one design of oil-blast circuit breakers, a radial oil-blast chamber is fastened to the lower ends of each breaker bushing. The chamber contains a stationary upper butt contact, a floating movable contact, and a hollow movable rod contact. As the breaker opens, the intermediate and lower contacts move downwards, first drawing an arc between the upper and intermediate contacts and later between the intermediate and lower contacts. The gas pressure developed in the upper chamber forces a blast of oil radially into the vent through the hollow rod. This blast of oil prevents the re-establishment of the arc after an early current zero. The gas pressure thus produced is 4-6 MPa, and the arc is allowed to exist for not more than 0.015-0.02 s (one cycle of an alternating current) at the rated interrupted current.

A still better performance is shown by oil circuit breakers incorporating arc-controlling devices in the form of ruptors or expulsion chambers. They use the general principle of extinguishing the arc quickly in a confined space by forcing oil through or along the arc by means of the gas pressure generated by the gas.

At voltages of 35 kV and higher preference is given to *air circuit breakers*. They may be air-break units using atmospheric air, and air-blast units employing compressed air. In air-blast circuit breakers one or both parting contacts are made hollow, and compressed air under a pressure of 0.7-2 MPa is forced through the space thus formed, and its blast blows the arc off the contact faces. As compared with

oil circuit breakers, they are lighter in weight and have a shorter opening time. Unfortunately, an air-blast circuit breaker needs an air compressor for its operation, and the compressed air must be duly purified and dried.

After it has opened the associated circuit, a circuit breaker remains under voltage which is a hazardous inconvenience because an operation of a circuit breaker is an indication that both the de-energized circuit and the breaker need inspection and, if necessary, servicing or even repair. To make it safe, it is customary to add an isolator (disconnecting switch) to each circuit breaker. With it, an operated circuit breaker can be readily disconnected from the source of high voltage.

17.5 Relays and Relay Protection

A *relay* is a device in which the sensing element responds to a change in some physical quantity (current, voltage, frequency, light intensity, temperature, pressure, etc.) by causing its control element to initiate the necessary switching operation in the controlled circuit and bring about the desired changes in the same or other physical quantity (current, voltage, etc.).

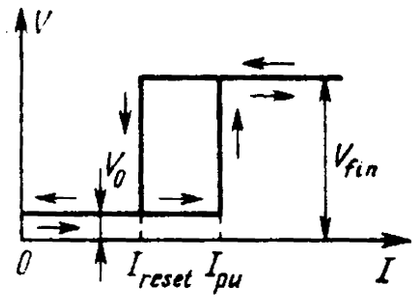

FIGURE 17.10

Electric protective relays protect apparatus and entire systems from dangerous conditions which may arise due to various causes (short circuits in electric apparatus, faults to ground, abnormal changes in voltage, reversal of power flow, and the like). To perform these functions, such relays are arranged in what are often extremely sophisticated protection systems. In addition to disconnecting a faulty piece of equipment from its power source, a relay protection system usually provides an indication of the fault, operates the associated circuit breakers so as to restore power to loads, effects a change-over (or transfer) to stand-by sources, and sheds some of the load if the supply system in question cannot deliver enough power.

It is usual to describe the performance of a relay in terms of its *control characteristic* which relates the controlled quantity (say, a voltage V) to the control quantity (say, a current I), such as shown in Fig. 17.10. So long as the control quantity remains below its *operate* or *pick-up value* (in our case, $I < I_{pu}$), the controlled quantity remains unchanged (in our case, $V = \text{const}$). When the control quantity becomes equal to its operate or pick-up value, the final control element of the relay brings about a stepwise change in the controlled quantity (by, say, closing or opening an electric circuit) so that it becomes equal to its *final* or *sealing value* (in our case,

V_{fin} or V_{seal}). Any further increase in the control quantity leaves the controlled quantity unchanged. The latter likewise remains unchanged so long as the control quantity exceeds what is known as its *reset* or *dropout value* (in our case, this is the reset or dropout current). At the instant when the control quantity drops to its reset value (in our case, $I = I_{\text{reset}}$), the relay causes the controlled quantity to fall back to its original value (in our case, V_0).

Taking our case, the ratio $I_{\text{reset}}/I_{\text{pu}} = k_{\text{reset}}$ is called the *reset ratio*. Depending on the principle of operation used and design, k_{reset} may be anywhere between 0.3 and 0.98.

To assure the reliable functioning of a relay, its *operating quantity* (not to be confused with the operate or pick-up quantity) must somewhat exceed the pickup or operate value. If the control quantity is current, the ratio $I_{\text{op}}/I_{\text{pu}} = k_{\text{safe}}$ is called the *safety factor*. Usually, k_{safe} ranges from 1 to 3.

According to the function(s) performed by relays, they are customarily classed into overcurrent and overvoltage relays, undercurrent and undervoltage relays, and differential relays. An *overcurrent* or *overvoltage relay* operates when the actuating quantity (current or voltage, respectively) exceeds its operate or pickup value. An *undercurrent* or *undervoltage relay* operates when the actuating quantity falls below the reset or dropout value. A *differential relay* functions by reason of the difference between two quantities of the same kind (two currents, two voltages, etc.).

The performance of a protective relay system is stated in terms of its selectivity, speed of response, reliability, and sensitivity.

The *selectivity of a protective relay system* consists in that a faulty piece of electric plant is disconnected from power sources by the nearest circuit breaker, so that the fault affects the normal supply of power to a minimum number of loads.

The *high speed of response* is essential so as to keep to a minimum the extent of damage caused to the affected circuit by the thermal action of electric current, to minimize the effect of an abnormal drop in voltage in the performance of other loads, to improve the quality of electric lighting, etc.

The *sensitivity of a protective relay system* is important because it is essential that the protective system should respond to a faulty condition at the earliest possible time, preferably at the very onset. In many cases, the sensitivity of a protective relay system is a most vital criterion of its usability.

The *reliability of a protective relay system* refers to its ability to operate under fault conditions without a hitch. A system using a smaller count of relays interacting in the simplest possible manner will be more reliable than a system using a greater number of relays interacting in a more elaborate fashion. One way to enhance the reliability of a protective relay system is through redundancy—

provision of a back-up protection system which would go into action, should the main one fail to do its job properly.

The requirements and features of protective relay systems are somewhat conflicting. For example, if we use simpler mechanisms in order to enhance system reliability, we will run the risk of impairing its sensitivity. By the same token, an improvement in selectivity through the use of time lag would impair the speed of response.

Yet, the use of time lag is one of the simplest ways to assure the desired selectivity. The necessary time lag, that is, the time elapsing between the instant when the sensing element of a relay recognizes a change in the actuating quantity (say, current) and the instant

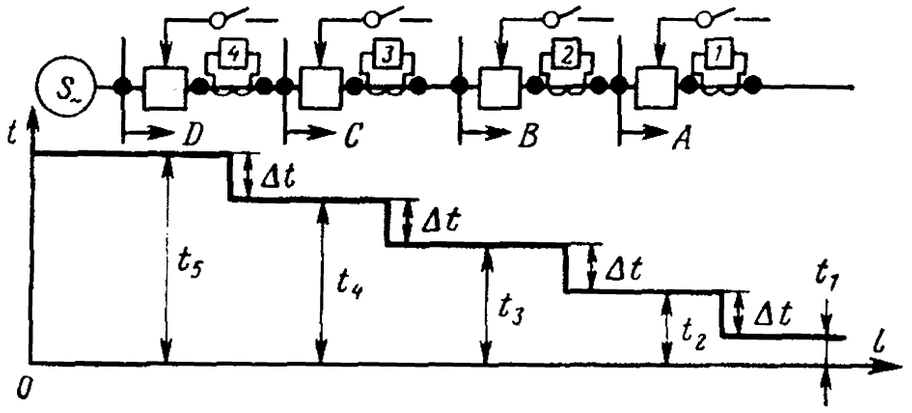

FIGURE 17.11

when the output element of the relay operates, can be provided by the natural sluggishness of the relay mechanism itself or by the inclusion of a special device or by the use of an auxiliary time relay which is part of the overall relay system. The time lag may be made dependent on or independent of the value of the actuating quantity (a current pulse in the case of a fault).

A simple example of selective relay protection is the use of several overcurrent relays some of which are made to respond with a time lag, as shown in Fig. 17.11. The figure shows a radial distribution network fed single-ended from a substation S. The desired selectivity is achieved through the use of what are called graded time-lags: the lags of the relays are set progressively longer for relays closer to the substation. The time lag of the relay protecting the farthest section A may be taken to be zero: its response time will be equal to the operate time of the relay proper, t_1 (for present-day relays, $t_1 = 0.02\text{-}0.12$ s).

Consider the simplest case of definite time-lag. In this case, the time lag of the relay protecting the farthest but one section B must be greater than t_1 by Δt called the *definite time-lag increment*. In the case of a fault within section A, all the relays on line AD will sense

the fault condition because the fault current flows through all of them. But their time-lags must be adjusted so that relay 1 operates and the arc in the breaker it has tripped is extinguished before any other relay operates. After relay 1 operates, the time count is discontinued in all the other relays, and all the line, except for section A, remains in service. Thus, the definite time-lag increment Δt must be somewhat longer than the total fault clearing time (usually 0.04-0.16 s). A major limitation of this simple selective protection scheme is the excessive time lag within the sections near the power source, as this leads to an impairment in the speed of response.

According to the actuating quantity used, relays may be classed into current relays, voltage relays, resistance relays (which respond to variations in the voltage-to-current ratio), and power-directional relays.

Direct-acting relays, as their name implies, cause the associated circuit-breaker to trip directly. In contrast, an *indirect-acting relays* do so through the intermediary of an auxiliary (control) current which energizes the trip coil of the associated circuit breaker.

Primary relays are connected directly in the circuits they are to protect. This simplifies the arrangement of a protection scheme, but since a primary relay is always under voltage, this poses the problem of keeping watch on its operational status.

Secondary relays are connected to the circuits to be protected via current and potential instrument transformers (see Sec. 8.17), so it is safe for attending personnel to service them. In most protection schemes, use is made of indirect-acting secondary relays.

The moving element of a relay may utilize any one of several effects of electric current. For example, the indirect-acting current relays may be the electromagnetic type and the induction type. In diagrammatic form, the arrangement of an electromagnetic instantaneous overcurrent relay is shown in Fig. 17.12. The relay has a core, 1, built up from electrical-sheet steel laminations and carrying a coil, 2, so that both make up an electromagnet. Between the poles of the electromagnet there is a shaft which mounts a Zee-shaped armature, 3, held in its initial position by a restoring spring, 4. When the armature is forced to move from its neutral position, the spring develops a restoring torque. One end of the spring is made fast to the armature shaft, and the other end is attached to an arm, 7, in turn joined to a pointer which sweeps over a scale, 8, and indicates the current that causes the relay to operate. When the relay is de-energized, the spring holds the armature against a limit stop, 9. When the torque developed by the current in coil 2 exceeds the restoring torque produced by the spring, the armature turns, and movable contacts, 5, come against fixed contacts, 6—the relay has operated. The operate current of the relay can be adjusted by varying the tension of the spring with arm, 7. This is an instantaneous relay,

but its contacts are designed to close low-power circuits—this is the reason why it may only be used as an indirect-acting relay. As the relay operates, its contacts complete the path for the current that energizes an *intermediate*, or *pilot*, *relay*. The latter has stronger contacts that complete the circuit for the current that energizes the trip coil of the associated circuit breaker. An intermediate, or pilot, relay may have several sets of make and break contacts, closing or opening the circuits to the trip coils of several circuit breakers.

If necessary, the pilot relay may be made to operate with a time delay. This time delay is obtained by putting a short-circuited turn on the relay core. The short-circuited turn slows down the build-up

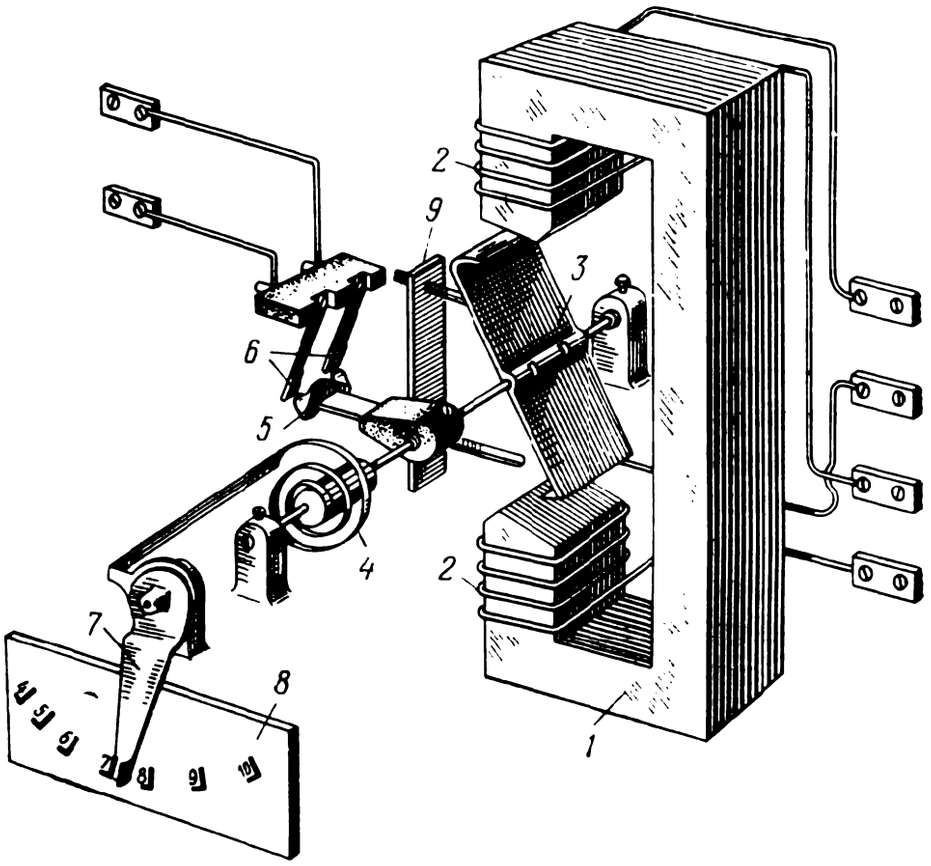

FIGURE 17.12

of the magnetic flux so that a time delay of 0.1-0.2 s is obtained. Where a longer time delay is required, resort is made to an external time-delay relay.

Induction relays are often used as adjustable-time-delay, direct-acting secondary relays. The moving element of such a relay is similar to that of an induction electricity meter (see Sec. 12.9). An induction relay has relatively strong contacts so its output unit may close the operating-current circuit of a circuit breaker directly. With this arrangement, the protection scheme needs neither a pilot relay nor a time-delay relay, but its selectivity and sensitivity are reduced.

A relay protection scheme has two groups of electric circuits: a.c. circuits connecting the relays to sources of data about the status of the equipment to be protected, and control current circuits responsible for the operation of circuit-opening devices in a desired sequence and with a desired selectivity.

Control-current sources (d.c. or a.c.) may be dependent on or independent of the operating duty and condition of the primary (power) circuits of the protected plant. Dependent control-current sources are potential transformers, current transformers, and special-purpose transformers. Independent control-current sources are storage batteries, but their use materially complicates the arrangement and maintenance of the protected equipment, for which reason they are only used at major and especially critical installations.

A relatively independent supply of control current is provided by banks of capacitors with a capacitance of 25-500 μF and a voltage of up to 400 V. Under normal conditions, the capacitors are charged by potential transformers via rectifiers. They can store their charge for several hours. In the case of a fault in the protected plant, they can supply the necessary control current for a time sufficient for all form of protection to operate. Importantly, capacitors will enable the protective gear to operate even in the case of a complete voltage failure in the a.c. line feeding them.

17.6 Contactors, Magnetic Starters. Controllers

A contactor is essentially a relay because it is an electromagnetic switching device which operates when the circuit feeding its control current is completed or broken. In contrast to relays, however, contactors are designed to handle substantially heavier currents, sometimes at a relatively high voltage. They are used to control large motors, heating appliances and similar high-power loads. Thus, contactors are direct-acting relays, and their electromagnets must be designed for longer plunger stroke (travel) and larger pull (or traction).

A contactor is operated by pressing a button in the control-current circuit (push-button control).

The arrangement of a d.c. electromagnetic contactor is shown in Fig. 17.13. When the control current energizes the operating coil, 1, of the contactor, its magnet core, 2, attracts a steel armature, 3. Turning on its pivot, 4, the armature closes the main contacts, 5, in the power circuit. There is a compression spring, 6, which causes the movable contact to seal reliably against the fixed contact. The power circuit is connected to the power-current terminals, 7, and includes the main (power) contacts, a flexible conductor, 8, and a magnetic blow-out coil, 9. The field of this coil is used to aid the interruption of the arc drawn between the main (power) contacts.

The contactor also has *auxiliary contacts*, 10, placed in the control-current circuit and intended to perform auxiliary control operations.

Alternating-current contactors are usually employed as part of *magnetic starters*. Most often, a magnetic starter combines one or several magnetic contactors, thermal relays, and control buttons. The circuit of a nonreversible a.c. magnetic starter is shown in Fig. 17.14. The starter is operated with two buttons, one labelled "Start" and the other, "Stop", which are ocated outside the box

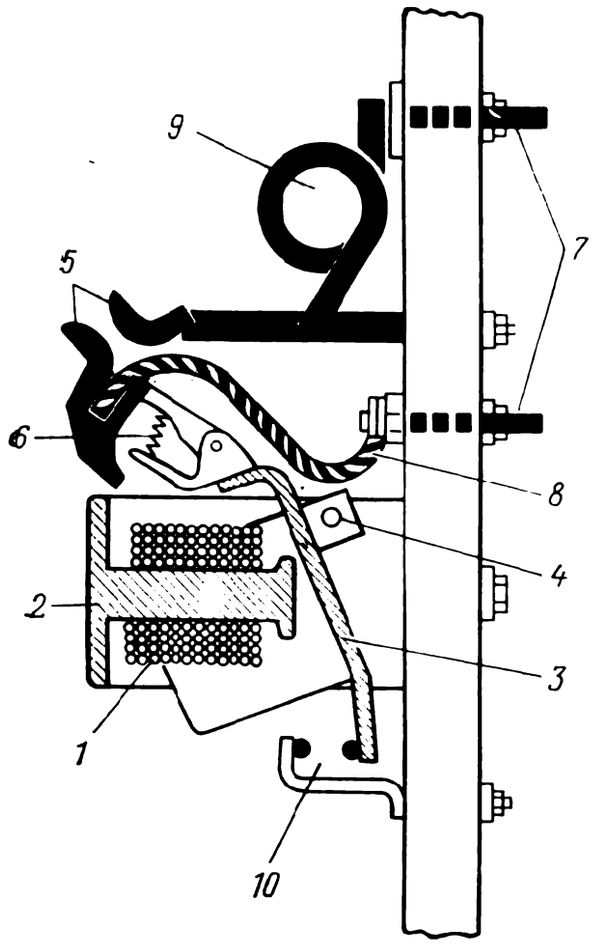

FIGURE 17.13

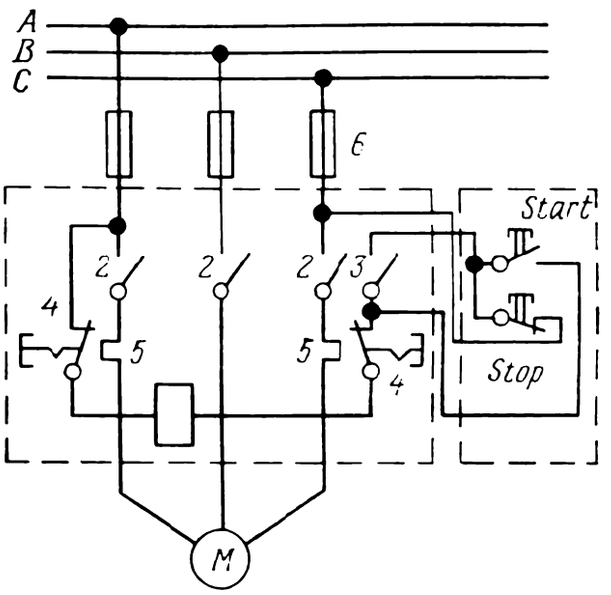

FIGURE 17.14

enclosing the starter. Pressure on the "Start" button completes the control-current circuit from the phase *A* wire of the supply line, via contacts 4 of one thermal relay 5, contactor coil 1, contacts 4 of a second thermal relay 5, the contacts of the "Start" and "Stop" buttons to the phase *C* wire of the supply line. With its operating coil energized, the electromagnet of the three-pole contactor attracts the armature which is made fast to the main (power) contacts 2 (giving two breaks per phase in the power-current circuit) and the auxiliary contacts 3 in the control-current circuit. The motor is started after all of these contacts have closed.

On closing, auxiliary contacts 3 enable the control current to by-pass the contacts of the "Start" button. In this way, they prevent

the interruption of the control current when the "Start" button is released once the motor is started.

When the operator presses the "Stop" button, this breaks the circuit feeding the control current to the operating coil 1 of the contactor, the armature drops out, the spring forces the power contacts, 2, to part, and the motor is stopped.

The two bimetallic thermal relays 5 provided in the magnetic starter provide overload protection (see Fig. 17.6). Due to a con-

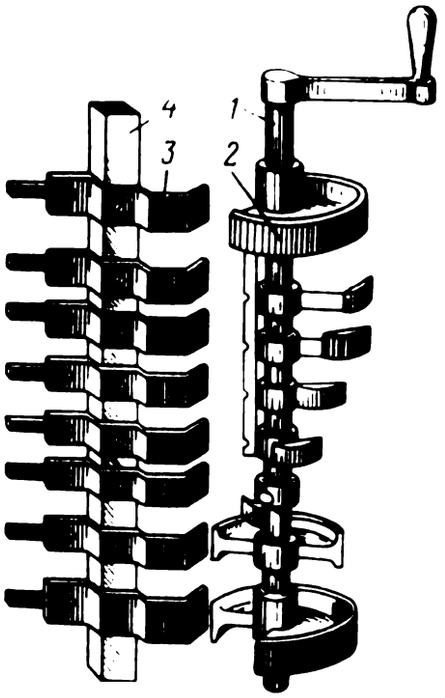

FIGURE 17.15

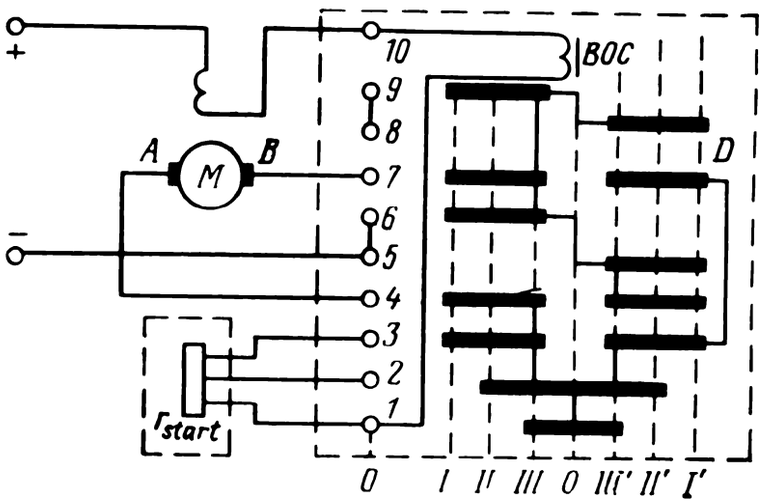

FIGURE 17.16

siderable thermal lag, however, these relays cannot protect the motor against short-circuit currents. This is done by installing fuses 6 in the power-current circuit.

A *controller* (or, more accurately, an *electric controller*) is a device or a group of devices that serve to govern in some predetermined manner the electric power delivered to the apparatus to which it is connected. The functions it can perform include starting, frequency and/or speed control, reversals, braking, and stopping. As often as not, a controller is actuated by rotation of a handwheel or a knob—a fact which simplifies the job of an electric locomotive's driver or an electric crane's operator.

The most commonly used form of electric controller is the *drum controller* shown in Fig. 17.15. It comprises a rotatable drum 1, which carries movable contact strips 2 (usually made of copper) usually displaced through different angles with respect to one another, with some of the strips electrically interconnected. Heavy contact fingers 3, secured to a stationary base 4, bear on the respective contact strips to make the required electrical connections as the drum is rotated.

The contact fingers are insulated from one another and soldered to the wires that connect the controller to the unit it controls.

Figure 17.16 shows a developed circuit diagram which illustrates the use of a drum controller to control a series-wound d.c. motor. Here the fixed contact fingers (at 3 in Fig. 17.15) are shown as a vertical row of circles labelled 1 through 10. The dashed box *D* encloses a developed circuit diagram of the controller drum; the strips represent the contact segments on the drum. The drum has seven fixed positions numbered I, II, III, 0, III′, II′, and I′. When

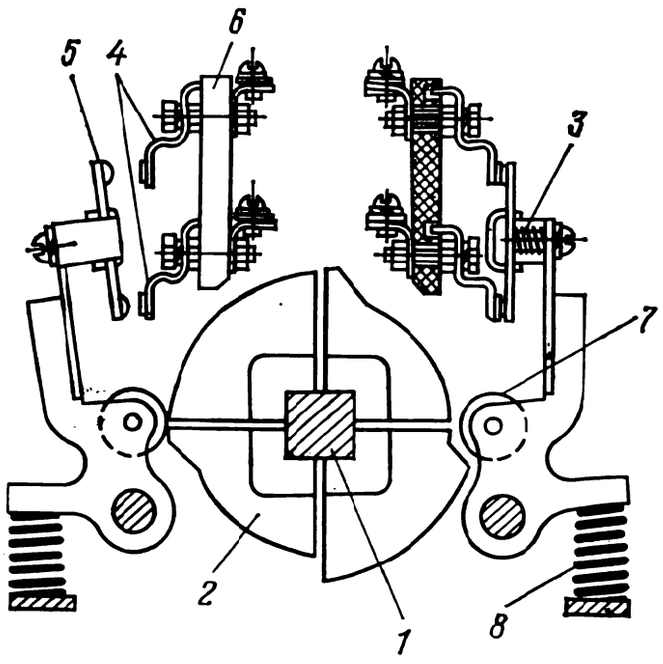

FIGURE 17.17

Control diagram of cam controller

Contact No.	Forward				Backward		
	I	II	III	0	III′	II′	I′
1				×			
2	×	×	×		×	×	×
3					×	×	×
4	×	×	×				
5	×	×				×	×
6	×						×

FIGURE 17.18

the drum is in the "0" position, the motor is de-energized because all contact fingers only touch the insulated surface of the drum. When the controller is moved to position I, the vertical line marked I in the diagram is lined up with the vertical row of contact fingers. Now the current path is completed through the series field winding of the motor, the magnetic blowout coil *BOC* (which serves to speed up the extinction of the arc striking between the movable contact segments and the stationary contact fingers of the controller), contact finger 1, the two halves of the starting rheostat r_{start}, contact finger 3, two contact segments on the drum, contact finger 4, brush A, the armature of motor M, brush B, contact finger 7, two contact segments on the drum, and finger 6.

It is easy to see that as the drum is moved to position II, its contact segments short out half the starting rheostat r_{start}. In position III, all of the starting rheostat is shorted out, so the full supply-line voltage is applied to the motor terminals.

Rotating the drum from position 0 to position I′, that is, in the opposite direction, this will reverse the direction of current flow

through the motor armature as compared with that existing in positions I through III, and the motor will run in the opposite direction.

If the starting rheostat is designed to carry the load current of the motor for a long time, the controller may also be used to adjust the speed of the motor.

Drum controllers may only be used for infrequent switching operations. More frequent operations are performed with *cam controllers* which are built around cam-type contactors. In sketch form, a cam controller for two circuits is shown in Fig. 17.17. Its shaft 1 mounts insulated control cams 2. The necessary pressure of bridge-type movable contacts 5 against the stationary contacts 4 set up on an insulating baseboard is provided by two springs 3. As the shaft is rotated, the lobe of a cam brings pressure to bear upon a roller 7, and this pushes away the movable contacts, thus breaking the controlled circuit at two points. With the further rotation of the shaft, the lobe of the cam comes off the roller, spring 8 rotates the arm carrying the movable contacts, and the circuit is closed.

The shaft can be fitted with a multiplicity of cam units so that many circuits can be controlled at the same time (up to 12 circuits in the controllers used on industrial electric locomotives).

The sequence in which a cam controller operates can be readily followed by reference to its control diagram such as shown in Fig. 17.18. The contacts are numbered vertically, and the crosses in the horizontal rows indicate the positions when the contacts are closed.

There are switches which are operated by some part or motion of a power-driven machine or equipment in order to alter connections in the electric circuits of the machine or equipment. Since the switching action takes place at the end or somewhere along the way of travel of the controlled machine, these devices are called limit switches. Basically, they are cam switches, with their cams breaking the power circuit when they run against, or are pressed by, a part of the driven machine.

We have described only the relatively simple types of electric apparatus. At present, most users employ specialized designs and makes adapted to the needs of a particular application or industry (mining, metal-cutting, building construction, etc.). They are described in detail in the literature on these subjects.

17.7 Power Supply Networks

Electric energy generated by large power stations is delivered to consumers' loads, often situated a long distance from a station, over power transmission lines operating at voltages in the range 35-750 kV. The transmission lines terminate at *large main substations*

from which the power is distributed to small secondary substations where the voltage is steppeb down to 6 or 10 kV (Fig. 17.19). A substation includes step-down transformers connected to the incoming transmission line via circuit breakers and isolators. On the H.V. side

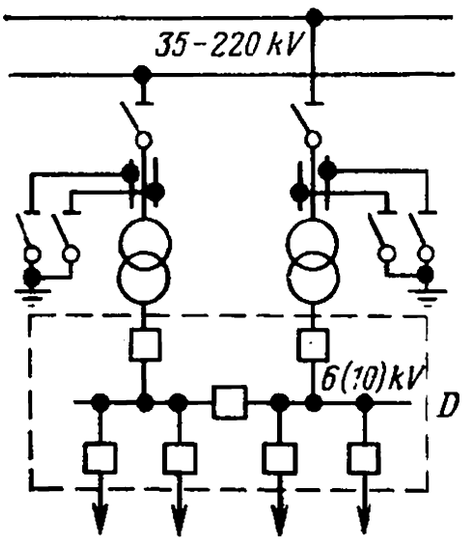

FIGURE 17.19

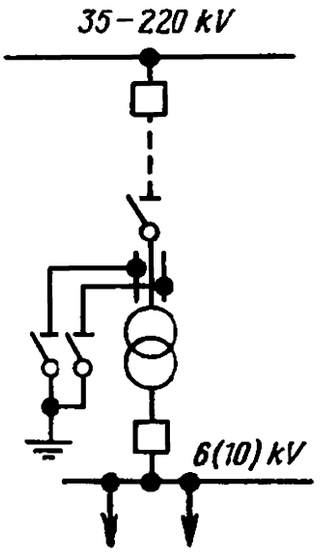

FIGURE 17.20

the transformers are fitted with short-circuiters which are used when repair or inspection has to be carried out.

On the L.V. (6-10 kV) side the transformers are connected to a *distributor*, *D*, which has sectionalized buses connected via circuit

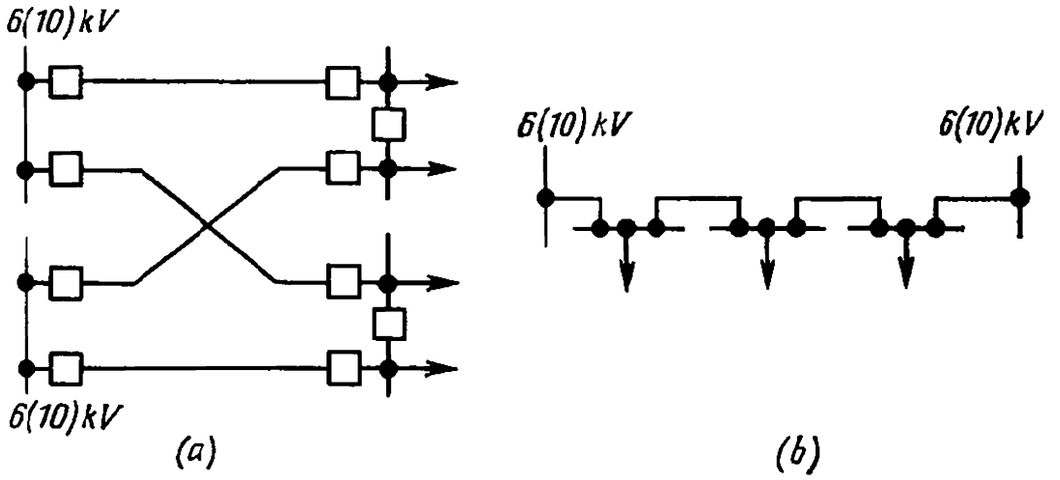

FIGURE 17.21

breakers to individual consumers. Distributors may also be set up in other parts of a distribution network where no voltage transformation may be needed.

Since most loads are designed for operation at a voltage of less than 1000 V, it is usual to locate *secondary transformer substations* nearby. These substations are equipped with step-down transformers and all the necessary switching apparatus.

Large consumer loads, such as factories, hotels, and the like, commonly take their supply directly from the primary system at one of the primary distribution voltages which is then stepped down by a substation located on the premises (Fig. 17.20).

In configuration, secondary distribution networks may be classed into *radial* (Fig. 17.21 a), *parallel* (Fig. 17.21 b), and *mixed*. The choice of a particular configuration depends on how a given consumer load is critical. In this respect, it is customary to divide them in the USSR into three categories: first, second, and third (in the descending order of importance).

Category I loads will be usually supplied over a radial distribution network in which the main power supply is backed up by a stand-by one. Loads in categories II and III may use simpler versions of the radial configuration. This also applies to the parallel and mixed network configurations.

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